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On Some Explicit Constructions of Finsler Metrics with Scalar Flag Curvature

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Abstract. We give an explicit construction of polynomial (*of arbitrary degree*) (α,β)-metrics with scalar flag curvature and determine their scalar flag curvature. These Finsler metrics contain all non-trivial projectively flat (α,β)-metrics of constant flag curvature.

1 Introduction

The flag curvature in Finsler geometry is an analogue of sectional curvature in Riemannian geometry which was first introduced by L. Berwald [1, 2]. It measures the rate of changes of the Finsler metric modulo the changes of the geometry of tangent spaces. For a Finsler manifold (M, F), the flag curvature is a function $K(P, y)$ of tangent planes $P \subset T_xM$ and directions $y \in P$. F is said to be of *scalar curvature* if the flag curvature $K(P, y) = K(x, y)$ is independent of flags P associated with any fixed flagpole y. One of important problems in Finsler geometry is to study Finsler metrics of scalar curvature because Finsler metrics of scalar curvature are the natural extension of Riemannian metrics of isotropic sectional curvature (which are of constant sectional curvature in dimension $n \geq 3$ by the Schur Lemma). See [3, 6, 7] for some recent developments.

(α,β)-metrics (for definition, see Section 2) form a special class of Finsler metrics. Most important, they are "computable" although the computation sometimes runs into very complicated situations.

The aim of this paper is to explicitly construct Finsler metrics of scalar curvature and determine their scalar flag curvature. For (α,β)-metrics in the polynomial form, we obtain the following:

Theorem 1.1 *Let $\phi(s)$ be a polynomial function defined by*

$$\phi(s) = 1 + 2^n s + 2n\Sigma_{k=0}^{n-1} \frac{(-1)^k C_k^{n-1} s^{2k+2}}{(2k+1)(2k+2)} \tag{1.1}$$

where

$$C_k^m := \frac{m(m-1)\cdots(m-k+1)}{k!}.$$

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Then the following polynomial (α,β) *metric on an the open subset at origin in* $\mathbb{R}^n$

$$F := \frac{(1+\langle a,x\rangle)^{2n}}{(1-|x|^2)^{n+1}}\sqrt{(1-|x|^2)|y|^2+\langle x,y\rangle^2} \times \phi\Big(\frac{(1-|x|^2)\langle a,y\rangle+(1+\langle a,x\rangle)\langle x,y\rangle}{(1+\langle a,x\rangle)\sqrt{(1-|x|^2)|y|^2+\langle x,y\rangle^2}}\Big) \tag{1.2}$$

is of scalar curvature with flag curvature

$$\begin{aligned} K = &-\frac{(n+1)|y|^2}{F^2\omega^2}+\frac{(n^2-1)\langle x,y\rangle^2}{F^2\omega^4}-\frac{\zeta^{2n-4}\psi^2\phi''}{2\theta F^3\omega^{2n+2}} \\ &+\frac{\zeta^{2n-2}}{4F^4\omega^{4n+4}}(2n\langle a,y\rangle\theta\phi\zeta+\phi'\psi)[4(n+1)F\langle x,y\rangle\omega^{2n}+3\zeta^{2n-2}] \\ &-(n\langle a,y\rangle\theta\phi\zeta+\phi'\psi)\frac{(2n-1)\langle a,y\rangle\zeta^{2n-3}}{F^3\omega^{2n+2}}, \end{aligned}$$

where $n\in\{0,1,2,\dots\}$, $a\in\mathbb{R}^n$ *is a constant vector with* $|a|<1$ *and*

$$\begin{aligned} \omega &:= \sqrt{1-|x|^2}, \\ \theta &:= \sqrt{(1-|x|^2)|y|^2+\langle x,y\rangle^2}, \\ \zeta &:= 1+\langle a,x\rangle, \\ \psi &:= \zeta^2|y|^2-2\langle a,y\rangle\langle x,y\rangle\zeta-\omega^2\langle a,y\rangle^2, \\ \phi^{(i)} &:= \phi^{(i)}\Big(\frac{(1-|x|^2)\langle a,y\rangle+(1+\langle a,x\rangle)\langle x,y\rangle}{(1+\langle a,x\rangle)\sqrt{(1-|x|^2)|y|^2+\langle x,y\rangle^2}}\Big), \end{aligned}$$

where $\phi^{(i)}$ *denotes i-order derivative for* $\phi(s)$.

Our construction includes the following two important special cases:

(a) when $n=0$, then

$$F=\frac{\sqrt{(1-|x|^2)|y|^2+\langle x,y\rangle^2}}{1-|x|^2}+\frac{\langle x,y\rangle}{1-|x|^2}+\frac{\langle a,y\rangle}{1+\langle a,x\rangle}$$

is of constant negative flag curvature $K=-\frac{1}{4}$ [10]. These (α,β)-metrics are generalized Funk metrics. In particular, when $a=0$, they are the well-known Funk metrics on the unit ball $\mathbb{B}^n(1)$;

(b) when $n=1$, then

$$F=\frac{[(1+\langle a,x\rangle)\big(\sqrt{(1-|x|^2)|y|^2+\langle x,y\rangle^2}+\langle x,y\rangle\big)+(1-|x|^2)\langle a,y\rangle]^2}{(1-|x|^2)^2\sqrt{(1-|x|^2)|y|^2+\langle x,y\rangle^2}}$$

is of zero flag curvature $K=0$. These (α,β)-metrics were constructed in [8]. In particular, when $a=0$, they are reduced to Berwald's metric [2].

From a classification theorem due to [5], (1.2) contains all non-trivial projectively flat (α, β)-metrics of constant flag curvature.

