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ON PAIRED ROOT SYSTEMS OF COXETER GROUPS

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Abstract

This paper examines a systematic method of constructing a pair of (inter-related) root systems for arbitrary Coxeter groups from a class of nonstandard geometric representations. This method can be employed to construct generalizations of root systems for a large family of linear groups generated by involutions. We then give a characterization of Coxeter groups, among these groups, in terms of such paired root systems. Furthermore, we use this method to construct and study the paired root systems for reflection subgroups of Coxeter groups.

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1. Introduction

A *Coxeter group* W is an abstract group generated by a set of involutions R, called its *Coxeter generators*, subject only to certain braid relations. Despite the simplicity of this definition, there is a rich theory for Coxeter groups with nontrivial applications in a multitude of areas of mathematics and physics. When studying Coxeter groups, one of the most powerful tools we have at our disposal is the notion of *root systems*. In the classical literature ([2, Ch. V, Section 4] or [13, Sections 5.3–5.4], for example), the root system of a Coxeter group W is a geometric construction arising from the *Tits representation* of W. The Tits representation of W is an embedding of W into the orthogonal group of a certain bilinear form on a suitably chosen vector space V subject to the requirement that the W-conjugates of elements of R are mapped to reflections with respect to certain hyperplanes in V. In the case that W is finite, V is Euclidean (of dimension equal to the cardinality of R), and the root system of W simply consists of representative normal vectors for these hyperplanes. Those elements of the root system corresponding to the elements of R are known as *simple roots*, and in most of the classical literature ([2, Ch. V, Section 4] and [13, Section 5.3], for example), the simple roots are linearly independent.

Similar constructions of root systems can be extended to infinite Coxeter groups and Kac–Moody Lie algebras. However, the actual constructions of root systems differ depending on whether the root systems are associated to Kac–Moody Lie algebras or infinite Coxeter groups. As discussed in the introduction of [12], while all definitions of root systems are related to a given bilinear form, the actual bilinear forms considered in the case of Kac–Moody Lie algebras are different from the ones in Coxeter groups. Furthermore, it is well-known ([6] and [7, Ch.3]) that within an arbitrary Coxeter group W, all of its reflection subgroups are themselves Coxeter groups, but in the literature ([7] or [8], for example), the construction of the root systems corresponding to such reflection subgroups as subsets of the root system of W requires special care. In particular, since a proper reflection subgroup may have strictly more Coxeter generators than the over-group (as seen in [12, Example 5.1]), the equivalent of the simple roots in these root systems need not be linearly independent, making the overall theory of root systems and root bases somewhat less uniform. Consequently, it seems profitable to develop a universal method for constructing root systems that is applicable to arbitrary Coxeter groups and their reflection subgroups, as well as to objects like Kac–Moody Lie algebras (in fact, to all groups with a so-called *root group datum*, as surveyed in [3]).

In [4, 5, 14] and [3], a number of more general notions of root systems have been studied. In these approaches, a pair of root systems are constructed in two vector spaces which are essentially algebraic duals of each other (apart from [14], the two vector spaces involved are explicitly required to be algebraic duals of each other, whereas in [14] the two vector spaces are linked by a nondegenerate bilinear pairing satisfying certain integrality conditions).

Recently, an approach taken in [10] and [11] generalizes those of [4, 5, 14] and [3]. In this approach, for an arbitrary Coxeter group, a pair of root systems are constructed in two vector spaces linked only by a bilinear pairing which does not require the nondegeneracy and integrality conditions of [14]. This particular approach allows more abstract geometry in the two root systems to take place (for example, the two representation spaces need not be algebraic duals of each other), whilst providing a unified theory of root systems, especially with respect to reflection subgroups (this last point is to be established in Section 3 of this present paper).

In this paper, we present a few results demonstrating the 'universalness' of the notion of root systems in [10] and [11]. In fact, this new approach applies to a large family of linear groups generated by involutions, and one of the key results of this paper (Theorem 2.8) shows that these groups are Coxeter groups only if the corresponding root systems decompose as disjoint unions of those roots generalizing the classical concept of *positive roots* and those roots generalizing the classical concept of *negative roots*. In fact, this result provides an alternative characterization for Coxeter groups, since it is well-known that for any Coxeter group we may construct a root system that decomposes in the same way. This alternative characterization is implicitly suggested in the work of Dyer [7], and we are very grateful to Dyer for a large number of helpful suggestions leading to the development of this generalized

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notion of root systems. Indeed, we have recently been made aware that Dyer has obtained results similar to those contained in Section 2 of the present paper in Section 2 of his preprint [9].

The main body of this paper is organized into two sections, namely, Sections 2 and 3. In Section 2 we develop a notion of root systems applicable to a large family of groups that are generated only by involutions, and we investigate when these root systems may decompose into disjoint unions of the so-called *positive roots* and the so-called *negative roots*, and we prove that such groups are Coxeter groups only if such decompositions take place (Theorems 2.4 and 2.8). In Section 3 we prove that the notion of root systems in [10] and [11] applies to all the reflection subgroups of any Coxeter group. In particular, we give a geometric characterization of the roots that correspond to the Coxeter generators of reflection subgroups (Propositions 3.16 and 3.24), and we show that these characterizations are precisely those allowing these roots to be *the simple roots* for the root systems of such reflection subgroups within the root systems of the respective over-groups.

