17

The Altarelli–Parisi equation

Although convenient, the use of OPE to deep inelastic scattering does not provide a transparent physical intuition of the parton model. An elegant reformulation of the moments which makes a close contact with the parton model picture is given by the Altarelli–Parisi equation [239] and the review in [48].

17.1 The non-singlet case

One can illustrate this equation by taking the simple example of the non-singlet structure functions, which one can write as an incoherent sum of quark parton densities $q_f(x)$. In presence of external fields, the quark parton densities acquire a Q^2 dependence, and the structure function can be written as:

$$F^{NS}_{2,n}(Q^2) = x \sum_f \delta^{NS}_f q_f(x, Q^2) \, , \tag{17.1}$$

where δ^{NS}_f are known coefficients. Using the QCD expression to lowest order, the corresponding moments read:

$$\mathcal{M}^{NS}_{2,n}(Q^2) = (\log(Q^2/\Lambda^2))^{\gamma^0_n/\beta_1} \, . \tag{17.2}$$

From its definition, the moments of the quark densities are:

$$\mathcal{M}_{f,n}(Q^2) \equiv \int_0^1 dx \; x^{n-1} q_f(x, Q^2) \, . \tag{17.3}$$

Using Eq. (17.2), one can derive the evolution equation:

$$\frac{\partial \mathcal{M}_{f,n}(t)}{\partial t} = -\gamma^0_n \left(\frac{\alpha_s}{\pi}\right)(t) \mathcal{M}_{f,n}(t) \, , \tag{17.4}$$

with $t \equiv (1/2)\log(Q^2/\nu^2)$. Defining:

$$\int_0^1 dz \; z^{n-1} P^{(0)}_{NS}(z) = -\gamma^0_n \, , \tag{17.5}$$

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and taking the Mellin transform of Eq. (17.3), one can deduce the Altarelli–Parisi equation [239]:

$$\frac{\partial q_f(x,t)}{\partial t} = \frac{\alpha_s(t)}{\pi} \int_x^1 \frac{dy}{y} P_{NS}^{(0)}\left(\frac{x}{y}\right) q_f(y,t)\,, \tag{17.6}$$

which one can symbolically write as:

$$\frac{\partial q_f}{\partial t} = \frac{\alpha_s(t)}{\pi} q_f \otimes P_{NS}^{(0)}\,. \tag{17.7}$$

In its infinitesimal form, the equation can be rewritten as:

$$q_f(x,t) + dq_f(x,t) = \int_0^1 dy \int_0^1 dz\, \delta(zy - x) q_f(x,t) \left\{\delta(z-1) + \left(\frac{\alpha_s}{\pi}\right) P_{NS}^{(0)}(z) dt\right\}\,. \tag{17.8}$$

One can interpret $P_{NS}^{(0)}(z)$ (*splitting functions*) as controlling the rate of change of the parton distribution probability with respect to t. One can check that Eq. (17.5) is satisfied if:

$$P_{NS}^{(0)}(z) = C_F \left\{\frac{3}{2}\delta(1-z) + \frac{1+z^2}{(1-z)_+}\right\}\,, \tag{17.9}$$

where for any function g:

$$\int_0^1 \frac{dz}{(1-z)_+} g(z) \equiv \int_0^1 \frac{dz}{(1-z)} [g(z) - g(1)]\,. \tag{17.10}$$

17.2 The singlet case

In the case of singlet structure functions, analogous relations can be obtained. The Altarelli–Parisi evolution-coupled equations are:[1]

$$\frac{\partial q_f(x,t)}{\partial t} = \frac{\alpha_s(t)}{\pi} \int_x^1 \frac{dy}{y} \left\{P_{qq}^{(0)}\left(\frac{x}{y}\right) q_f(y,t) + P_{qg}^{(0)}\left(\frac{x}{y}\right) G(y,t)\right\}\,, \tag{17.11}$$

$$\frac{\partial G(x,t)}{\partial t} = \frac{\alpha_s(t)}{\pi} \int_x^1 \frac{dy}{y} \left\{P_{gp}^{(0)}\left(\frac{x}{y}\right) q_f(y,t) + P_{gg}^{(0)}\left(\frac{x}{y}\right) G(y,t)\right\}\,. \tag{17.12}$$

The splitting functions are:

$$\begin{aligned}
P_{qq}^{(0)} &= P_{NS}^{(0)}\,,\\
P_{qg}^{(0)} &= \frac{T}{n_f}(x^2 + (1-x)^2)\,,\\
P_{gq}^{(0)} &= C_F \frac{1}{x}(1 + (1-x)^2)\,,\\
P_{gg}^{(0)} &= 2C_G \left(\frac{1-x}{x} + x(1-x) + \frac{x}{(1-x)_+}\right) + \delta(1-x)\frac{(11C_G - 4T)}{6}\,,
\end{aligned} \tag{17.13}$$

[1] Recall our definition of t which is $1/2$ of the one used in the original paper.