In the proof of Theorem 1.1, we use some interesting properties lying in examples (a) and (b) (*cf.* Section 2, Lemma 2.5). Hence these projectively flat Finsler metrics are a natural extension of examples (a) and (b).

Our approach is to find the new solutions of ordinary differential equations related to projective flatness of (α, β)-metrics (see Theorem 2.1). A Finsler metric $F = F(x, y)$ on an open subset $\mathcal{U} \subset \mathbb{R}^n$ is said to be *projectively flat* if all geodesics are straight in $\mathcal{U}$. Recall that all projectively flat Finsler metrics are of scalar flag curvature [4, Proposition 6.1.3]. In fact, we obtain more Finsler metrics of scalar curvature and determine their flag curvature by using Lemma 2.5 (see Section 5). These Finsler metrics are close to examples (a) and (b) in the some sense.

2 (α, β)-metrics

Consider the function

$$F := \alpha\phi(s), \quad s = \frac{\beta}{\alpha}, \tag{2.1}$$

where $\phi = \phi(s)$ is a positive C^∞ function on $(-b_o, b_o)$ satisfying

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad |s| \le b \le b_o.$$

Then by Lemma 1.1.2 in [4], F is a Finsler metric if $\|\beta_x\|_\alpha < b_o$ for any $x \in M$. A Finsler metric in the form (2.1) is called an *(α, β)-metric*.

The following result tells us a approach to find projectively flat (α, β)-metric. We will manufacture Finsler metrics of scalar curvature using it.

Theorem 2.1 ([11, Theorem 1.2]) *Let $F = \alpha\phi(\beta/\alpha)$ be an (α, β)-metric on an open subset $\mathcal{U} \subset \mathbb{R}^n$. Assume that $\phi = \phi(s)$ satisfies*

$$\phi(s) - s\phi'(s) = (p + rs^2)\phi''(s) \quad (r \neq 0)$$

where p, r are constants. Define

$$\alpha := \rho(h)\alpha_\mu, \quad \beta := C_2\rho(h)^{r+1}dh$$

where

$$\alpha_\mu = \frac{\sqrt{|y|^2 + \mu(|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 + \mu|x|^2}$$

where

$$\rho(t) = \Big[-\frac{2r(C_2)^2}{p}\Big(C_3 + \eta t - \frac{1}{2}\mu t^2\Big)\Big]^{-\frac{1}{2r}}$$

$$h := \frac{1}{\sqrt{1 + \mu|x|^2}}\Big\{C_1 + \langle a, x\rangle + \frac{\eta|x|^2}{1 + \sqrt{1 + \mu|x|^2}}\Big\}$$

where η and C_i are constants ($C_2 > 0$) and $a \in \mathbb{R}^n$ is a constant vector. Then F is projectively flat.

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In order to find projectively flat (α, β)-metrics we consider the following ordinary differential equation

$$\begin{cases} \phi - s\phi' = (p + rs^2)\phi'' \\ \phi(0) = 1, \phi'(0) = \epsilon. \end{cases} \tag{2.2}$$

Lemma 2.2 *When $rp \neq 0$, then the solution of (2.2) is*

$$\phi_{r,p} = 1 + \epsilon s + \frac{1}{p}\int_0^s \int_0^\tau \left(1 + \frac{r}{p}\sigma^2\right)^{-\frac{1}{2r}-1} d\sigma\, d\tau \tag{2.3}$$

Furthermore, if $f = f(s)$ satisfies

$$\begin{cases} f''(s) = \frac{1}{p}\left(1 + \frac{r}{p}s^2\right)^{-\frac{1}{2r}-1} \\ f(0) = 1, f'(0) = \epsilon. \end{cases}$$

Then $f = \phi_{r,p} = 1 + \epsilon s + \frac{1}{p}\int_0^s \int_0^\tau \left(1 + \frac{r}{p}\sigma^2\right)^{-\frac{1}{2r}-1} d\sigma\, d\tau$.

Proof We have the expansion

$$(1 + x)^\lambda = \Sigma_{k=0}^\infty C_k^\lambda x^k, \quad x \in (-1, 1),$$

where

$$C_k^\lambda = \frac{\lambda(\lambda - 1)\cdots(\lambda - k + 1)}{k!}.$$

By simple calculations, we have

$$\frac{\lambda}{k+1}C_k^{\lambda-1} = C_{k+1}^\lambda,$$

$$\phi_{r,p}(s) = 1 + \epsilon s + \Sigma_{k=0}^\infty \frac{C_k^{\lambda-1}t^k s^{2k+2}}{p(2k+1)(2k+2)},$$

$$\phi'_{r,p}(s) = \epsilon + \Sigma_{k=0}^\infty \frac{C_k^{\lambda-1}t^k s^{2k+1}}{p(2k+1)},$$

where $\lambda = -\frac{1}{2r}$, $t = \frac{r}{p}$. Thus we have

$$\phi_{r,p}(0) = 1, \quad \phi'_{r,p}(0) = \epsilon,$$

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and

$$\begin{aligned}
\phi_{r,p}(s) - s\phi'_{r,p}(s) &= 1 + \Sigma_{k=0}^{\infty}\Big[\frac{C_k^{\lambda-1}t^k}{p(2k+1)(2k+2)} - \frac{C_k^{\lambda-1}t^k}{p(2k+1)}\Big]s^{2k+2}\\
&= 1 + \Sigma_{k=0}^{\infty}\frac{kC_k^{\lambda-1}t^{k+1}s^{2k+2}}{k+1}\\
&= 1 + \Sigma_{k=0}^{\infty}C_k^{\lambda}(ts^2)^n\\
&= (1+ts^2)^{\lambda}\\
&= \Big(1+\frac{r}{p}s^2\Big)^{-\frac{1}{2r}}\\
&= (p+rs^2)\frac{1}{p}\Big(1+\frac{r}{p}s^2\Big)^{-\frac{1}{2r}-1}\\
&= (p+rs^2)\phi''_{r,p}(s).
\end{aligned}$$