NOTATION 1.1. If A is a subset of a real vector space then we define the *positive linear cone* of A, denoted $\mathrm{PLC}(A)$, to be the set

$$\Big\{ \sum_{a\in A} c_a a \;\Big|\; c_a \geq 0 \text{ for all } a \in A \text{ with } c_{a'} > 0 \text{ for some } a' \in A \text{ and } c_a = 0 \text{ for all but finitely many } a \in A \Big\}.$$

Furthermore, we define $-A := \{-v \mid v \in A\}$. Also, if B is a subset of a group G then $\langle B\rangle$ denotes the subgroup of G generated by B.

2. Decomposition of root systems and Coxeter datum

Let V_1 and V_2 be vector spaces over the real field $\mathbb{R}$ equipped with a bilinear pairing $\langle\, ,\, \rangle\colon V_1 \times V_2 \to \mathbb{R}$. Let S be an indexing set, and suppose that $\Pi_1 := \{\alpha_s \mid s \in S\} \subseteq V_1$ and $\Pi_2 := \{\beta_s \mid s \in S\} \subseteq V_2$ such that the maps $s \mapsto \alpha_s$ and $s \mapsto \beta_s$ give bijections $S \to \Pi_1$ and $S \to \Pi_2$ respectively. Further, suppose that Π_1 and Π_2 satisfy the following conditions:

(D1) $\langle \alpha_s, \beta_s\rangle = 1$ for all $s \in S$;

(D2) (i) $0 \notin \mathrm{PLC}(\Pi_1)$ and $0 \notin \mathrm{PLC}(\Pi_2)$.

(ii) $\alpha_s \notin \mathrm{PLC}(\Pi_1 \setminus \{\alpha_s\})$ and $\beta_s \notin \mathrm{PLC}(\Pi_2 \setminus \{\beta_s\})$ for each $s \in S$.

We call $\mathscr{C} := (S, V_1, V_1, \Pi_1, \Pi_2, \langle\, ,\, \rangle)$ a *datum*.

Observe that condition (D2) (i) implies that for each $s \in S$, we shall have both $\alpha_s \notin \mathrm{PLC}(-\Pi_1 \setminus \{-\alpha_s\})$ and $\beta_s \notin \mathrm{PLC}(-\Pi_2 \setminus \{-\beta_s\})$. We remark that there can be examples for which Π_1 (respectively Π_2) is linearly dependent, in which case necessarily some α_s (respectively β_s) will be expressible as a linear combination of $\Pi_1 \setminus \{\alpha_s\}$ (respectively $\Pi_2 \setminus \{\beta_s\}$) with coefficients of mixed signs. Furthermore, (D1) and (D2) require that any finite subset of Π_1 (respectively, Π_2) to be a set of representatives of

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the extreme rays of a pointed polyhedral cone (where pointed means a cone intersects its negative only at the origin). It should also be observed that no distinct elements of Π_1 (respectively, Π_2) are proportional.

Definition 2.1. For $s \in S$, define $\rho_1(s) \in \mathrm{GL}(V_1)$ and $\rho_2(s) \in \mathrm{GL}(V_2)$ by the rules

$$\rho_1(s)(x) := x - 2\langle x, \beta_s\rangle \alpha_s$$

for all $x \in V_1$, and

$$\rho_2(s)(y) := y - 2\langle \alpha_s, y\rangle \beta_s$$

for all $y \in V_2$. Further, we define, for each $i \in \{1, 2\}$:

$$\begin{aligned} R_i &:= \{\rho_i(s) \mid s \in S\}; \\ W_i &:= \langle R_i \rangle; \\ \Phi_i &:= W_i \Pi_i; \\ \Phi_i^+ &:= \Phi_i \cap \mathrm{PLC}(\Pi_i); \end{aligned}$$

and

$$\Phi_i^- := -\Phi_i^+.$$

For each $i \in \{1, 2\}$, and for each $s \in S$, we call $\rho_i(s)$ the *reflections* corresponding to s in V_i. We call Φ_i the *root system* for the *Weyl group* W_i realized in V_i, and we call Π_i the set of *simple roots* in Φ_i. Furthermore, we call Φ_i^+ the set of *positive roots* in Φ_i and Φ_i^- the set of *negative roots* in Φ_i.

Definition 2.2. Let $\mathscr{C} = (S, V_1, V_2, \Pi_1, \Pi_2, \langle\, ,\, \rangle)$ be a datum, and $c := (c_s)_{s\in S}$ a family of positive real numbers. Then $\mathscr{C}_c := (S, V_1, V_2, \Pi_1', \Pi_2', \langle\, ,\, \rangle)$ is a new datum satisfying conditions (D1) and (D2), where $\Pi_1' := \{c_s\alpha_s \mid s \in S\}$ and $\Pi_2' := \{c_s^{-1}\beta_s \mid s \in S\}$. We say that $\mathscr{C}_c$ is obtained by *rescaling* $\mathscr{C}$ by c.

Remark 2.3. For each $i \in \{1, 2\}$ and each $s \in S$ note that $\rho_i(s)$ is an involution with a -1-eigenvector of multiplicity 1. Also it is clear that if $\mathscr{C}$ is a datum and $\mathscr{C}_c$ is a rescaling of it, then they give rise to the same maps in R_i (and hence they both correspond to the same group W), however rescaling a datum may give rise to different Φ_i. Furthermore, it is a consequence of condition (D2) that $\Phi_i^+ \cap \Phi_i^- = \emptyset$. Use $\uplus$ to denote disjoint unions, which gives the following result.

Theorem 2.4. *Given conditions (D1) and (D2), the following are equivalent*:

(i) $\Phi_1 = \Phi_1^+ \uplus \Phi_1^-$.
(ii) $\Phi_2 = \Phi_2^+ \uplus \Phi_2^-$.
(iii) *For all $s, t \in S$ the following three conditions are satisfied*:

(D3) $\langle \alpha_s, \beta_t\rangle \leq 0$ *and* $\langle \alpha_t, \beta_s\rangle \leq 0$ *whenever* $s \neq t$.
(D4) $\langle \alpha_s, \beta_t\rangle = 0$ *if and only if* $\langle \alpha_t, \beta_s\rangle = 0$.

