HOLOMORPHIC FUNCTIONAL CALCULI
AND SUMS OF COMMUTING OPERTORS

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Let S and T be commuting operators of type ω and type ϖ in a Banach space $\mathcal{X}$. Then the pair has a joint holomorphic functional calculus in the sense that it is possible to define operators $f(S, T)$ in a consistent manner, when f is a suitable holomorphic function defined on a product of sectors. In particular, this gives a way to define the sum $S + T$ when $\omega + \varpi < \pi$. We show that this operator is always of type μ where $\mu = \max\{\omega, \varpi\}$. We explore when bounds on the individual functional calculi of S and T imply bounds on the functional calculus of the pair (S, T), and some implications for the regularity problem of when $\|(S + T)u\|$ is equivalent to $\|Su\| + \|Tu\|$.

1. INTRODUCTION

The class of operators of type ω is important in functional analysis and partial differential equations. Well known examples include elliptic operators on various domains [12, 22], m-accretive operators and m-sectorial operators [13]. In the paper [17], one of the authors developed a functional calculus for operators of type ω on Hilbert spaces, which was extended to include more general Banach spaces in [4]. This calculus provides a unified theory by which fractional powers, exponentials, logarithms, imaginary powers, and other functions, including unbounded functions, of these operators can be defined and their properties investigated. The objectives have been first to show that functions of operators of type ω are closed, second to say something about their domains, and third to give sufficient conditions for the boundedness of the functional calculus. Further work on functional calculi is included in [2, 8, 10, 18, 19]. There is also related material in [5] and [6].

There has been much work done on functions of two or several commuting operators. One important result is the maximal regularity theorem of Dore and Venni [7] and Giga and Sohr [12]. They prove that if S and T are two commuting operators on a UMD space which have bounded imaginary powers, then $\|Su\| \leqslant c\|Su + Tu\|$ for all $u \in \mathcal{D}(S) \cap \mathcal{D}(T)$, or, in other words, $S/(S + T)$ is a bounded operator. In [3], Baillon and

Clement construct a pair of closed commuting operators on a Hilbert space whose sum is not closed. In this paper we extend the techniques developed in [17] and [4] to construct joint functional calculi for several commuting operators of type ω. It turns out that, when defined via this functional calculus, functions of several commuting operators of type ω are always closed. However, questions of boundedness, and hence domain, depend on the underlying Banach space. For ease of exposition we develop our ideas for pairs of commuting operators, though most of our results can easily be extended to n-tuples of operators. Some of this work was investigated by one of the authors in his thesis [1]. An alternative development of a joint functional calculus of a pair of operators of type ω is given in [16].

An outline of this paper is as follows. In Section 2 we give definitions and conventions, and show how results for the functional calculus of a single operator of type ω acting on any Banach space $\mathcal{X}$ can be extended to a joint functional calculus for pairs of commuting operators.

In Section 3 we specialise to consider the sum $S + T$ of two commuting operators S and T of type ω. The sum defined by the functional calculus is always of type ω. It may happen that the sum defined in this way has a strictly larger domain $\mathcal{D}(S + T)$ than the the intersection $\mathcal{D}(S) \cap \mathcal{D}(T)$, as follows from the example in Section 6.

In Section 4, we show, for operators on Hilbert spaces, that a pair (T, S) of commuting operators of type ω which individually admit bounded holomorphic functional calculi, admits a bounded joint functional calculus. In fact more holds, and in Theorem 4.1 we show that even if only one operator T has a bounded functional calculus, and if $\mu > \omega$, then

$$(1) \qquad \big\| f(T, S) \big\| \leqslant c \, \sup \big\{ \big\| f(z, S) \big\| : |\arg z| < \mu \big\}.$$

In Section 5 we consider commuting operators S and T of type ω acting on $L^p(\Omega)$, $1 < p < \infty$. It is unknown whether (1) holds in this case when T has a bounded functional calculus. Nevertheless we use a different argument to prove that the pair (T, S) has a bounded joint functional calculus whenever both T and S individually have bounded holomorphic functional calculi.

In Section 6 we give an example of two commuting operators of type ω each of which admits a bounded holomorphic functional calculus, though the joint functional calculus of the pair is not bounded.

Before attending to the technical matters of the rest of this paper, the authors thank Alan Pryde and Xuan Duong for a number of helpful discussions.

2. FUNCTIONAL CALCULUS

We begin with a few definitions and conventions. For $0 \leqslant \omega < \pi$, let S_ω denote the open sector defined by

$$S_\omega = \left\{ z \in \mathbb{C} : |\arg(z)| < \omega \right\},$$

with closure given by $\bar{S}_\omega$. An operator T acting on a complex Banach space $\mathcal{X}$ is said to be of *type* ω if T is closed, $\sigma(T)$ is contained in $\bar{S}_\omega$, and for each μ in (ω, π), there exists c_μ such that

$$\left\| (T - zI)^{-1} \right\| \leqslant c_\mu |z|^{-1} \text{ for all } z \in \mathbb{C} \setminus \bar{S}_\mu.$$

We say that two operators of type ω *commute* if their resolvents commute.