It follows that $\phi_{r,p}$ satisfies (2.2). ∎

Lemma 2.3 *Suppose that $\phi_{r,p}$ is given in Lemma 2.2. Then there exists $b_o > 0$ such that*

$$\phi_{r,p}(s) > 0, \quad \phi_{r,p}(s) - s\phi'_{r,p}(s) + (b^2 - s^2)\phi''_{r,p}(s) > 0$$

where s and b are arbitrary numbers with $|s| \leq b \leq b_o$.

Proof Note that $\phi_{r,p}(0) = 1$. Hence we have $b_1 > 0$ satisfying

$$\phi_{r,p}|_{(-b_1,b_1)} > 0.$$

From the proof of Lemma 2.2, we have the following

$$\phi_{r,p}(s) - s\phi'_{r,p}(s) = \Big(1+\frac{r}{p}s^2\Big)^{-\frac{1}{2r}},$$

$$\phi''_{r,p}(s) = \frac{1}{p}\Big(1+\frac{r}{p}s^2\Big)^{-\frac{1}{2r}-1}.$$

Thus we obtain

$$\phi_{r,p}(s) - s\phi'_{r,p}(s) + (b^2 - s^2)\phi''_{r,p}(s) = \Big(1+\frac{r}{p}s^2\Big)^{-\frac{1}{2r}-1}\Big(1+\frac{b^2}{p}+\frac{r-1}{p}s^2\Big).$$

Set

$$b_2 = \begin{cases}\sqrt{-\frac{p}{r}} & \text{if } \frac{p}{r} < 0,\\ \infty & \text{if } \frac{p}{r} > 0.\end{cases}$$

Then

$$1 + \frac{r}{p}s^2 > 0$$

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for $|s| < b_2$. On the other hand,

$$1+\frac{b^2}{p}+\frac{r-1}{p}s^2 > \begin{cases} 1+\frac{b^2}{p} & \text{if } \frac{r-1}{p}s^2>0,\\ 1+\frac{rb^2}{p} & \text{if } \frac{r-1}{p}s^2<0,\end{cases}$$

for $|s| < b$. It follows that

$$1+\frac{b^2}{p}+\frac{r-1}{p}s^2 \geq \min\Big\{1+\frac{b^2}{p}, 1+\frac{rb^2}{p}\Big\}.$$

We put

$$b_3 = \begin{cases} \sqrt{-p} & \text{if } p<0,\\ \infty & \text{if } p>0.\end{cases}$$

Then

$$1+\frac{b^2}{p}+\frac{r-1}{p}s^2 > 0$$

whenever $|s| < b < \min\{b_2, b_3\}$. Finally we take

$$b_o = \min\{b_1, b_2, b_3\}.$$

Our b_o satisfies the requirement of Lemma 2.3. ■

Together with Lemma 1.1.2 in [4], we have the following:

Corollary 2.4 *Suppose that $\phi_{r,p}$ is given in Lemma 2.2. Then there exists $b_o > 0$ such that*

$$F := \alpha\phi_{r,p}(\beta/\alpha)$$

is a Finsler metric when $\|\beta\|_\alpha < b_o$.

A function f defined on TM can be locally expressed as $f(x^1, \dots, x^n; y^1, \dots, y^n)$. We employ the convention that f_{x^j} denotes the partial derivative of f with respect to the coordinate x^j. In the following we explore some interesting properties of θ, ω, ζ and ψ in Theorem 1.1.

Lemma 2.5 *Suppose that θ, ω, ζ and ψ are defined on $T\mathbb{B}^n(1)\backslash\{0\}$ in Theorem 1.1. Then*

(i) $\theta_{x^j} y^j = 0$;
(ii) $(\frac{\omega^2\langle a,y\rangle+\zeta\langle x,y\rangle}{\zeta\theta})_{x^j} y^j = \frac{\psi}{\zeta^2\theta}$;
(iii) $\psi_{x^j} y^j = 0$;
(iv) $\psi = \zeta^2 F_a(\theta - \langle x, y\rangle - \frac{\omega^2\langle a,y\rangle}{\zeta})$

where $F_a := \frac{\theta+\langle x,y\rangle}{\omega^2} + \frac{\langle a,y\rangle}{\zeta}$ is the generalized Funk metric.

Proof By direct calculations one obtains

$$(\omega^2)_{x^j} y^j = -2\langle x, y\rangle, \tag{2.4}$$

$$\langle x, y\rangle_{x^j} y^j = |y|^2, \tag{2.5}$$

$$\zeta_{x^j} y^j = \langle a, y\rangle. \tag{2.6}$$

(i)
$$\begin{aligned} 2\theta\theta_{x^j} y^j = (\theta^2)_{x^j} y^j &= (\omega^2|y|^2 + \langle x, y\rangle^2)_{x^j} y^j \\ &= -2\langle x, y\rangle|y|^2 + 2\langle x, y\rangle|y|^2 = 0. \end{aligned}$$

It follows that $\theta_{x^j} y^j = 0$.