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(D5) *Either* $\langle \alpha_s, \beta_t\rangle\langle \alpha_t, \beta_s\rangle = \cos^2(\pi/m_{st})$ *for some integer* $m_{st} \geq 2$, *or else* $\langle \alpha_s, \beta_t\rangle\langle \alpha_t, \beta_s\rangle \geq 1$.

PROOF. It is enough to show that (i) and (iii) are equivalent. Indeed, once this is proven then by duality the equivalence of (ii) and (iii) will follow easily, since (iii) is self-dual.

In showing that (i) and (iii) are equivalent, we first observe that (iii) implies (i). Indeed, given conditions (D3), (D4), and (D5) of the present paper then the datum $\mathscr{C} = (S, V_1, V_2, \Pi_1, \Pi_2, \langle , \rangle)$ forms a Coxeter datum in the sense of [11], and hence (i) follows immediately from Lemma 3.2 of [11]. So it only remains to prove that (i) implies (iii), and before we do so we introduce the following construction. Consider the datum $\mathscr{C} = (S, V_1, V_2, \Pi_1, \Pi_2, \langle\ ,\ \rangle)$ satisfying conditions (D1) and (D2). For any subset S' of S, one has a datum of the same type

$$\mathscr{C}^{S'} := (S', V_1, V_2, \Pi_1^{S'}, \Pi_2^{S'}, \langle\ ,\ \rangle),$$

where $\Pi_1^{S'} := \{\alpha_r \mid r \in S'\} \subseteq \Pi_1$, and $\Pi_2^{S'} := \{\beta_r \mid r \in S'\} \subseteq \Pi_2$. Clearly, $\mathscr{C}^{S'}$ also satisfies conditions (D1) and (D2). For a part (x) (one of (i) to (iii)) of Theorem 2.4, let $(x)^{S'}$ denote the corresponding assertion for the datum $\mathscr{C}^{S'}$. Note that any one of the assertions in Theorem 2.4 holds for $\mathscr{C}$ if and only if it holds for any (equivalently, all) rescalings $\mathscr{C}_c$ of $\mathscr{C}$.

Our strategy in proving that (i) implies (iii) is as follows: for all $S' \subseteq S$ with $|S'| = 2$,

(S1) first we show that $(i)^S$ implies that $(iii)^{S'}$;

(S2) then by concentrating in quadratic spaces, we prove that $(i)^{S'}$ implies that $(iii)^{S'}$;

(S3) finally, we prove that $(iii)^{S'}$ implies $(iii)^S$ (which is trivial, since $(iii)^{S'}$ for all $S' \subseteq S$ with $|S'| = 2$ is simply $(iii)^S$ itself).

Consequently, we only need to complete (S1) and (S2).

For (S1) suppose that $(i)^S$ holds and suppose for a contradiction that $(i)^{S'}$ does not hold for some two-element subset $S' = \{s, t\}$ of S. Then there must exist λ, μ both strictly positive such that

$$\lambda\alpha_s - \mu\alpha_t \in \langle \rho_1(s), \rho_1(t)\rangle\{\alpha_s, \alpha_t\}.$$

Since $(i)^S$ holds, it follows that either

$$\lambda\alpha_s - \mu\alpha_t = \sum_{r\in S} c_r\alpha_r, \text{ where } c_r \geq 0 \text{ for all } r \in S, \tag{2.1}$$

or else

$$\lambda\alpha_s - \mu\alpha_t = -\sum_{r\in S} c_r\alpha_r, \text{ where } c_r \geq 0 \text{ for all } r \in S. \tag{2.2}$$

If (2.1) is the case, then $(\lambda - c_s)\alpha_s = (\mu + c_t)\alpha_t + \sum_{r\in S\setminus\{s,t\}} c_r\alpha_r$. If $\lambda - c_s > 0$ then the above yields a contradiction to (D2) since then $\alpha_s \in \mathrm{PLC}(\Pi_1 \setminus \{\alpha_s\})$; on the other hand if $\lambda - c_s \leq 0$ then we again have a contradiction to (D2) since then $0 \in \mathrm{PLC}(\Pi_1)$. If (2.2) is the case, then an entirely similar argument produces the same contradiction to (D2), and thus (S1) is completed.

For (S2), let $S' = \{s, t\} \subseteq S$ be an arbitrary two-element set, and suppose that $(i)^{S'}$ holds. We need to show that then conditions $(D3)^{S'}$, $(D4)^{S'}$, and $(D5)^{S'}$ all hold.

Since $\rho_1(t)\alpha_s \in \langle\rho_1(s), \rho_1(t)\rangle\{\alpha_s, \alpha_t\}$ and $\rho_1(s)\alpha_t \in \langle\rho_1(s), \rho_1(t)\rangle\{\alpha_s, \alpha_t\}$, it follows from $(i)^{S'}$ and the definitions of $\rho_1(s)$ and $\rho_1(t)$ that $\langle\alpha_s, \beta_t\rangle \le 0$ and $\langle\alpha_t, \beta_s\rangle \le 0$ whenever s, t are distinct. Whence $(D3)^{S'}$ holds.