For $\mu, \nu < \pi$, and functions f and g which are holomorphic on S_μ and $S_\mu \times S_\nu$ respectively, set

$$\| f \|_\infty = \sup \left\{ |f(z)| : z \in S_\mu \right\}, \text{ and}$$

$$\| g \|_\infty = \sup \left\{ |g(z, w)| : (z, w) \in S_\mu \times S_\nu \right\}.$$

As usual, denote the Banach algebras of bounded holomorphic functions on S_μ and on $S_\mu \times S_\nu$ by $H^\infty(S_\mu)$ and $H^\infty(S_\mu \times S_\nu)$. Also set

$$\Psi(S_\mu) = \left\{ \psi \in H^\infty(S_\mu) : |\psi(z)| \leqslant c \frac{|z|^s}{1 + |z|^{2s}} \text{ for some } s > 0 \right\},$$

$$\Psi(S_\mu \times S_\nu) = \left\{ \psi \in H^\infty(S_\mu \times S_\nu) : |\psi(z, w)| \leqslant c \frac{|zw|^s}{(1 + |z|^{2s})(1 + |w|^{2s})} \right\}.$$

Let $\omega < \mu < \pi$, and $\varpi < \nu < \pi$, and let T and S be commuting one-to-one operators of type ω and ϖ respectively, with dense domains and ranges in a complex Banach space $\mathcal{X}$. If $f \in H^\infty(S_\mu \times S_\nu)$ and h is defined by

$$h(z, w) = \phi(z, w) f(z, w), \qquad \text{where}$$

$$\phi(z, w) = \frac{zw}{(z + 1)^2 (w + 1)^2},$$

then h is in $\Psi(S_\mu \times S_\nu)$. Thus, $h(T, S)$ is a well defined bounded operator given by the following integral, which converges absolutely in operator norm,

$$h(T, S) = \left(\frac{1}{2\pi i} \right)^2 \iint_{\partial S_\mu \times \partial S_\nu} h(z, w) \frac{1}{z - T} \frac{1}{w - S} \, dz dw.$$

We define then $f(T, S)$ by

$$f(T, S) = \left(T(T + I)^{-2} S(S + I)^{-2} \right)^{-1} h(T, S).$$

Now $\left(T(T+I)^{-2}S(S+I)^{-2}\right)^{-1}$ is a closed operator with dense domain and $h(T,S)$ preserves this domain, thus $f(T,S)$ is a closed densely defined operator. Using the one variable methods from [17, 4], one can show that the definition of $f(T,S)$ does not depend on our choice of ϕ.

By using functions in $\Psi(S_\mu \times S_\nu)$ with sufficient decay, one also defines unbounded functions of T and S such as polynomials. In fact the one variable methods extend to proving the following theorem.

FUNCTIONAL CALCULUS THEOREM. *Let T and S be commuting one-to-one operators of type ω and ϖ respectively, with dense domains and ranges in a complex Banach space $\mathcal{X}$. Then for each $\mu > \omega$ and $\nu > \varpi$ there exists a unique holomorphic joint functional calculus of T and S with the property that, for all holomorphic functions f on $S_\mu \times S_\nu$ with polynomial bounds at ∞ and rational bounds at 0, $f(T,S)$ is a closed operator with dense domain $\mathcal{D}\big(f(T,S)\big)$, and*

$$f(T,S) + g(T,S) = \big(f+g\big)(T,S)\Big|_{\mathcal{D}(f(T,S))} \qquad \text{and}$$
$$g(T,S)f(T,S) = \big(fg\big)(T,S)\Big|_{\mathcal{D}(f(T,S))}.$$

We say that (T,S) has a *bounded $H^\infty(S_\mu \times S_\nu)$ functional calculus* if, for all $f \in H^\infty(S_\mu \times S_\nu)$, $f(T,S) \in \mathcal{L}(\mathcal{X})$ and

$$\big\|f(T,S)\big\| \leqslant c \, \|f\|_\infty$$

for some constant c independent of f. Note that this implies that T and S have bounded $H^\infty(S_\mu)$ and $H^\infty(S_\nu)$ functional calculi respectively. We also have a Convergence Lemma for commuting operators which is useful for obtaining bounds for functions of operators.

CONVERGENCE LEMMA. *Suppose $\omega < \mu < \pi$, and $\varpi < \nu < \pi$, and let T and S be commuting one-to-one operators of type ω and ϖ respectively, with dense domains and ranges in a complex Banach space $\mathcal{X}$. Let $\{f_\alpha\}$ be a uniformly bounded net in $H^\infty(S_\mu \times S_\nu)$. Suppose the net $\{f_\alpha(T,S)\}$ is a uniformly bounded net of operators and $\{f_\alpha\}$ converges uniformly on compact subsets of S_μ to f. Then $f(T)$ is bounded, $f_\alpha(T)x \to f(T)x$ for all $x \in \mathcal{X}$, and $\|f(T)\| \leqslant \sup\limits_\alpha \|f_\alpha(T)\|$.*

3. THE SUM $S + T$

Included in the functional calculus is the operator $S+T$ defined by $S+T = s(S,T)$ where $s(z,w) = z+w$. By the Functional Calculus Theorem, $\mathcal{D}(S) \cap \mathcal{D}(T) = \mathcal{D}(S+T) \cap \mathcal{D}(S)$, and thus $\mathcal{D}(S) \cap \mathcal{D}(T) \subseteq \mathcal{D}(S+T)$.

Using the joint functional calculus we show that in any Banach space, this operator $S+T$ is of type ω under the following natural conditions.

THEOREM 3.1. *Suppose S and T are one-to-one operators of type ω and ϖ respectively, which act on a complex Banach space $\mathcal{X}$, with $\omega < \pi/2$, $\omega \leqslant \varpi$ and $\omega + \varpi < \pi$. If the the resolvents of S and T commute, then the operator $S + T$ defined by the joint functional calculus of S and T is of type ϖ.*

PROOF: Let $\nu > \varpi$. Choose μ so that $\omega < \mu < \pi - \varpi$. One has, whenever $|\arg z| \geqslant \nu$,

$$
\frac{1}{z - (S + T)} = \frac{1}{z - S} + \frac{T}{(z - S) + (z - (S + T))}
$$
$$
= \frac{1}{z - S} + \left(\frac{T}{z - T}\right)\frac{1}{z - S} + \left(\frac{T}{z - T}\right)\frac{S}{(z - S)(z - (S + T))}.
$$