(ii)
$$\begin{aligned} &\Big(\frac{\omega^2\langle a, y\rangle + \zeta\langle x, y\rangle}{\zeta\theta}\Big)_{x^j} y^j \\ &\quad = \Big(\frac{1}{\theta}\Big)_{x^j} y^j \frac{\omega^2\langle a, y\rangle + \zeta\langle x, y\rangle}{\zeta} + \frac{1}{\theta}\Big(\frac{\omega^2\langle a, y\rangle + \zeta\langle x, y\rangle}{\zeta}\Big)_{x^j} y^j \\ &\quad = \frac{1}{\theta\zeta^2}(-2\langle x, y\rangle\langle a, y\rangle\zeta + \zeta^2|y|^2 - \omega^2\langle a, y\rangle^2) = \frac{\psi}{\zeta^2\theta} \end{aligned}$$

from (i) and (2.4)–(2.6).

(iii) By using (2.4), (2.5) and (2.6), we have

$$\begin{aligned} \psi_{x^j} y^j &= (-2\langle x, y\rangle\langle a, y\rangle\zeta + \zeta^2|y|^2 - \omega^2\langle a, y\rangle^2)_{x^j} y^j \\ &= 2\zeta\langle a, y\rangle|y|^2 - 2\langle x, y\rangle\langle a, y\rangle^2 - 2\zeta\langle a, y\rangle|y|^2 + 2\langle x, y\rangle\langle a, y\rangle^2 \\ &= 0. \end{aligned}$$

(iv) It is easy to see that

$$\omega^2 F_0^2 - \langle x, y\rangle F_0 - |y|^2 = 0,$$

where F_0 is the well-known Funk metric. It follows that

$$\begin{aligned} \frac{\psi}{\zeta^2} &= |y|^2 - 2\langle x, y\rangle\frac{\langle a, y\rangle}{\zeta} - \omega^2\frac{\langle a, y\rangle^2}{\zeta^2} \\ &= |y|^2 - 2\langle x, y\rangle(F_a - F_0) - \omega^2(F_a - F_0)^2 \\ &= |y|^2 + 2\langle x, y\rangle F_0 - \omega^2 F_0^2 - 2\langle x, y\rangle F_a - \omega^2 F_a^2 + 2\omega^2 F_a F_0 \\ &= F_a[-2\langle x, y\rangle - \omega^2 F_a + 2\omega^2 F_0] \\ &= F_a[-2\langle x, y\rangle - \omega^2 F_a + 2(\theta + \langle x, y\rangle)] \\ &= F_a(2\theta - \omega^2 F_a) \\ &= F_a\Big[2\theta - \omega^2\Big(\frac{\theta + \langle x, y\rangle)}{\omega^2} + \frac{\langle a, y\rangle}{\zeta}\Big)\Big] \\ &= F_a\Big(\theta - \langle x, y\rangle - \frac{\omega^2\langle a, y\rangle}{\zeta}\Big). \end{aligned}$$

■

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3 Scalar Flag Curvature

In this section we determine the projective factor and flag curvature of a class of projectively flat (α, β)-metrics, generalizing expressions previously only known in the case of generalized Funk metrics [10] and Mo–Shen–Yang metrics [8].

Recall that for a Finsler metric F on an open subset $\mathcal{U} \subset \mathbb{R}^n$, F is *projectively flat* if and only if its geodesic coefficients G^i satisfy the following [9]

$$G^i = Py^i.$$

The scalar function P is called the *projective factor* of F.

Proposition 3.1 *Suppose that Finsler metric*

$$F = \frac{\zeta^{-\frac{1}{r}}\theta}{\omega^{-\frac{1}{r}+2}} \phi\Big(\frac{\omega^2\langle a, y\rangle + \zeta\langle x, y\rangle}{\zeta\theta}\Big) \tag{3.1}$$

is projectively flat where $\phi = \phi_{r,-r}$ satisfying (2.2). *Then F has projective factor*

$$P = \frac{(2r-1)\langle x, y\rangle}{2r\omega^2} + (\phi'\psi - \frac{1}{r}\langle a, y\rangle\theta\phi\zeta)\frac{\zeta^{-\frac{1}{r}-2}}{2F\omega^{-\frac{1}{r}+2}} \tag{3.2}$$

and scalar flag curvature

(3.3)
$$\begin{aligned} K = &-\Big(1 - \frac{1}{2r}\Big)\frac{|y|^2}{F^2\omega^2} + \Big(\frac{1}{4r^2} - 1\Big)\frac{\langle x, y\rangle^2}{F^2\omega^4} - \frac{\zeta^{-\frac{1}{r}-4}\psi^2\phi''}{2\theta F^3\omega^{-\frac{1}{r}+2}} \\ &+ \frac{\zeta^{-\frac{1}{r}-2}}{4F^4\omega^{-\frac{2}{r}+4}}\Big(-\frac{1}{r}\langle a, y\rangle\theta\phi\zeta + \phi'\psi\Big)\Big[4\Big(-\frac{1}{2r} + 1\Big)F\langle x, y\rangle\omega^{-\frac{1}{r}} + 3\zeta^{-\frac{1}{r}-2}\Big] \\ &+ \Big(-\frac{1}{2r}\langle a, y\rangle\theta\phi\zeta + \phi'\psi\Big)\frac{(r+1)\langle a, y\rangle\zeta^{-\frac{1}{r}-3}}{rF^3\omega^{-\frac{1}{r}+2}}. \end{aligned}$$