To prove $(D4)^{S'}$, suppose that $\langle\alpha_s, \beta_t\rangle = 0$. Since

$$\begin{aligned}\rho_1(t)\rho_1(s)\alpha_t = \rho_1(t)(\rho_1(s)\alpha_t) &= -\alpha_t - 2\langle\alpha_t, \beta_s\rangle\alpha_s + 4\langle\alpha_t, \beta_s\rangle\langle\alpha_s, \beta_t\rangle\alpha_t \\ &= -\alpha_t - 2\langle\alpha_t, \beta_s\rangle\alpha_s \\ &\in \langle\rho_1(s), \rho_1(t)\rangle\{\alpha_s, \alpha_t\},\end{aligned}$$

it follows from $(i)^{S'}$ and the fact $\langle\alpha_t, \beta_s\rangle \le 0$ just proved above that

$$\langle\alpha_t, \beta_s\rangle = 0.$$

Whence $(D4)^{S'}$ holds.

To prove $(D5)^{S'}$, first note that we may assume that $\langle\alpha_s, \beta_t\rangle\langle\alpha_t, \beta_s\rangle \ne 0$, for otherwise $\langle\alpha_s, \beta_t\rangle\langle\alpha_t, \beta_s\rangle = \cos^2(\pi/2)$, trivially satisfying (D5). By rescaling, we may further assume that $\langle\alpha_s, \beta_t\rangle = \langle\alpha_t, \beta_s\rangle = -\gamma$ for some γ. Further, we may assume that $\gamma \ge 0$ in view of $(D3)^{S'}$ above. Then the matrices M of $\rho_1(s)\rho_1(t)$ and M' of $\rho_1(t)\rho_1(s)$ in its action on the subspace with basis $\{\alpha_s, \alpha_t\}$ are

$$M = \begin{pmatrix} -1 & -2\langle\alpha_t, \beta_s\rangle \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2\langle\alpha_s, \beta_t\rangle & -1 \end{pmatrix} = \begin{pmatrix} 4\gamma^2 - 1 & -2\gamma \\ 2\gamma & -1 \end{pmatrix}, \tag{2.3}$$

and

$$M' = \begin{pmatrix} 1 & 0 \\ -2\langle\alpha_s, \beta_t\rangle & -1 \end{pmatrix}\begin{pmatrix} -1 & -2\langle\alpha_t, \beta_s\rangle \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2\gamma \\ -2\gamma & 4\gamma^2 - 1 \end{pmatrix}. \tag{2.4}$$

It follows from (2.3), (2.4) and an easy induction on $n \in \mathbb{N}$ that

$$M^n = \begin{pmatrix} c_{2n+1} & -c_{2n} \\ c_{2n} & -c_{2n-1} \end{pmatrix}, \tag{2.5}$$

and

$$M'^n = \begin{pmatrix} -c_{2n-1} & c_{2n} \\ -c_{2n} & c_{2n+1} \end{pmatrix}, \tag{2.6}$$

where

$$c_n = \begin{cases} n & \gamma = 1 \\ \dfrac{\sinh n\theta}{\sinh\theta} \quad (\text{where } \theta = \cosh^{-1}\gamma) & \gamma > 1 \\ \dfrac{\sin n\theta}{\sin\theta} \quad (\text{where } \theta = \cos^{-1}\gamma) & 0 \le \gamma < 1. \end{cases} \tag{2.7}$$

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Since α_s has the same coefficient in $(\rho_1(s)\rho_1(t))^n\alpha_s$ and $\rho_1(t)(\rho_1(s)\rho_1(t))^n\alpha_s$ (also, α_t has the same coefficient in $(\rho_1(t)\rho_1(s))^n\alpha_t$ and $\rho_1(s)(\rho_1(t)\rho_1(s))^n\alpha_t$), it follows from (2.5), (2.6), and (2.7) that

$$\langle\rho_1(s),\rho_1(t)\rangle\{\alpha_s,\alpha_t\} = \{\pm(c_n\alpha_s + c_{n\pm1}\alpha_t) \mid n \in \mathbb{N}\}. \tag{2.8}$$

If $0 \le \gamma < 1$ then $\gamma = \cos\theta$ where $0 < \theta \le \pi/2$. Let m be the largest integer such that

$$0 < \theta < 2\theta < \cdots < m\theta \le \pi.$$

Note that $m \ge 2$. Now if $m\theta \ne \pi$ then $\pi < (m+1)\theta < 2\pi$. Consequently, $c_m > 0$ and $c_{m+1} < 0$. If $m = 2r$ is even, then in view of (2.5)

$$(\rho_1(s)\rho_1(t))^r\alpha_s = c_{m+1}\alpha_s + c_m\alpha_t,$$

contradicting $(i)^{S'}$. If $m = 2r-1$ is odd, then in view of (2.5)

$$(\rho_1(s)\rho_1(t))^r\alpha_t = -c_{m+1}\alpha_s - c_m\alpha_t,$$

again contradicting $(i)^{S'}$. Thus, if $0 \le \gamma < 1$ then $\gamma = \cos(\pi/m)$ for some integer m with $m \ge 2$.

If $\gamma \ge 1$, then it follows form (2.7) and (2.8) that $(i)^{S'}$ imposes no restriction on $\langle\alpha_s,\beta_t\rangle\langle\alpha_t,\beta_s\rangle$ if $\langle\alpha_s,\beta_t\rangle\langle\alpha_t,\beta_s\rangle \ge 1$. Whence $(D5)^{S'}$ holds, and this completes (S2) and indeed the whole theorem is established. □

We acknowledge that Dyer has obtained similar observations in [9] (Lemma 2.3 and Theorem 2.10). A consequence of Theorem 2.4 is that if any of the equivalent conditions in it is satisfied then W_1 and W_2 are isomorphic Coxeter groups. We would also like to thank the referee for suggesting to us the current proof of Theorem 2.4 which has replaced a much longer proof.