Thus, since $(z - S)^{-1}$ and $T(z - T)^{-1}$ are bounded, to show that

$$
\left\|\frac{1}{z - (S + T)}\right\| \leqslant \frac{c}{|z|}, \quad |\arg z| \geqslant \nu,
$$

one need only show that

(2) $$\left\|\frac{S}{(z - S)(z - (S + T))}\right\| \leqslant \frac{c}{|z|}, \quad |\arg z| \geqslant \nu.$$

To estimate this norm we first note that

(3) $$\frac{S}{(z - S)(z - (S + T))} = \frac{1}{2\pi i}\int_{\partial S_\mu}\frac{\zeta}{(z - \zeta)(z - (\zeta + T))(\zeta - S)}d\zeta.$$

Now split ∂S_μ into two pieces by letting $\Gamma_1 = \partial S_\mu \cap \{\zeta : |\zeta| \leqslant 2|z|\}$ and $\Gamma_2 = \partial S_\mu \cap \{\zeta : |\zeta| > 2|z|\}$. Since S and T are one-to-one operators of type ω and ϖ and $\omega < \mu < \pi - \varpi$ one has for $\zeta \in \Gamma_1$ that

$$
\left\|\left(z - (\zeta + T)\right)^{-1}\right\|, \ |z - \zeta|^{-1} \leqslant \frac{c}{|z|}.
$$

Further, for $\zeta \in \Gamma_2$, the norm of the operator in the integrand of (3) is bounded by $c|\zeta|^{-2}$. Using these estimates one obtains

$$
\left\|\frac{S}{(z - S)(z - (S + T))}\right\| \leqslant c\int_{\Gamma_1}\left|\frac{1}{(z - \zeta)}\right|\left\|\frac{1}{z - (\zeta + T)}\right\|\left\|\frac{\zeta}{\zeta - S}\right\||d\zeta|
$$
$$
+ c\int_{\Gamma_2}\left\|\frac{\zeta}{(z - \zeta)(z - (\zeta + T))(\zeta - S)}\right\||d\zeta|
$$
$$
\leqslant \frac{c}{|z|^2}\int_{\Gamma_1}|d\zeta| + c\int_{\Gamma_2}|\zeta|^{-2}|d\zeta|
$$
$$
\leqslant \frac{c}{|z|}.
$$

Thus (2) holds and hence the proof is complete.   □

The operator $S + T$ defined by the functional calculus may have domain $\mathcal{D}(S + T)$ strictly bigger than $\mathcal{D}(S) \cap \mathcal{D}(T)$, or in other words, the operator $S/(S + T)$ may be unbounded. See Section 6 for an example. It follows from the results of Dore and Venni, as extended by Giga and Sohr, that $S/(S + T)$ is necessarily bounded if either

(i) $\mathcal{X}$ is a Hilbert space and one of the operators has a bounded functional calculus, or

(ii) $\mathcal{X}$ is a UMD space and both operators have bounded functional calculi.

See the next two sections.

Such results in other spaces have been proved in recent years by Lancien, Lancien and Le Merdy [14, 15] using the discrete quadratic estimates of [10].

4. HILBERT SPACES

The following result shows how well behaved Hilbert spaces are with regard to joint functional calculi. The techniques used in the proof are similar to those used to prove the sufficiency of quadratic estimates in [17].

THEOREM 4.1.   *Let $\omega < \mu < \pi$, and $\varpi < \nu < \pi$, and let T and S be commuting one-to-one operators of type ω and ϖ respectively, with dense domains and ranges in a complex Hilbert space $\mathcal{H}$. If T has a bounded $H^\infty(S_\mu)$ functional calculus, then*

$$\left\| f(T, S) \right\| \leqslant c \sup\left\{ \left\| f(z, S) \right\| : z \in S_\mu \right\}.$$

PROOF: By the results in [17], there exist constants q, q^* and a non-zero function $\psi \in \Psi(S_\nu)$ such that for every $x \in \mathcal{H}$,

$$\left\{ \int_0^\infty \left\| \psi(tT)x \right\|^2 \frac{dt}{t} \right\}^{1/2} \leqslant q \| x \|$$

and for T^*, the adjoint of T,

$$\left\{ \int_0^\infty \left\| \psi^*(tT^*)x \right\|^2 \frac{dt}{t} \right\}^{1/2} \leqslant q^* \| x \|$$

where $\psi^*(z) = \overline{\psi(\overline{z})}$. Let θ be any function in $\Psi(S_\nu)$ such that

$$\int_0^\infty \theta(t)\psi^2(t) \frac{dt}{t} = 1.$$

For $f \in H^\infty(S_\mu \times S_\nu)$ and $0 < \varepsilon < R < \infty$, define $f_{\varepsilon,R} \in \Psi(S_\nu)$ by

$$f_{\varepsilon,R}(z, w) = \int_\varepsilon^R f(z, w)\theta(tz)\psi^2(tz) \frac{dt}{t}.$$

Therefore, for every $x, y \in \mathcal{H}$, we have

$$\left| \langle f_{\varepsilon, R}(T, S)x, y \rangle \right| = \left| \int_{\varepsilon}^{R} \langle f(T, S)\theta(tT)\psi(tT)x, \psi^*(tT^*)y \rangle \frac{dt}{t} \right|$$
$$\leqslant q\, q^* \sup_t \left\| f(T, S)\theta(tT) \right\| \left\| x \right\| \left\| y \right\|.$$

Since $\theta(tz)$ is uniformly bounded with respect to t in $L^1(\partial S_\mu, |dz|/|z|)$, one obtains

$$\sup_t \left\| f(T, S)\theta(tT) \right\| = \sup_t \left\| \frac{1}{2\pi i} \int_{\partial S_\mu} \theta(tz) f(z, S) \frac{1}{z - T} \, dz \right\|$$
$$\leqslant c \sup \left\{ \left\| f(z, S) \right\| : z \in S_\mu \right\},$$

where c depends on T, ν and θ but not f. The result now follows by the Convergence Lemma.   ⬚

With this theorem we have the following two corollaries, the first of which is immediate.