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where: $T \equiv n_f/2$, $C_F = (N^2-1)/(2N)$ and $C_G = N$. In the limit $C_F = C_G = 2T$ (supersymmetry of the massless QCD lagrangian where both gluons and Weyl fermions transform according to the regular representation of the group), one has the remarkable relation:

$$P_{qq}^{(0)} + P_{qg}^{(0)} = 2n_f P_{gq}^{(0)} + P_{gg}^{(0)} . \tag{17.14}$$

By taking the difference of Eq. (17.11) for q_i and q_j, where $q_{i,j}$ are any quark or anti-quark densities, the gluon term drops out, and one recovers the previous simple result for the non-singlet (or valence) evolution equations:

$$\frac{\partial V_{ij}}{\partial t} = \frac{\alpha_s(t)}{\pi} V_{ij} \otimes P_{qq}^{(0)} \quad : \qquad V_{ij}(x,t) \equiv q_i(x,t) - q_j(x,t) . \tag{17.15}$$

Defining:

$$\Sigma(x,t) \equiv \sum_f q_f(x,t) = \sum_{\text{flavours}} [q(x,t) + \bar{q}(x,t)] , \tag{17.16}$$

one can also obtain the evolution equations in terms of two independent densities:

$$\begin{aligned} \frac{\partial \Sigma}{\partial t} &= \frac{\alpha_s(t)}{\pi} \left\{ \Sigma \otimes P_{qq}^{(0)} + G \otimes 2n_f P_{qg}^{(0)} \right\} , \\ \frac{\partial G}{\partial t} &= \frac{\alpha_s(t)}{\pi} \left\{ \Sigma \otimes P_{gq}^{(0)} + G \otimes P_{gg}^{(0)} \right\} , \end{aligned} \tag{17.17}$$

which are convenient to work with in the phenomenological analysis. The solution for a quark i can be reconstructed by splitting it into its non-singlet $q_i - \Sigma/2$ and singlet $\Sigma/2$ components.

17.3 Some physical interpretations and factorization theorem

We have seen in Eq. (17.8) that one can interpret the (*splitting functions*) $P_{NS}^{(0)}(z)$ as controlling the rate of change of the parton distribution probability with respect to t. This can be understood by considering the scattering of an off-shell photon on the parton as depicted in the different diagrams in Fig. 17.1.

Diagram 17.1a shows the free quark diagram in the parton model with a certain probability $q_f(x)$ of having a fraction of the proton momentum $q_f(x)$. After a time t, the quark may radiate into gluons as depicted in different diagrams shown in 17.1b and 17.1c. One can show that, in the axial (physical) gauge, only diagram 17.1b contributes to the cross-section and gives a term proportional to t:

$$\sigma(\gamma^* q \to q + g) \simeq \frac{\alpha_s(t)^2}{e} \pi [t \ P(x) + f(x)] . \tag{17.18}$$

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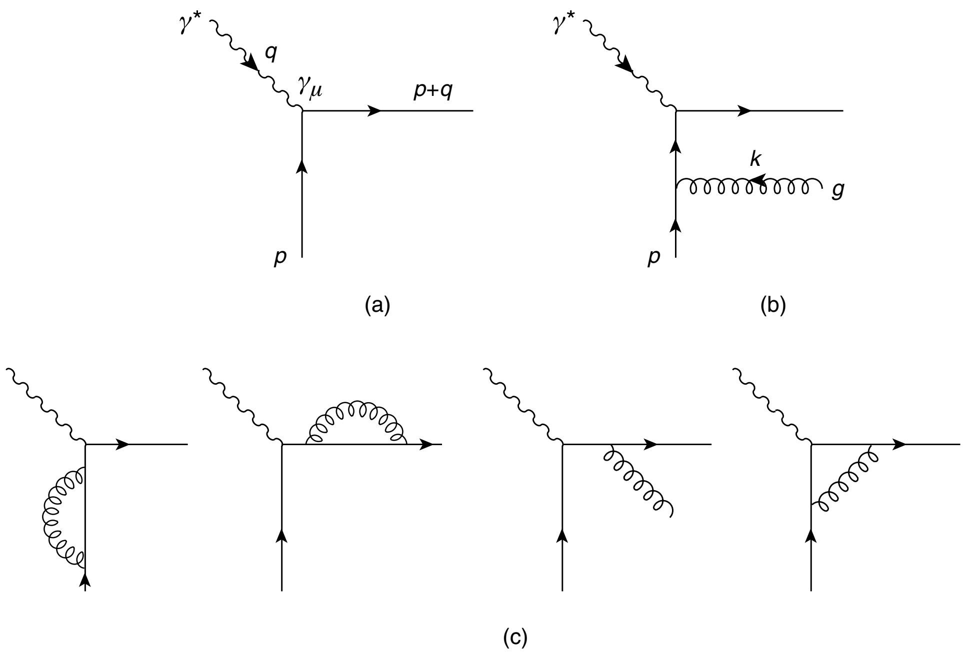

Fig. 17.1. Scattering of an off-shell photon on the parton.