Notation 2.5. For $w_i \in W_i$ (for each $i \in \{1,2\}$), let $\mathrm{ord}(w_i)$ denote the order of w_i in W_i. For those $s,t \in S$ with $\langle\alpha_s,\beta_t\rangle\langle\alpha_t,\beta_s\rangle \ge 1$, extend the definition of m_{st} (given in Theorem 2.4) by setting $m_{st} = \infty$.

Proposition 2.6. *Suppose that one of the (equivalent) statements of Theorem 2.4 is satisfied. Then* $\mathrm{ord}(\rho_i(s)\rho_i(t)) = m_{st}$.

Proof. If one of the (equivalent) statements of Theorem 2.4 is satisfied, then we see that $\mathscr{C} := (S, V_1, V_2, \Pi_1, \Pi_2, \langle,\rangle)$ forms a Coxeter datum in the sense of [11], and thus the required result follows immediately from Proposition 2.8 of [11]. □

We point out that a Coxeter datum in the sense of [11] automatically satisfies the conditions (D1)–(D5) of the present paper. Indeed, the only possible difference of these two formulations is that in (D2) of the present paper we require a seemingly extra condition that $\alpha_s \notin \mathrm{PLC}(\Pi_1 \setminus \{\alpha_s\})$ and $\beta_s \notin \mathrm{PLC}(\Pi_2 \setminus \{\beta_s\})$ for each $s \in S$. However, it can be checked that this condition is an immediate consequence of (C1), (C2), and (C5) of a Coxeter datum in the sense of [11] (in fact, this is just [11, Lemma 2.5]). Thus, the following result is obtained.

PROPOSITION 2.7. *The following are equivalent:*

(i) $\mathscr{C} := (S, V_1, V_2, \Pi_1, \Pi_2, \langle\ ,\ \rangle)$ *satisfies one of the (equivalent) statements of Theorem 2.4 ;*

(ii) $\mathscr{C} := (S, V_1, V_2, \Pi_1, \Pi_2, \langle\ ,\ \rangle)$ *is a Coxeter datum in the sense of* [11]. □

Next we have a result which enables us to give a characterization of Coxeter groups, among a large family of linear groups that are generated by involutions, in terms of their root systems.

THEOREM 2.8. *Let S, Π_1, and Π_2 be the same as at the beginning of this section, and let R_1, W_1, Φ_1, R_2, W_2, and Φ_2 be as in Definition 2.1. Let (W, R) be a Coxeter system in the sense of* [2] *or* [13], *with W being an abstract group generated by a set of involutions $R := \{r_s \mid s \in S\}$ subject only to the condition that for $s, t \in S$ the order of $r_s r_t$ is either equal to m if $\langle\alpha_s, \beta_t\rangle\langle\alpha_t, \beta_s\rangle = \cos^2(\pi/m)$, or else equal to infinity. Then $\Phi_1 = \Phi_1^+ \uplus \Phi_1^-$, or equivalently, $\Phi_2 = \Phi_2^+ \uplus \Phi_2^-$ only if there exist isomorphisms $f_1 \colon W \to W_1$ and $f_2 \colon W \to W_2$ such that $f_1(r_s) = \rho_1(s)$ and $f_2(r_s) = \rho_2(s)$ for all $s \in S$.*

PROOF. Follows immediately from Proposition 2.7 above and [11, Theorem 2.10]. □

REMARK 2.9. Theorem 2.8 shows that if $\Phi_1 = \Phi_1^+ \uplus \Phi_1^-$, or equivalently, $\Phi_2 = \Phi_2^+ \uplus \Phi_2^-$ then (W_1, R_1) and (W_2, R_2) are Coxeter systems isomorphic to (W, R). It is well-known in the literature that all Coxeter groups have root systems decomposable into a disjoint union of positive roots and negative roots ([1, Proposition 4.2.5] or [13, Section 5.4], for example). Furthermore, given an arbitrary Coxeter system (W', R'), it follows from [10] and [11] that we could associate a Coxeter datum $\mathscr{C}' := (S', V_1', V_2', \Pi_1', \Pi_2', \langle\ ,\ \rangle')$ to (W', R'), such that the paired root systems Φ_1' and Φ_2' arising from this particular Coxeter datum admit decompositions $\Phi_1' = \Phi_1'^+ \uplus \Phi_1'^-$ and $\Phi_2' = \Phi_2'^+ \uplus \Phi_2'^-$. These facts combined with Theorem 2.8 yield that if a linear group is generated by involutions, then it is a Coxeter group if and only if it has a root system decomposable into a disjoint union of positive roots and negative roots.

Let W and R be as in Theorem 2.8, we call (W, R) the *abstract Coxeter system* corresponding to $\mathscr{C}$ with W being the corresponding *abstract Coxeter group*. We see immediately from the above theorem that f_1 and f_2 give rise to faithful W-actions on V_1 and V_2 in the natural way with $wx := (f_1(w))(x)$ and $wy := (f_2(w))(y)$ for all $w \in W$, $x \in V_1$, and $y \in V_2$.

To close this section we include the following useful result taken from [11].

LEMMA 2.10.

(i) $\langle\ ,\ \rangle$ *is W-invariant, that is, $\langle wx, wy\rangle = \langle x, y\rangle$ for all $w \in W$, $x \in V_1$, and $y \in V_2$.*

(ii) *There is a W-equivariant bijection $\phi \colon \Phi_1 \to \Phi_2$ satisfying $\phi(\alpha_s) = \beta_s$ for all $s \in S$.*

(iii) *Let ϕ be as in (ii) above, and let $x, x' \in \Phi_1$. Then $\langle x, \phi(x')\rangle = 0$ if and only if $\langle x', \phi(x)\rangle = 0$.*

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PROOF. (i) Lemma 2.13 of [11].