Proof For a projectively flat Finsler metric F, its projective factor P and scalar flag curvature K are given by

$$P = \frac{F_{x^j}y^j}{2F}, \quad K = \frac{P^2 - P_{x^j}y^j}{F^2}. \tag{3.4}$$

Write $\lambda = -\frac{1}{2r}$. Then

$$F = \frac{\zeta^{2\lambda}\theta}{\omega^{2\lambda+2}}\phi\Big(\frac{\omega^2\langle a, y\rangle + \zeta\langle x, y\rangle}{\zeta\theta}\Big),$$

that is

$$(\omega^2)^{\lambda+1}F = \zeta^{2\lambda}\theta\phi. \tag{3.5}$$

By using (3.5) and Lemma 2.5, we have

$$-2\langle x, y\rangle(\lambda+1)\omega^{2\lambda}F + \omega^{2\lambda+2}F_{x^j}y^j = 2a\zeta^{2\lambda-1}\langle a, y\rangle\theta\phi + \zeta^{2\lambda-2}\phi'\psi. \tag{3.6}$$

It follows that

$$F_{x^j}y^j = \frac{1}{\omega^{2\lambda+2}}\{2\langle x, y\rangle(\lambda+1)\omega^{2\lambda}F + \zeta^{2\lambda-2}[2\lambda\langle a, y\rangle\theta\phi\zeta + \phi'\psi]\}. \tag{3.7}$$

Substituting (3.7) into the first formula of (3.4) yields (3.2). We rewrite as

$$F\omega^{2\lambda+2}P = \langle x, y\rangle(\lambda+1)\omega^{2\lambda}F + \zeta^{2\lambda-2}\Big[\lambda\langle a, y\rangle\theta\phi\zeta + \frac{\phi'}{2}\psi\Big]. \tag{3.8}$$

From (3.5) and (3.6) we have

$$(F\omega^{2\lambda+2})_{x^j}y^j = \zeta^{2\lambda-2}[2\lambda\langle a, y\rangle\theta\phi\zeta + \phi'\psi].$$

Together with (2.4) we obtain

$$(F\omega^{2\lambda})_{x^j}y^j = 2\omega^{2\lambda-2}F\langle x, y\rangle + \zeta^{2\lambda-2}\omega^{-2}[2\lambda\langle a, y\rangle\theta\phi\zeta + \phi'\psi].$$

On the other hand, by using Lemma 2.5, we have

$$\begin{aligned}&\zeta^{2\lambda-2}\Big[\lambda\langle a, y\rangle\theta\phi\zeta + \frac{\phi'}{2}\psi\Big]_{x^j}y^j\\ &\qquad = \lambda(2\lambda-1)\zeta^{2\lambda-2}\langle a, y\rangle^2\theta\phi + (2\lambda-1)\zeta^{2\lambda-3}\langle a, y\rangle\phi'\psi + \frac{1}{2\theta}\zeta^{2\lambda-4}\phi''\psi^2.\end{aligned}$$

Differentiating (3.8) with respect to x_j and contracting with y^j yields

$$\begin{aligned}&\zeta^{2\lambda-2}(2\lambda\langle a, y\rangle\theta\phi\zeta + \phi'\psi)P + F\omega^{2\lambda+2}P_{x^j}y^j\\ &\quad = (\lambda+1)[2F\omega^{2\lambda-2}\langle x, y\rangle + \zeta^{2\lambda-2}\omega^{-2}(2\lambda\langle a, y\rangle\theta\phi\zeta + \phi'\psi)]\langle x, y\rangle\\ &\qquad + (\lambda+1)F\omega^{2\lambda}|y|^2 + \lambda(2\lambda-1)\zeta^{2\lambda-2}\langle a, y\rangle^2\theta\phi(2\lambda-1)\zeta^{2\lambda-3}\langle a, y\rangle\phi'\psi\\ &\qquad + \frac{1}{2\theta}\zeta^{2\lambda-4}\phi''\psi^2.\end{aligned}$$

It follows that

$$\begin{aligned}P_{x^j}y^j = {}&\frac{2(\lambda+1)\langle x, y\rangle^2}{\omega^4} + \frac{(\lambda+1)\langle x, y\rangle\zeta^{2\lambda-2}}{F\omega^{2\lambda+4}}(2\lambda\langle a, y\rangle\theta\phi\zeta + \phi'\psi)\\ &+ \frac{(\lambda+1)|y|^2}{\omega^2} + \frac{\lambda(2\lambda-1)\langle a, y\rangle^2\zeta^{2\lambda-2}}{F\omega^{2\lambda+2}}\theta\phi\\ &+ \frac{(2\lambda-1)\langle a, y\rangle\zeta^{2\lambda-3}}{F\omega^{2\lambda+2}}\phi'\psi\\ &+ \frac{\zeta^{2\lambda-4}}{2F\theta\omega^{2\lambda+2}}\phi''\psi^2 - \frac{\zeta^{2\lambda+2}}{F\omega^{2\lambda+2}}(2\lambda\langle a, y\rangle\theta\phi\zeta + \phi'\psi).\end{aligned} \tag{3.9}$$

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By using (3.2) we obtain

$$P^2 = \frac{(\lambda+1)^2\langle x,y\rangle^2}{\omega^4} + \frac{(\lambda+1)\langle x,y\rangle\zeta^{2\lambda-2}}{F\omega^{2\lambda+4}}(2\lambda\langle a,y\rangle\theta\phi\zeta + \phi'\psi) + \frac{\zeta^{4\lambda-4}}{4F^2\omega^{4\lambda+4}}(2\lambda\langle a,y\rangle\theta\phi\zeta + \phi'\psi)^2.$$