COROLLARY 4.2. *Let T and S be as in Theorem 4.1, and suppose further that T and S have bounded $H^\infty(S_\mu)$ and $H^\infty(S_\nu)$ functional calculi respectively. Then (T, S) has a bounded $H^\infty(S_\mu \times S_\nu)$ functional calculus.*

COROLLARY 4.3. *Let T and S be as in Theorem 4.1, and suppose further that $\omega + \varpi < \pi$. Then $S + T$, defined by the joint functional calculus, is a closed operator of type $\max\{\omega, \varpi\}$. Moreover, if T has a bounded $H^\infty(S_\mu)$ functional calculus then $\| Su \| \leqslant c\| Su + Tu \|$ for all $u \in \mathcal{D}(S) \cap \mathcal{D}(T)$.*

PROOF: By the Functional Calculus Theorem and Theorem 3.1, $S + T$ is a closed operator of type $\max\{\omega, \varpi\}$. Let $f(z, w) = w(z + w)^{-1}$. Then $f \in H^\infty(S_\mu \times S_\mu)$, where $\max\{\omega, \varpi\} < \mu < \pi$. So, by Theorem 4.1, $f(T, S)$ is bounded and, from the properties of the functional calculus of T and S one has

$$\| Su \| = \left\| f(T)(S + T)u \right\| \leqslant c\| Su + Tu \|$$

for all $u \in \mathcal{D}(S) \cap \mathcal{D}(T)$.   ⬚

Dore and Venni proved, in their paper [7], a similar result by a completely different technique, under the additional assumptions that S and T are invertible.

REMARK. The methods used in Theorem 4.1 can be used to show that on a Hilbert space, an operator T of type ω which admits a bounded $H^\infty(S_\mu)$ functional calculus, admits a bounded operator-valued functional calculus as well. More precisely, if $\mathcal{A}$ is the commutant of T and $F(z) \in H^\infty(S_\mu, \mathcal{A})$ then by following the proof of Theorem 4.1 replacing f by F one obtains a bounded operator $F(T)$ with bound depending on the uniform norm of F. More generally, even if T does not admit a bounded $H^\infty(S_\mu)$ functional calculus and acts on a general Banach space $\mathcal{X}$, one can show that there exists

a unique operator-valued functional calculus, that is a linear mapping from $H^\infty(S_\mu, \mathcal{A})$ into the closed operators on $\mathcal{X}$ which takes F to $F(T)$. More information about such functional calculi is contained in [1].

5. JOINT FUNCTIONAL CALCULI IN L^p SPACES

In this section we are concerned with operators acting on $L^p(\Omega)$, $1 < p < \infty$, where Ω is a σ-finite measure space. Our treatment here closely follows the work in one variable in [4]. Let $\omega < \mu < \pi$ and $\varpi < \nu < \pi$. Throughout this section (T, S) denotes a pair T and S of commuting one-to-one operators of type ω and ϖ respectively, with dense domains and ranges acting on $L^p(\Omega)$. Let p' denote the conjugate index of p, $\langle \cdot, \cdot \rangle$ the bilinear pairing between $L^p(\Omega)$ and $L^{p'}(\Omega)$, and T' and S' the operators on $L^{p'}(\Omega)$ which are dual to T and S with respect to this pairing.

Let $\psi \in \Psi(S_\mu \times S_\nu)$ and consider the following conditions which the pair (T, S) and (T', S') may satisfy:

$(F_{\mu,\nu})$ $\qquad \left\| f(T, S) \right\| \leqslant c \left\| f \right\|_\infty$, for all $f \in H^\infty(S_\mu \times S_\nu)$, and

(W_ψ) $\qquad \displaystyle\int_0^\infty \int_0^\infty \left| \langle \psi(tT, sS)u, v \rangle \right| \frac{dt\,ds}{ts} \leqslant c \left\| u \right\|_p \left\| v \right\|_{p'}$,

(S_ψ) $\qquad \left\| \left[\displaystyle\int_0^\infty \int_0^\infty \left| \psi(tT, sS)u(\cdot) \right|^2 \frac{dt\,ds}{ts} \right]^{1/2} \right\|_p \leqslant c \left\| u \right\|_p$,

(S_ψ^*) $\qquad \left\| \left[\displaystyle\int_0^\infty \int_0^\infty \left| \psi(tT', sS')v(\cdot) \right|^2 \frac{dt\,ds}{ts} \right]^{1/2} \right\|_{p'} \leqslant c \left\| v \right\|_{p'}$,

for all $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$, where c denotes a constant which may be different for different conditions.

These conditions are natural generalisations of the conditions for a single operator of type ω which were investigated by Cowling et alia [4]. The functional condition $(F_{\mu,\nu})$ means that the pair of operators (T, S) has a bounded $H^\infty(S_\mu \times S_\nu)$ functional calculus, and the conditions (W_ψ) and (S_ψ) are weak and strong quadratic estimates respectively.

For a single operator in a Hilbert space, it was shown by McIntosh in [17] that the strong quadratic estimate, together with the dual statement, is equivalent to the functional condition, and to the weak quadratic estimate. In [4] it was shown for general Banach spaces, that the functional condition is equivalent to the weak quadratic estimate, while for L^p spaces, it is also equivalent to the strong quadratic estimate together with the dual statement.

In this section we shall show that (T, S) satisfying the functional condition is equivalent to T and S satisfying individual functional conditions. Before proving this we shall list several results for pairs of operators. The proofs of these results are not given and are similar to the proofs for a single operator given in [4].

(a) If (T, S) satisfies condition $(F_{\mu,\nu})$ then (T', S') satisfies condition $(F_{\mu,\nu})$.

(b) If (T, S) satisfies condition $(F_{\mu,\nu})$ and $\psi \in \Psi(S_\mu \times S_\nu)$ then (T, S) satisfies condition (W_ψ).