(ii) Proposition 3.18 of [11].

(iii) Corollary 3.25 of [11]. □

For the remainder of this paper, the notation ϕ will be fixed for the bijection $\Phi_1 \to \Phi_2$ in Lemma 2.10(iii).

3. Reflection subgroups and canonical generators in Coxeter groups

Given a Coxeter group W and its Coxeter generators R, a subgroup W' of W is called a *reflection subgroup* if W' is generated by those elements of the form wrw^{-1} (where $w \in W$ and $r \in R$). It is well-known that W' is a Coxeter group. In this section we study the paired root systems for W' as a subset of the paired root systems for W. In the spirit of the previous section, our investigation of the paired root systems for W' is based on a Coxeter datum $\mathscr{C}'$ closely related to the Coxeter datum for the over-group W. In particular, we show that the Coxeter generators of W' are characterized by this Coxeter datum $\mathscr{C}'$. In addition to obtaining certain geometric insights of reflection subgroups of Coxeter groups, these investigations also establish the fact that the method of constructing paired root systems via Coxeter data applies to paired root systems for reflection subgroups of a Coxeter group, either on their own or as subsets of the paired root systems of the over-group.

Suppose that $\mathscr{C} := (S, V_1, V_2, \Pi_1, \Pi_2, \langle\ ,\ \rangle)$ satisfies conditions (D1)–(D5) of Section 2 inclusive (or in view of Proposition 2.7, we could equivalently suppose that $\mathscr{C}$ is a Coxeter datum in the sense of [11]). Let (W, R) be the abstract Coxeter system associated to the Coxeter datum $\mathscr{C}$, and keep all the notation of the previous section.

Let $T := \bigcup_{w\in W} wRw^{-1}$, and call it the *set of reflections* in W. For $s \in S$ and $w \in W$, observe that for each $x \in V_1$ and $y \in V_2$, Lemma 2.10 yields that

$$\begin{aligned} wr_sw^{-1}x = w(w^{-1}x - 2\langle w^{-1}x, \beta_s\rangle\alpha_s) &= x - 2\langle w^{-1}x, \beta_s\rangle w\alpha_s \\ &= x - 2\langle x, \phi(w\alpha_s)\rangle w\alpha_s, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} wr_sw^{-1}y = w(w^{-1}y - 2\langle \alpha_s, w^{-1}y\rangle\beta_s) &= y - 2\langle w\alpha_s, y\rangle w\beta_s \\ &= y - 2\langle \phi^{-1}(w\beta_s), y\rangle w\beta_s. \end{aligned} \tag{3.2}$$

Now suppose that $\alpha \in \Phi_1$ and $\beta \in \Phi_2$ are arbitrary. Then $\alpha = w_1\alpha_s$ and $\beta = w_2\beta_t$ for some $w_1, w_2 \in W$, and $s, t \in S$. It follows from (3.1) and (3.2) that we can unambiguously define $r_\alpha, r_\beta \in T$, the *reflection corresponding to* α and β respectively, by

$$r_\alpha = r_{w_1\alpha_s} := w_1r_sw_1^{-1}, \tag{3.3}$$

and

$$r_\beta = r_{w_2\beta_t} := w_2r_tw_2^{-1}, \tag{3.4}$$

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with

$$r_\alpha x = x - 2\langle x, \phi(\alpha)\rangle\alpha$$

for all $x \in V_1$ and

$$r_\beta y = y - 2\langle \phi^{-1}(\beta), y\rangle\beta$$

for all $y \in V_2$.

Definition 3.1.

(i) A subgroup W' of W is called a *reflection subgroup* if W' is generated by the reflections that it contains, that is, $W' = \langle W' \cap T\rangle$.

(ii) For each $i \in \{1, 2\}$, a subset Φ'_i of Φ_i is called a *root subsystem* if $r_x y \in \Phi'_i$ whenever $x, y \in \Phi'_i$.

(iii) If W' is a reflection subgroup, set $\Phi_i(W') := \{x \in \Phi_i \mid r_x \in W'\}$ for each $i \in 1, 2$.

Lemma 3.2. *Let W' be a reflection subgroup of W. Then for each $i \in \{1, 2\}$*

$$W'\Phi_i(W') = \Phi_i(W').$$

Proof. We prove that $W'\Phi_1(W') = \Phi_1(W')$ here, and we stress that the other half follows in the same way. Let $w \in W'$. By definition, we have $w = t_1t_2 \cdots t_n$ where $t_1, t_2, \ldots, t_n \in W' \cap T$. Now let $x \in \Phi_1(W')$ be arbitrary. Then $r_x \in W'$, and hence $r_{t_n x} = t_n r_x t_n \in W'$, which in turn yields that $t_n x \in \Phi_1(W')$. Then it follows that $t_{n-1}t_n x \in \Phi_1(W')$ and so on. Thus, $wx = t_1 \cdots t_n x \in \Phi_1(W')$. Since $x \in \Phi_1(W')$ is arbitrary, it follows that $w\Phi_1(W') \subseteq \Phi_1(W')$. Finally, replacing $w \in W'$ by w^{-1} and we may see that $\Phi_1(W') \subseteq w\Phi_1(W')$. □

Remark 3.3. Let W' be a reflection subgroup. For each $i \in \{1, 2\}$, it follows from the above lemma that $\Phi_i(W')$ is a root subsystem of Φ_i (in fact, it is the maximal root subsystem in Φ_i with respect to inclusion whose corresponding reflections generate W'), and we call it *the root subsystem corresponding to W'*.