Together with (3.9) yields

$$\begin{aligned} P^2 - P_{x^j}y^j = &-\frac{(\lambda+1)|y|^2}{\omega^2} + \frac{(\lambda^2-1)\langle x,y\rangle^2}{\omega^4} - \frac{\zeta^{2\lambda-4}}{2F\theta\omega^{2\lambda+2}}\phi''\psi^2 \\ &+ \frac{(\lambda+1)\langle x,y\rangle\zeta^{2\lambda-2}}{F\omega^{2\lambda+4}}(2\lambda\langle a,y\rangle\theta\phi\zeta + \phi'\psi) \\ &+ \frac{3\zeta^{4\lambda-4}}{4F^2\omega^{4\lambda+4}}(2\lambda\langle a,y\rangle\theta\phi\zeta + \phi'\psi)^2 \\ &- \frac{(2\lambda-1)\langle a,y\rangle\zeta^{2\lambda-3}}{F\omega^{2\lambda+2}}(\phi'\psi + \lambda\langle a,y\rangle\theta\phi\zeta). \end{aligned}$$

Plugging this into the second formula of (3.4) yields (3.3). ■

Remark We have two special cases of Proposition 3.1:

(1) When ϕ is given in (1.1) and $n = 0$ then F is the generalized Funk metric with projective factor

$$P = \frac{1}{2}\Big[\frac{\sqrt{(1-|x|^2)|y|^2 + \langle x,y\rangle^2}}{1-|x|^2} + \frac{\langle x,y\rangle}{1-|x|^2} - \frac{\langle a,y\rangle}{1+\langle a,x\rangle}\Big]$$

and constant flag curvature $K = -\frac{1}{4}$;

(2) When ϕ is given in (1.1) and $n = 1$, then F was constructed in [8] with projective factor

$$P = \frac{\sqrt{(1-|x|^2)|y|^2 + \langle x,y\rangle^2}}{1-|x|^2} + \frac{\langle x,y\rangle}{1-|x|^2}$$

and zero flag curvature.

4 Proof of Theorem 1.1

In the following we find the polynomial solutions of (2.2).

Lemma 4.1 *If $r = -p = -\frac{1}{2n}$, $\epsilon = 2^n$, $n \in \mathbb{N}$, then the solution of* (2.2) *is*

$$\phi(s) = 1 + 2^n s + 2n\Sigma_{k=0}^{n-1}\frac{(-1)^k C_k^{n-1} s^{2k+2}}{(2k+1)(2k+2)}.$$

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Proof Let

$$f(s) = 1 + 2^n s + 2n\Sigma_{k=0}^{n-1}\frac{(-1)^k C_k^{n-1} s^{2k+2}}{(2k+1)(2k+2)}.$$

It is easy to show $f(0) = 1$, and

$$f'(s) = 2^n + 2n\Sigma_{k=0}^{n-1}\frac{(-1)^k C_k^{n-1} s^{2k+1}}{(2k+1)}.$$

It follows that $f'(0) = \epsilon$, and

$$f''(s) = 2n\Sigma_{k=0}^{n-1}(-1)^k C_k^{n-1} s^{2k} = \frac{1}{p}\Big(1 + \frac{r}{p}s^2\Big)^{-\frac{1}{2r}-1}.$$

Now our conclusion can be obtained from Lemma 2.2 immediately. ■

Proof of Theorem 1.1 Let us take a look at the special case of Theorem 2.1: when $C_1 = C_2 = 1, C_3 = \eta = 0, \mu = -1, r = -p$,

$$h = \frac{\zeta}{\omega}, \quad \rho(h) = h^{-\frac{1}{r}}.$$

Thus we have

$$\alpha = \frac{\zeta^{-\frac{1}{r}}\theta}{\omega^{2-\frac{1}{r}}}, \quad \beta = \frac{\zeta^{-\frac{1}{r}}(\omega^2\langle a, y\rangle + \zeta\langle x, y\rangle)}{\zeta\omega^{2-\frac{1}{r}}}.$$

It follows that $F := \alpha\phi(\frac{\beta}{\alpha})$ satisfies (3.1) and

$$\alpha = \frac{\zeta^{-\frac{1}{r}}}{\omega^{-\frac{1}{r}}}\bar{\alpha}, \quad \beta = \frac{\zeta^{-\frac{1}{r}}}{\omega^{-\frac{1}{r}}}\bar{\beta},$$

where

$$\bar{\alpha} = \frac{\theta}{\omega^2}, \quad \bar{\beta} = \frac{\langle x, y\rangle}{\omega^2} + \frac{\langle a, y\rangle}{\zeta}.$$

A direct calculation yields

$$\|\bar{\beta}\|_{\bar{\alpha}} = \|\beta\|_{\alpha} = 1 - \frac{\omega^2(1 - |a|^2)}{\zeta} \tag{4.1}$$

(*cf.* [4, p. 9]). By Corollary 2.4, there exists $b_0 > 0$ such that

$$F := \alpha\phi_{-\frac{1}{2n},\frac{1}{2n}}\Big(\frac{\beta}{\alpha}\Big)$$

is a Finsler metric when $\|\beta\|_{\alpha} < b_o$. Note that $|a| < 1$. Together with (4.1) there exists an open subset at the origin in $\mathbb{R}^n$, denoted by V, such that $\|\beta\|_{\alpha} < b_o$ whenever

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$x \in V$. It follows that F is a Finsler metric on V. Now Theorem 1.1 can be obtained from Proposition 3.1, Lemma 4.1 and Theorem 2.1 immediately. ■

5 More Constructions

In the rest of this paper we shall find more Finsler metrics of scalar curvature and determine their flag curvature.