$P(x)$ is well-defined perturbatively, while $f(x)$ depends on the IR regularization procedure. One can generalize the above procedure to sum up the contributions of an arbitrary number of gluons. In the leading log approximation, one can show [240] that only the ladder graphs in Fig. 17.2 contribute and lead to the factorization theorem:

$$\sigma(\gamma^* q \to q + g) \sim \left(\frac{\alpha_s(t)}{\pi}\right)^n \ln^n \frac{Q^2}{\nu^2} \, . \qquad (17.19)$$

It is also important to recall that the splitting functions $P_{ab}^{(0)}$ are universal (anomalous dimension of the RGE) and consequently the parton densities depend only on the the target and are independent of the nature and polarization of the probe (vector, axial-vector, . . .).

17.4 Polarized parton densities

The previous approach can be generalized to parton densities of definite helicity in a polarized target. The quark and gluon densities $q_{i\pm}$ and $G_{\pm}$, with helicity $\pm$ in a target of definite polarization, are related to the unpolarized ones as:

$$p_{A+}(x,t) + p_{A-}(x,t) = p_A(x,t) \;: \qquad s(p_A \equiv q_i,\; G) \, . \qquad (17.20)$$

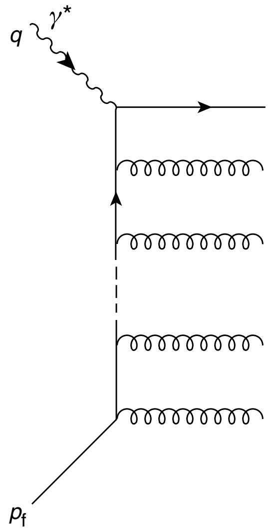

Fig. 17.2. Ladder diagrams contributing at the leading log approximation.

The corresponding evolution equations can be written to leading order as:

$$\frac{\partial p_{A\pm}}{\partial t} = \frac{\alpha_s(t)}{\pi} \left\{ \sum_B \left[p_{B_+} \otimes P^{(0)}_{A_\pm B_+} + p_{B_-} \otimes P^{(0)}_{A_\pm B_-} \right] \right\} . \tag{17.21}$$

Parity and probability conservation gives:

$$\begin{aligned} P^{(0)}_{A_\pm B_-} &= P^{(0)}_{A_- B_\mp} , \\ P^{(0)}_{A_+ B_+} + P^{(0)}_{A_- B_+} &= P^{(0)}_{A_+ B_-} + P^{(0)}_{A_- B_-} , \end{aligned} \tag{17.22}$$

which imply that $(p_{A_+} + p_{A_-}) = p_A$ and $(p_{A_+} - p_{A_-}) = \Delta p_A$ evolve separately for any A. The evolution equation for the difference is:

$$\frac{\partial p_{A\pm}}{\partial t} = \frac{\alpha_s(t)}{\pi} \sum_B \Delta p_B \otimes \Delta P^{(0)}_{AB} . \tag{17.23}$$

The splitting function:

$$\Delta P^{(0)}_{AB} \equiv P^{(0)}_{A_+ B_+} - P^{(0)}_{A_- B_+} , \tag{17.24}$$

measures the tendency of a parton A to remember the polarization of its parent B. From the helicity conservation at the quark gluon vertex, it follows that the non-singlet kernel is the same as in the unpolarized case:

$$\Delta P^{(0)}_{qq} \equiv P^{(0)}_{q_+ q_+} - P^{(0)}_{qq} . \tag{17.25}$$

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One also finds:

$$\begin{aligned}
\Delta P_{qg}^{(0)} &= \frac{T}{n_f}\frac{1}{2}(x^2 - (1-x)^2)\,, \\
\Delta P_{gq}^{(0)} &= \frac{C_F}{2}\frac{1}{x}(1 - (1-x)^2)\,, \\
\Delta P_{gg}^{(0)} &= \frac{C_G}{2}\left((1+x^4)\left(\frac{1}{x} + \frac{1}{(1-x)_+}\right) - \frac{(1-x)^3}{x}\right) \\
&\quad + \delta(1-x)\frac{(11C_G - 4T)}{12}\,. \qquad (17.26)
\end{aligned}$$

In this case, all charge moments are well defined, as the total helicity is finite though the total number of gluons and quark pairs is infinite. To leading order, the net helicity is also conserved, such that $\Delta P_{qg}^{(0)}$ and $\Delta P_{qq}^{(0)}$ are zero.

The previous evolution equations for parton densities with definite helicities are sufficient for the prediction of scaling violations in leptoproduction on a longitudinal polarized target. Additional information is needed for a transversely polarized target.

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