Remark 3.4. For each $i \in \{1, 2\}$, it has been observed in [11] that a nontrivial scalar multiple of an element of Φ_i can still be an element of Φ_i (see the example immediately after [11, Definition 3.1] and [11, Lemma 3.20]). Therefore, unlike in the classical setting of [13], we do not have a bijection from T to either Φ_1^+ or Φ_2^+. Furthermore, for each $i \in \{1, 2\}$, suppose that x is a root in Π_i such that cx is yet another root in Φ_i for some positive scalar $c \neq 1$. Let $r = r_x = r_{cx}$ denote the reflection in x (also in cx), and let $W' = \langle r\rangle = \{1, r\}$ be the reflection subgroup it generates. Then $\{\pm x\}$, $\{\pm cx\}$, $\{\pm x, \pm cx\}$ are three pairwise distinct root subsystems of Φ_i as defined in Definition 3.1(ii). Moreover, $\Phi_i(W')$ as defined in Definition 3.1(iii) certainly contains the root subsystem $\{\pm x, \pm cx\}$. In [11, Proposition 3.17], it has been observed that if $c\alpha_s \in \Phi_1$ for some nonzero constant c, then $c^{-1}\beta_s \in \Phi_2$. Thus, any reflection subgroup W' of W may give rise to a root subsystem which is not canonically associated to a Coxeter datum.

It can be readily observed that any rescaling $\mathscr{C}_c$ of $\mathscr{C}$ and $\mathscr{C}$ itself have the same abstract Coxeter system (W, R), and these data bear the same action of W on both V_1 and V_2. In particular, any reflection subgroup W' of W has a corresponding root subsystem afforded by some (in general, noncanonically associated) Coxeter datum $\mathscr{C}_{W'}$, that any second Coxeter datum arising in this way occurs by rescaling $\mathscr{C}_{W'}$, and that any root subsystem of Φ_i for each $i \in \{1, 2\}$ whose corresponding reflections generate W' (in particular, the maximal one $\Phi_i(W')$) is a union of certain root subsystems of Φ_i associated to Coxeter datums which are rescalings of $\mathscr{C}_{W'}$. We thank the referee for directing our attention to this subtlety. For this section, we give a method of computing one such datum for each reflection subgroup of an arbitrary Coxeter group.

Definition 3.5. For each $i \in \{1, 2\}$, define an equivalence relation $\sim_i$ on Φ_i as follows: if $z_1, z_2 \in \Phi_i$, then $z_1 \sim_i z_2$ if and only if z_1 and z_2 are (nonzero) scalar multiples of each other. For each $z \in \Phi_i$, write $\widehat{z}$ for the equivalence class containing z and write $\widehat{\Phi_i} = \{\widehat{z} \mid z \in \Phi_i\}$.

Remark 3.6. Observe that W has a natural action on $\widehat{\Phi_i}$ (for each $i \in \{1, 2\}$) given by $w\widehat{z} = \widehat{wz}$ for all $w \in W$ and $z \in \Phi_i$. Furthermore, given $z, z' \in \Phi_i$, the corresponding reflections r_z and $r_{z'}$ are equal if and only if $\widehat{z} = \widehat{z'}$.

Definition 3.7. For $i \in \{1, 2\}$, and for each $w \in W$, define

$$N_i(w) = \{\widehat{z} \mid z \in \Phi_i^+ \text{ and } wz \in \Phi_i^-\}.$$

Note that for $w \in W$, the set $N_i(w)$ $(i = 1, 2)$ can be alternatively characterized as $\{\widehat{z} \mid z \in \Phi_i^- \text{ and } wz \in \Phi_i^+\}$. Hence $\widehat{z} \in N_i(w)$ if and only if precisely one element of the set $\{z, wz\}$ is in Φ_i^+.

Notation 3.8. Let $\ell : W \to \mathbb{N}$ denote the *length function* with respect to (W, R), that is, for $w \in W$,

$$\ell(w) = \min\{n \in \mathbb{N} \mid \ w = r_1 r_2 \cdots r_n, \text{ where } r_1, r_2, \ldots, r_n \in R\}.$$

A mild generalization of the techniques used in ([13, Section 5.6]) then yields the following connection between the length function and the functions N_1 and N_2.

Lemma 3.9 [11, Lemma 3.8]. (i) $N_1(r_s) = \{\widehat{\alpha_s}\}$ *and* $N_2(r_s) = \{\widehat{\beta_s}\}$ *for all* $s \in S$.

(ii) *Let* $w \in W$. *Then* $N_1(w)$ *and* $N_2(w)$ *both have cardinality* $\ell(w)$.

(iii) *Let* $w_1, w_2 \in W$ *and let* $\dotplus$ *denote set symmetric difference. Then for each* $i \in \{1, 2\}$,

$$N_i(w_1 w_2) = w_2^{-1} N_i(w_1) \dotplus N_i(w_2).$$

□

Remark 3.10. The proof of Lemma 3.9 (and its analog in [13] and most other places in the literature) can be greatly simplified by using functions called *cocycles* and *coboundaries* as introduced in [7] and [8, 2.1]. We thank the referee for pointing this out.

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The last lemma enables us to deduce the following generalization of [11, Lemma 3.2(ii)].

Proposition 3.11. *For each $i \in \{1,2\}$, let $w \in W$ and $x \in \Phi_i^+$ be arbitrary. If $\ell(wr_x) > \ell(w)$ then $wx \in \Phi_i^+$, whereas if $\ell(wr_x) < \ell(w)$ then $wx \in \Phi_i^-$.*

Proof. We prove the statement that $\ell(wr_x) > \ell(w)$ if and only if wx is positive in the case $x \in \Phi_1$, and again we stress that a similar argument also shows the desired result holds in Φ_2.