Lemma 5.1 *If $r = -p = -\frac{1}{2n-1}$, then the solution of* (2.2) *is*

$$\phi(s) = \epsilon s + \frac{(2n-1)!!}{(2n-2)!!}\Big[\sqrt{1-s^2} + s\arcsin s - \sum_{k=1}^{n-1}\frac{(2k-2)!!}{(2k+1)!!}(1-s^2)^{\frac{2k+1}{2}}\Big].$$

Proof For any $\sigma \in \mathbb{R}$, we have the following

$$(5.1)\qquad (3+2\sigma)\int(1\pm x^2)^{1+\sigma}\,dx - 2(1+\sigma)\int(1\pm x^2)^{\sigma}\,dx = x(1\pm x^2)^{1+\sigma} + C_1,$$

and

$$(5.2)\qquad \int x(1\pm x^2)^{\sigma}\,dx = \pm\frac{1}{2(1+\sigma)}(1\pm x^2)^{1+\sigma} + C_2$$

where C_1 and C_2 are constants.

When $n = 1$, it is direct calculations similar to the proof of Lemma 4.1.

In the general case, taking

$$\sigma = \frac{2n-5}{2} \neq -\frac{3}{2} \text{ and } \sigma = \frac{2n-3}{2} \neq -1$$

in (5.1) and (5.2) respectively we obtain the following formulas

$$(5.3)\qquad \int(1-x^2)^{\frac{2n-3}{2}}\,dx = \frac{1}{2(n-1)}x(1-x^2)^{\frac{2n-3}{2}} + \frac{2n-3}{2n-2}\int(1-x^2)^{\frac{2n-5}{2}}\,dx,$$

and

$$(5.4)\qquad \int x(1-x^2)^{\frac{2n-3}{2}}\,dx = -\frac{1}{2n-1}(1-x^2)^{\frac{2n-1}{2}}.$$

Let us assume that Lemma 5.1 is true for $n-1$. By using (2.3), (5.3) and (5.4), we obtain

$$
\begin{aligned}
&\phi_{-\frac{1}{2n-1},\frac{1}{2n-1}}\\
&= 1+\epsilon s+(2n-1)\int_0^s\int_0^\tau (1-\sigma^2)^{\frac{2n-3}{2}}\,d\sigma\,d\tau\\
&= 1+\epsilon s+(2n-1)\int_0^s\Big[\frac{\sigma(1-\sigma^2)^{\frac{2n-3}{2}}}{2(n-1)}\Big|_0^\tau+\frac{2n-3}{2n-2}\int_0^\tau(1-\sigma^2)^{\frac{2n-5}{2}}\,d\sigma\Big]\,d\tau\\
&= 1+\epsilon s+\frac{2n-1}{2n-2}\Big[-\frac{1}{2n-1}(1-\sigma^2)^{\frac{2n-1}{2}}\Big]_0^s\\
&\qquad+\frac{(2n-1)(2n-3)}{2n-2}\int_0^s\int_0^\tau(1-\sigma^2)^{\frac{2n-3}{2}}\,d\sigma\,d\tau\\
&= 1+\epsilon s+\frac{1}{2n-2}-\frac{(1-\sigma^2)^{\frac{2n-1}{2}}}{2n-2}\\
&\qquad+\frac{(2n-1)(2n-3)}{2n-2}\int_0^s\int_0^\tau(1-\sigma^2)^{\frac{2n-5}{2}}\,d\sigma\,d\tau\\
&= \frac{2n-1}{2n-2}\Big[1-(2n-3)\int_0^s\int_0^\tau(1-\sigma^2)^{\frac{2n-5}{2}}\,d\sigma\,d\tau\Big]+\epsilon s-\frac{(1-s^2)^{\frac{2n-1}{2}}}{2n-2}\\
&= \frac{2n-1}{2n-2}\Big[\phi_{-\frac{1}{2n-3},\frac{1}{2n-3}}-\epsilon s\Big]+\epsilon s-\frac{(1-s^2)^{\frac{2n-1}{2}}}{2n-2}\\
&= \frac{2n-1}{2n-2}\Big\{\frac{(2n-3)!!}{(2n-4)!!}\Big[\sqrt{1-s^2}+s\arcsin s-\sum_{k=1}^{n-2}\frac{(2k-2)!!}{(2k+1)!!}(1-s^2)^{\frac{2k+1}{2}}\Big]\Big\}\\
&\qquad+\epsilon s-\frac{(1-s^2)^{\frac{2n-1}{2}}}{2n-2}\\
&= \epsilon s+\frac{(2n-1)!!}{(2n-2)!!}\Big[\sqrt{1-s^2}+s\arcsin s-\sum_{k=1}^{n-1}\frac{(2k-2)!!}{(2k+1)!!}(1-s^2)^{\frac{2k+1}{2}}\Big].
\end{aligned}
$$

■

Lemma 5.2 *If $r=-p=\frac{1}{2n}$, $n\in\mathbb{N}$, then the solution of (2.2) is*

$$\phi(s)=\epsilon s+\frac{(2n-1)!!}{(2n-2)!!}\Big[1+\frac{1}{2}s\ln\frac{1-s}{1+s}-\sum_{k=1}^{n-1}\frac{(2k-2)!!}{(2k+1)!!}\frac{1}{(1-s^2)^k}\Big].$$

Proof Similar to the proof of Lemma 5.1 where we take $\sigma=-n\neq-1$ in (5.1) and (5.2). ■

Lemma 5.3 *If $r=-p=\frac{1}{2n-1}$, $n\in\mathbb{N}$, then the solution of (2.2) is*

$$\phi(s)=\epsilon s+\frac{(2n-2)!!}{(2n-3)!!}\Big[\frac{1-2s^2}{2\sqrt{1-s^2}}-\sum_{k=2}^{n-1}\frac{(2k-3)!!}{(2k)!!}\frac{1}{(1-s^2)^{\frac{2k-1}{2}}}\Big]$$

for $n\geq 2$.