(c) If (T, S) satisfies (W_ψ) for some ψ in $\Psi_{\kappa-,\lambda-}$ (defined below), where $\omega < \kappa < \pi$ and $\varpi < \lambda < \pi$ then (T, S) satisfies $(F_{\eta,\zeta})$, where $\kappa < \eta < \pi$ and $\lambda < \zeta < \pi$.

(d) If (T, S) satisfies (S_ψ) and (T', S') satisfies (S_ϕ^*), where ψ, $\phi \in \Psi(S_\mu \times S_\nu)$ then (T, S) satisfies $(W_{\psi\bar\phi})$.

The Mellin transforms of $\phi \in \Psi(S_\mu)$ and $\psi \in \Psi(S_\mu \times S_\nu)$ are defined as follows:

$$\widehat{\phi}_e(x) = \int_0^\infty \psi(t) t^{-ix}\, \frac{dt}{t}, \qquad\qquad \text{for all } x \in \mathbb{R}, \text{ and}$$

$$\widehat{\psi}_e(x, y) = \int_0^\infty \int_0^\infty \psi(t, s) t^{-ix} s^{-iy}\, \frac{dt\, ds}{ts}, \qquad \text{for all } (x, y) \in \mathbb{R}^2.$$

Let $\Psi_\kappa(S_\mu)$ and $\Psi_{\kappa,\lambda}(S_\mu \times S_\nu)$ be the subspaces of $\Psi(S_\mu)$ and $\Psi(S_\mu \times S_\nu)$ respectively, consisting of those functions ϕ and ψ respectively, for which there exists $c > 0$ such that

$$\left|\widehat{\phi}_e(x)\right| \geqslant c\, e^{-\kappa|x|}, \qquad\qquad \text{for all } x \in \mathbb{R}, \text{ and}$$

$$\left|\widehat{\psi}_e(x, y)\right| \geqslant c\, e^{-\kappa|x|-\lambda|y|}, \qquad \text{for all } (x, y) \in \mathbb{R}^2, \text{ and let}$$

$$\Psi_{\kappa-} = \bigcap_{0<\mu<\kappa} \Psi_\kappa(S_\mu) \qquad \text{and} \qquad \Psi_{\kappa-,\lambda-} = \bigcap_{\substack{0<\mu<\kappa\\0<\nu<\lambda}} \Psi_{\kappa,\lambda}(S_\mu \times S_\nu).$$

EXAMPLE 5.1. *Let $0 < \theta < \pi$ and let*

$$g_\theta(z) = \frac{e^{i\theta}}{z - e^{i\theta}} - \frac{e^{-i\theta}}{z - e^{-i\theta}}.$$

Then g_θ, $g_\theta^2 \in \Psi_{\theta-}$. Moreover, if $f(z, w) = g_\theta(z)g_\theta(w)$ then f, $f^2 \in \Psi_{\theta-,\theta-}$.

The following lemma is a two dimensional randomisation lemma (see Stein [23] for more details).

LEMMA 5.2. *Suppose that $1 < p < \infty$ and that $\{u_{k,l}\}$ is a double sequence of functions in $L^p(\Omega)$. Then*

$$\left\|\left(\sum_{k,l} |u_{k,l}|^2\right)^{1/2}\right\|_p \leqslant C \sup_{|\alpha_k|\leqslant 1} \sup_{|\beta_l|\leqslant 1} \left\|\sum_{k,l} \alpha_k \beta_l u_{k,l}\right\|_p,$$

where C is a constant.

PROOF: Let $\{r_k\}$ denote the Rademacher functions on $[0,1]$. Then, by the results in [**23**] we have:

$$
\begin{aligned}
\left\|\left(\sum_{k,l}|u_{k,l}|^2\right)^{1/2}\right\|_p^p &= \int_\Omega \left(\sum_{k,l}|u_{k,l}(x)|^2\right)^{p/2} d\mu(x) \\
&= \int_\Omega \left(\int_0^1\int_0^1\left|\sum_{k,l}r_k(t)r_l(s)u_{k,l}(x)\right|^2 dt\,ds\right)^{p/2} d\mu(x) \\
&\leqslant C\int_\Omega\int_0^1\int_0^1\left|\sum_{k,l}r_k(t)r_l(s)u_{k,l}(x)\right|^p dt\,ds\,d\mu(x) \\
&= C\int_0^1\int_0^1\left\|\sum_{k,l}r_k(t)r_l(s)u_{k,l}(x)\right\|_p^p dt\,ds \\
&\leqslant C\sup_{|\alpha_k|\leqslant 1}\sup_{|\beta_l|\leqslant 1}\left\|\sum_{k,l}\alpha_k\beta_l u_{k,l}\right\|_p^p.
\end{aligned}
$$

<div align="right">□</div>

THEOREM 5.3. *Suppose* $\psi \in \Psi(S_\mu)$, $\phi \in \Psi(S_\nu)$, T *satisfies* $(F_{\mu'})$, *where* $\omega < \mu' < \mu$, *and* S *satisfies* $(F_{\nu'})$, *where* $\varpi < \nu' < \nu$. *Then* (T,S) *satisfies* $(S_{\psi\phi})$.