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Proof When $n = 2$, it is direct calculations similar to the proof of Lemma 4.1.

In the general case, similar to the proof of Lemma 5.1 where we take $\sigma = -\frac{2n+1}{2} \neq -1$ and $\sigma = -\frac{2n-1}{2} \neq -1$ in (5.1) and (5.2) respectively. ■

Theorem 5.4 *Let $\phi(s)$ be a function defined by*

$$\phi(s) = \epsilon s + \frac{(2n-1)!!}{(2n-2)!!}\Big[\sqrt{1-s^2} + s\arcsin s - \sum_{k=1}^{n-1}\frac{(2k-2)!!}{(2k+1)!!}(1-s^2)^{\frac{2k+1}{2}}\Big].$$

Then the polynomial (α, β) metric on an open subset at the origin in $\mathbb{R}^n$ given by

$$F := \frac{(1+\langle a,x\rangle)^{2n-1}}{(1-|x|^2)^{\frac{2n+1}{2}}}\sqrt{(1-|x|^2)|y|^2 + \langle x,y\rangle^2} \times \phi\Big(\frac{(1-|x|^2)\langle a,y\rangle + (1+\langle a,x\rangle)\langle x,y\rangle}{(1+\langle a,x\rangle)\sqrt{(1-|x|^2)|y|^2+\langle x,y\rangle^2}}\Big)$$

is of scalar curvature with flag curvature

$$\begin{aligned} K = &-\frac{(2n+1)|y|^2}{2F^2\omega^2} + \frac{(2n+1)(2n-3)\langle x,y\rangle^2}{4F^2\omega^4} - \frac{\zeta^{2n-5}\psi^2\phi''}{2\theta F^3\omega^{2n+1}} \\ &+ \frac{\zeta^{2n-3}}{4F^4\omega^{4n+2}}\big((2n-1)\langle a,y\rangle\theta\phi\zeta + \phi'\psi\big)\,[2(2n+1)F\langle x,y\rangle\omega^{2n-1} + 3\zeta^{2n-3}] \\ &- \Big(\frac{2n-1}{2}\langle a,y\rangle\theta\phi\zeta + \phi'\psi\Big)\frac{(2n-2)\langle a,y\rangle\zeta^{2n-4}}{F^3\omega^{2n+1}} \end{aligned}$$

where $n \in \{0,1,2,\dots\}$, $a \in \mathbb{R}^n$ is a constant vector with $|a| < 1$.

Proof Similar to the proof of Theorem 1.1 where we use Lemma 5.1 instead of Lemma 4.1. ■

Theorem 5.5 *Let $\phi(s)$ be a function defined by*

$$\phi(s) = \epsilon s + \frac{(2n-1)!!}{(2n-2)!!}\Big[1 + \frac{1}{2}s\ln\frac{1-s}{1+s} - \sum_{k=1}^{n-1}\frac{(2k-2)!!}{(2k+1)!!}\frac{1}{(1-s^2)^k}\Big].$$

Then the following (α, β) metric on an open subset at the origin in $\mathbb{R}^n$

$$F := \frac{\sqrt{(1-|x|^2)|y|^2+\langle x,y\rangle^2}}{(1-|x|^2)^{1-n}}\phi\Big(\frac{\langle x,y\rangle}{\sqrt{(1-|x|^2)|y|^2+\langle x,y\rangle^2}}\Big)$$

is of scalar curvature with flag curvature

$$\begin{aligned} K = &\frac{(n-1)|y|^2}{F^2\omega^2} + \frac{(n^2-1)\langle x,y\rangle^2}{F^2\omega^4} - \frac{\psi^2\phi''}{2\theta F^3\omega^{2-2n}} \\ &+ \frac{\phi'\psi}{4F^4\omega^{4-4n}}[4(1-n)F\langle x,y\rangle\omega^{-2n} + 3] \end{aligned}$$

where $n \in \{0,1,2,\dots\}$.

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Proof Similar to the proof of Theorem 1.1 where we use Lemma 5.2 instead of Lemma 4.1. ■

Theorem 5.6 *Let $\phi(s)$ be a function defined by*

$$\phi(s) = \epsilon s + \frac{(2n-2)!!}{(2n-3)!!}\Big[\frac{1-2s^2}{2\sqrt{1-s^2}} - \sum_{k=2}^{n-1}\frac{(2k-3)!!}{(2k)!!}\frac{1}{(1-s^2)^{\frac{2k-1}{2}}}\Big].$$

Then the (α,β) metric on an open subset at the origin in $\mathbb{R}^n$ given by

$$F := \frac{\sqrt{(1-|x|^2)|y|^2+\langle x,y\rangle^2}}{(1-|x|^2)^{\frac{3-2n}{2}}}\phi\Big(\frac{\langle x,y\rangle}{\sqrt{(1-|x|^2)|y|^2+\langle x,y\rangle^2}}\Big)$$

is of scalar curvature with flag curvature

$$K = \frac{(2n-3)|y|^2}{2F^2\omega^2} + \frac{(2n+1)(2n-3)\langle x,y\rangle^2}{4F^2\omega^4} - \frac{\psi^2\phi''}{2\theta F^3\omega^{3-2n}} + \frac{\phi'\psi}{4F^4\omega^{6-4n}}[2(3-2n)F\langle x,y\rangle\omega^{1-2n}+3]$$

where $n \in \{0,1,2,\dots\}$.

Proof Similar to the proof of Theorem 1.1 where we use Lemma 5.3 instead of Lemma 4.1. ■

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