PROOF: Let $h \in C_0^\infty(\mathbb{R})$ be such that $\mathrm{supp}(h) \subset [-2,2]$, and

$$
\sum_{k=-\infty}^{\infty} h_k(s)^2 \equiv 1,
$$

where $h_k(s) = h(s-k)$. Then

$$
\begin{aligned}
&\left\{\int_0^\infty\int_0^\infty|\psi(tT)\phi(sS)u|^2\frac{dt}{t}\frac{ds}{s}\right\}^{1/2} \\
&= \left\{\frac{1}{4\pi^2}\int_{-\infty}^\infty\int_{-\infty}^\infty\left|\widehat{\psi_e}(\kappa)T^{i\kappa}\widehat{\phi_e}(\lambda)S^{i\lambda}u\right|^2 d\kappa\,d\lambda\right\}^{1/2} \\
&= \left\{\frac{1}{4\pi^2}\sum_{k=-\infty}^\infty\sum_{l=-\infty}^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty\left|h_k(\kappa)\widehat{\psi_e}(\kappa)T^{i\kappa}h_l(\lambda)\widehat{\phi_e}(\lambda)S^{i\lambda}u\right|^2 d\kappa\,d\lambda\right\}^{1/2} \\
&\leqslant \frac{1}{2\pi}\sum_{k,l}\left\{\int_{-\infty}^\infty\int_{-\infty}^\infty\left|h_k(\kappa)\widehat{\psi_e}(\kappa)T^{i\kappa}h_l(\lambda)\widehat{\phi_e}(\lambda)S^{i\lambda}u\right|^2 d\kappa\,d\lambda\right\}^{1/2}.
\end{aligned}
$$

Thus

$$
\left\|\left\{\int_0^\infty\int_0^\infty|\psi(tT)\phi(sS)u|^2\frac{dt}{t}\frac{ds}{s}\right\}^{1/2}\right\|_p \leqslant C\sum_{k,l}\left\|\left\{\sum_{n,m}|a_{k,n}(T)b_{l,m}(S)u|^2\right\}^{1/2}\right\|_p
$$

where

$$a_{k,n}(z) = \int_{-\infty}^{\infty} h_k(\kappa)\widehat{\psi_e}(\kappa)e^{-in\kappa}z^{i\kappa}\,d\kappa \qquad \text{for all } z \in S_\mu, \text{ and}$$

$$b_{l,m}(w) = \int_{-\infty}^{\infty} h_l(\lambda)\widehat{\phi_e}(\lambda)e^{-im\lambda}w^{i\lambda}\,d\lambda \qquad \text{for all } w \in S_\nu.$$

In Lemma 6.5 of [4] it is shown that

$$\sum_n \sup_{\zeta \in S_\mu} \sum_k \left|a_{k,n}(\zeta)\right| < \infty, \text{ and } \sum_m \sup_{w \in S_\nu} \sum_l \left|b_{l,m}(w)\right| < \infty.$$

Let $\|u\|_{p,\mu,\nu} = \sup\left\{\|a(T)b(S)u\| : a \in H^\infty(S_\mu),\ b \in H^\infty(S_\nu)\ \|a\|_\infty, \|b\|_\infty \leqslant 1\right\}$. Using Lemma 5.2 and Lemma 6.5 of [4], we have:

$$\sum_{k,l}\left\|\left(\sum_{n,m}\left|a_{k,n}(T)b_{l,m}(S)u\right|^2\right)^{1/2}\right\|_p \leqslant C\sum_{k,l}\sup_{|\alpha_n|,|\beta_m|\leqslant 1}\left\|\sum_{n,m}\alpha_n a_{k,n}(T)\beta_m b_{l,m}(S)u\right\|_p$$

$$\leqslant C\sum_{k,l}\sup_{z\in S_{\mu'},w\in S_{\nu'}}\sum_{n,m}\left|a_{k,n}(z)b_{l,m}(w)\right|\,\|u\|_{p,\nu',\mu'}$$

$$\leqslant C'\|u\|_{p,\nu',\mu'}$$

$$\leqslant C'\|u\|_p.$$

□

We come now to the main result in this section.

THEOREM 5.4. *Suppose $\omega < \kappa < \eta < \pi$, T satisfies (F_κ), $\varpi < \lambda < \zeta < \pi$, and S satisfies (F_λ). Then (T,S) satisfies $(F_{\eta,\zeta})$.*

PROOF: Let $\kappa < \theta < \eta$ and $\lambda < \vartheta < \zeta$. Take any $\psi \in \Psi_{\theta-}$ and $\phi \in \Psi_{\vartheta-}$ such that $\psi^2\phi^2 \in \Psi_{\theta-,\vartheta-}$. Then, since T satisfies (F_κ) and S satisfies (F_λ), it follows from Theorem 5.3 and (a) that (T,S) satisfies $(S_{\psi\phi})$ and (T',S') satisfies $(S_{\overline{\psi\phi}})$. Therefore, by (d), (T,S) satisfies $(W_{\psi^2\phi^2})$. Hence, by (c), (T,S) satisfies $(F_{\eta,\zeta})$. □

REMARK. An alternative proof of Theorem 5.4 appears in [10]. It uses discrete quadratic estimates directly, and so does not need the argument in Theorem 5.3 of getting from discrete estimates to continuous ones.

An immediate consequence is the result of Dore and Venni [7] and Giga and Sohr [12], in the special case when $\mathcal{X} = L^p(\Omega)$ and T and S have bounded functional calculi. (They proved it when $\mathcal{X}$ is a UMD space and T, S have bounded imaginary powers.) See also [21] and [20].

COROLLARY 5.5. *Let T and S be commuting one-to-one operators in $L^p(\Omega)$ of type ω and ϖ respectively. Suppose $\omega < \kappa < \pi$, T satisfies (F_κ), $\varpi < \lambda < \pi$, S satisfies (F_λ), and $\kappa + \lambda < \pi$. Then $T + S$ is closed. Moreover $\|Su\| \leqslant c\|(S+T)u\|$ for all $u \in \mathcal{D}(S) \cap \mathcal{D}(T)$.*

PROOF: Let $f(z, w) = w(z + w)^{-1}$. Then $f \in H^\infty(S_\mu \times S_\nu)$, where $\kappa < \mu < \pi$, $\lambda < \nu < \pi$ and $\mu + \nu < \pi$. Therefore, by Theorem 5.4, $f(T, S)$ is a bounded operator, and so by a similar argument as in the proof of Corollary 4.3, the result follows. □

Another consequence of Theorem 5.4 is the following result of Giga, Giga and Sohr [11], in the special case when $\mathcal{X} = L^p(\Omega)$ and T and S have bounded functional calculi. (They proved it when $\mathcal{X}$ is a UMD space and T and S have bounded imaginary powers.)

PROPOSITION 5.6. *Let T and S be commuting one-to-one operators in $L^p(\Omega)$ of type ω and ϖ respectively. Suppose T satisfies (F_κ), S satisfies (F_λ), $\omega < \kappa$, $\varpi < \lambda$, $\kappa + \lambda < \pi$, and $0 < \alpha < 1$. Then the norms $\|T^\alpha u\| + \|S^\alpha u\|$, $\|(T^\alpha + S^\alpha)u\|$, and $\|(T + S)^\alpha u\|$ are all equivalent on $\mathcal{D}(T^\alpha) \cap \mathcal{D}(S^\alpha)$.*

PROOF: Let $f_\alpha(z, w) = z^\alpha(z + w)^{-\alpha}$. Then $f_\alpha \in H^\infty(S_\mu \times S_\nu)$ where $\kappa < \mu < \pi$ and $\lambda < \nu < \pi$. Therefore, by Theorem 5.4, $f_\alpha(T, S)$ is a bounded operator, and so

$$\|T^\alpha u\| \leqslant \|f_\alpha(T, S)(T + S)^\alpha u\| \leqslant \|f_\alpha\|_\infty \|(T + S)^\alpha u\|.$$

Consideration of other functions such as $w^\alpha(z + w)^{-\alpha}$ and $(z + w)^\alpha(z^\alpha + w^\alpha)^{-1}$ gives the result. □

6. COUNTEREXAMPLE

One might expect that a pair (T, S) of commuting operators of type ω admits a bounded joint functional calculus when T and S individually have bounded holomorphic functional calculi. The following counterexample shows that this need not be true, even for nice operators, if the underlying Banach space is bad enough.

We now construct an example of two operators with bounded holomorphic functional calculi, but for which the joint functional calculus is not bounded because of the nature of the Banach space in which they act. Indeed the estimate of Dore and Venni is not valid in this case.

The Banach space is $\mathcal{X} = \mathcal{K}(l_2)$, namely the space of all compact linear operators on the Hilbert sequence space l_2 with the operator norm. The operators S and T are defined by $S(K) = AK$ and $T(K) = KA$ where A is the unbounded self-adjoint operator in l_2 defined by $(Au)_j = 2^j u_j$ for all $u \in \mathcal{D}(A)$, where

$$\mathcal{D}(A) = \left\{ u \in l_2 : \sum_{j=1}^\infty 2^{2j}|u_j|^2 < \infty \right\}.$$

The operators S and T have domains

$$\mathcal{D}(S) = \{ K \in \mathcal{X} : \mathcal{R}(K) \subset \mathcal{D}(A) \text{ and } AK \in \mathcal{X} \}, \text{ and}$$
$$\mathcal{D}(T) = \{ K \in \mathcal{X} : KA \in \mathcal{X} \}.$$

It is straightforward to verify that S and T are both closed one-to-one operators with dense domains and dense ranges. Moreover, if $\zeta \in \rho(A)$ then $\zeta \in \rho(S)$ and $\zeta \in \rho(T)$ because

$$(S - \zeta I)^{-1}(K) = (A - \zeta I)^{-1}K \qquad \text{and}$$
$$(T - \zeta I)^{-1}(K) = K(A - \zeta I)^{-1},$$

for all $K \in \mathcal{X}$, and indeed $\left\|(S-\zeta I)^{-1}\right\| = \left\|(T-\zeta I)^{-1}\right\| = \left\|(A-\zeta I)^{-1}\right\|$. Therefore both S and T are of type 0 in $\mathcal{X}$. Clearly S and T commute (in the sense that their resolvents commute).

By similar reasoning we see that S and T each have a bounded $H^\infty(S_\mu)$ functional calculus for all $\mu > 0$. Indeed $f(S)(K) = f(A)K$ and $f(T)K = Kf(A)$ for all $K \in \mathcal{X}$, and so

$$\left\|f(S)\right\| = \left\|f(T)\right\| = \left\|f(A)\right\| = \|f\|_\infty$$

for all $f \in H^\infty(S_\mu)$.

Let us now consider the action of $S/(S+T)$ on the particular element $K = (K_{jk}) \in \mathcal{X}$ defined by

$$K_{jk} = \begin{cases} \dfrac{1}{k-j} & \text{if } j \neq k,\ 1 \leqslant j \leqslant N,\ 1 \leqslant k \leqslant N \\ 0 & \text{otherwise} \end{cases}$$

where N is a large positive integer. Note that K is essentially a finite Toeplitz matrix corresponding to the function $f(\theta) = i(\pi - \theta)$ on $0 < \theta < 2\pi$, and so $\|K\| \leqslant \pi$.

Let $Z = (Z_{jk}) = \big(S/(S+T)\big)(K)$. Then

$$Z_{jk} = \begin{cases} \dfrac{2^j}{(2^j + 2^k)(k-j)} & \text{if } j \neq k,\ 1 \leqslant j \leqslant N,\ 1 \leqslant k \leqslant N \\ 0 & \text{otherwise,} \end{cases}$$

and $\|Z\| \geqslant \big(\sqrt{2}/3\big) \log(N/2) - 1$, as is seen by applying Z to the vector $v = (v_j)$ where $v_j = 1$ if $1 \leqslant j \leqslant N$ and $v_j = 0$ if $j > N$.

Therefore, for all $n > 0$, there exists $K \in \mathcal{D}\big(S/(S+T)\big) \in \mathcal{X}$ such that

$$\left\| \frac{S}{S+T}(K) \right\| \geqslant n\|K\|.$$

Hence $S/(S+T)$ is not a bounded operator. Consequently (T, S) does not have an $H^\infty(S_\mu \times S_\mu)$ functional calculus, at least when $\mu < \pi/2$. (It is easy to adapt the example to treat the case $\mu \geqslant \pi/2$.)

On defining $L = (L_{jk}) \in \mathcal{X}$ by

$$
L_{jk} = \begin{cases} \dfrac{1}{(2^j + 2^k)(k - j)} & \text{if } j \neq k, \, 1 \leqslant j \leqslant N, \, 1 \leqslant k \leqslant N \\ 0 & \text{otherwise,} \end{cases}
$$

we see that

$$
\left\| S(L) \right\| \geqslant \frac{1}{\pi} \left(\frac{\sqrt{2}}{3} \log\left(\frac{N}{2} \right) - 1 \right) \left\| (S + T)(L) \right\|
$$

so that the estimate of Dore and Venni does not hold in the space $\mathcal{X}$.

This example is based on previous counterexamples of McIntosh. See, for example [19]. We also note that the imaginary powers of ST^{-1} form a contractive C_0-group but $(I + ST^{-1})^{-1}$ is unbounded. This provides an example of a C_0-group with an analytic generator which is not an operator of type ω; see also [9].

REFERENCES

[1] D. Albrecht, *Functional calculi of commuting unbounded operators*, Ph.D. thesis (Monash University, Melbourne, Australia, 1994).

[2] P. Auscher, A. McIntosh and A. Nahmod, 'Holomorphic functional calculi of operators, quadratic estimates and interpolation', *Indiana Univ. Math. J.* **46** (1997), 375–403.

[3] J.B. Baillon and Ph. Clement, 'Examples of unbounded imaginary powers of operators', *J. Funct. Anal.* **100** (1991), 419–434.

[4] M.G. Cowling, I. Doust, A. McIntosh and A. Yagi, 'Banach space operators with a bounded H^∞ functional calculus', *J. Austral. Math. Soc. Ser. A* **60** (1996), 51–89.

[5] R. deLaubenfels, 'Unbounded holomorphic functional calculus and abstract Cauchy problems for operators with polynomial bounded resolvents', *J. Funct. Anal.* **114** (1993), 348–394.

[6] R. deLaubenfels, *Existence families, functional calculi, and evolution*, Lecture notes in mathematics **1570** (Springer-Verlag, Berlin, Heidelberg, New York, 1994).

[7] G. Dore and A. Venni, 'On the closedness of the sum of two closed operators', *Math. Z.* **196** (1987), 189–201.

[8] E. Franks, 'Modified Cauchy kernels and functional calculus for operators on Banach space', *J. Austral. Math. Soc. Ser. A* **63** (1997), 91–99.

[9] E. Franks, 'Operators of type ω and strip type', preprint (1997).

[10] E. Franks and A. McIntosh, 'Discrete quadratic estimates and holomorphic functional calculi of operators in Banach spaces', Macqarie Mathematics Report 95/192 (1995).

[11] M. Giga, Y. Giga and H. Sohr, 'L^p estimates for the Stokes system', in *Functional Analysis and Related Topics (Berlin)*, (H. Komatsu, Editor), Lecture notes in mathematics **1540** (Springer-Verlag, Berlin, Heidelberg, New York, 1991), pp. 55–67.

[12] Y. Giga and H. Sohr, 'Abstract L^p estimates for the Cauchy problem with applications to the Navier-Stokes equations in exterior domains', *J. Funct. Anal.* **102** (1991), 72–94.

[13] T. Kato, *Perturbation Theory for Linear Operators*, (2nd edition) (Springer-Verlag, Berlin, Heidelberg, New York, 1976).

[14]   F. Lancien, G. Lancien and C. Le Merdy, 'A joint functional calculus for sectorial oper-
        ators with commuting resolvents', *Proc. London Math. Soc.* (to appear).

[15]   G. Lancien and C. Le Merdy, 'A generalized H^∞ functional calculus for operators on
        subspaces of L^p and application to maximal regularity', (preprint).

[16]   C. Li and A. McIntosh, 'Clifford algebras and H^∞ functional calculi of commuting op-
        erators', in *Clifford algebras in analysis and related topics*, (John Ryan, Editor) (CRC
        Press, Boca Raton, FL., 1995), **pp.** 89–101.

[17]   A. McIntosh, 'Operators which have an H^∞ functional calculus', in *Miniconference on
        Operator Theory and Partial Differential Equations*, Proc. Centre Math. Analysis **14**
        (A.N.U., Canberra, Australia, 1986), **pp.** 210–231.

[18]   A. McIntosh and T. Qian, 'Convolution singular integrals operators on Lipschitz curves',
        in *Proceedings of the Special Year on Harmonic Analysis at Nankai Institute of Math-
        ematics, Tianjin, China*, Lecture Notes in Mathematics **1494** (Springer-Verlag, Berlin,
        Heidelberg, New York, 1991), **pp.** 142–162.

[19]   A. McIntosh and A. Yagi, 'Operators of type-ω without a bounded H^∞ functional cal-
        culus', in *Miniconference on Operators in Analysis*, Proc. Centre. Math. Analysis **24**
        (A.N.U., Canberra, Australia, 1989), **pp.** 159–172.

[20]   S. Monniaux, 'A new approach to the Dore–Venni theorem', *Math. Nachr.* (to appear).

[21]   J. Prüss and H. Sohr, 'On operators with bounded imaginary powers in Banach spaces',
        Math. Z. **203** (1990), 429–452.

[22]   R. Seeley, 'Norms and domains of the complex powers A_b^z', *Amer. J. Math.* **93** (1971),
        299–309.

[23]   E.M. Stein, *Singular integrals and differentiability properties of functions* (Princeton Uni-
        versity Press, Princeton, N.J., 1970).

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