Observe that the second statement follows from the first, applied to wr_x in place of w: if $\ell(wr_x) < \ell(w)$ then $\ell((wr_x)r_x) > \ell(wr_x)$, forcing $(wr_x)x = w(r_x x) = -wx \in \Phi_1^+$, that is, $wx \in \Phi_1^-$.

Now we prove the first statement in Φ_1. Proceed by induction on $\ell(w)$, the case $\ell(w) = 0$ being trivial. If $\ell(w) > 0$, then there exists $s \in S$ with $\ell(r_s w) = \ell(w) - 1$, and hence

$$\ell((r_s w)r_x) = \ell(r_s(wr_x)) \geq \ell(wr_x) - 1 > \ell(w) - 1 = \ell(r_s w).$$

Then the inductive hypothesis yields that $(r_s w)x \in \Phi_1^+$. Suppose for a contradiction that $wx \in \Phi_1^-$. Then $\widehat{wx} \in N_1(r_s)$ and Lemma 3.9(i) yields that $wx = -\lambda\alpha_s$ for some $\lambda > 0$. But then $r_s wx = \lambda\alpha_s$, and hence $(r_s w)r_x(r_s w)^{-1} = r_s$ by calculations similar to (3.3) and (3.4). But this yields that $wr_x = r_s w$, contradicting $\ell(wr_x) > \ell(w) > \ell(r_s w)$, as desired. □

Definition 3.12. For each $w \in W$, define

$$\overline{N}(w) := \{t \in T \mid \ell(wt) < \ell(w)\}.$$

If $t \in T$ then $t = wr_s w^{-1}$ for some $w \in W$ and $s \in S$, and hence it follows from calculations like (3.3) and (3.4) that $t = r_{w\alpha_s} = r_{w\beta_s}$. We conclude from this and Proposition 3.11 the following result.

Proposition 3.13. *Let $w \in W$. Then*

$$\overline{N}(w) = \{r_x \mid \widehat{x} \in N_i(w)\}$$

for each $i \in \{1,2\}$. □

Definition 3.14. Suppose that W' is a reflection subgroup. Then we define

$$S(W') := \{\, t \in T \mid \overline{N}(t) \cap W' = \{t\}\}$$

and

$$\Delta_i(W') := \{x \in \Phi_i^+ \mid r_x \in S(W')\}$$

for each $i \in \{1,2\}$.

For a reflection subgroup W', the set $S(W')$ is called the *canonical generators* of W' in [7], and it is well-known that $(W', S(W'))$ is a Coxeter system. Indeed, the following lemma is obtained.

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LEMMA 3.15 [7]. *Let W' be a reflection subgroup of W.*

(i) *If $t \in W' \cap T$, then there exist $m \in \mathbb{N}$ and $t_0, \ldots, t_m \in S(W')$ such that $t = t_m \cdots t_1 t_0 t_1 \cdots t_m$.*
(ii) *$(W', S(W'))$ is a Coxeter system.*

PROOF. (i) [7, Lemma (1.7)(ii)].
(ii) [7, Theorem (1.8)(i)]. □

Observe that for a reflection subgroup W' we can equivalently define $\Delta_i(W')$ by requiring

$$\Delta_i(W') := \{x \in \Phi_i^+ \mid N_i(r_x) \cap \widehat{\Phi_i(W')} = \{\widehat{x}\}\}. \tag{3.5}$$

Suppose that $\Delta'_1 \subseteq \Phi_1^+$ and $\Delta'_2 \subseteq \Phi_2^+$ are two sets of roots satisfying $\phi(\Delta'_1) = \Delta'_2$ (where ϕ is as in Lemma 2.10(iii)). Furthermore, suppose that Δ'_1 and Δ'_2 satisfy:

(i′) $\langle x, \phi(x')\rangle \leq 0$, for all distinct $x, x' \in \Delta'_1$;
(ii′) $\langle x, \phi(x')\rangle\langle x', \phi(x)\rangle \in \{\cos^2(\pi/m) \mid m \in \mathbb{N}, m \geq 2\} \cup [1, \infty)$, for all $x, x' \in \Delta'_1$ with $r_x \neq r_{x'}$.

It follows from Lemma 2.10 that

$$\langle x, \phi(x)\rangle = 1, \quad \text{for all } x \in \Delta'_1. \tag{3.6}$$

For each $i \in \{1, 2\}$, since $\Delta'_i \subseteq \mathrm{PLC}(\Pi_i)$, it follows that $0 \notin \mathrm{PLC}(\Delta'_i)$. Note that it can be readily checked from conditions (i′), (ii′), and (3.6) that $x \notin \mathrm{PLC}(\Delta'_1 \setminus \{x\})$ and $\phi(x) \notin \mathrm{PLC}(\Delta'_2 \setminus \{\phi(x)\})$ for all $x \in \Delta'_1$. Furthermore, Lemma 2.10(iii) ensures that $\langle x, \phi(x')\rangle = 0$ whenever $\langle x', \phi(x)\rangle = 0$ for all $x, x' \in \Delta'_1 \subseteq \Phi_1$, Thus, Δ'_1 and Δ'_2 satisfy conditions (D1)–(D5) of the present paper inclusive. If we let S' be an indexing set for both Δ'_1 and Δ'_2 then

$$\mathscr{C}' := (S', \mathrm{span}(\Delta'_1), \mathrm{span}(\Delta'_2), \Delta'_1, \Delta'_2, \langle\ ,\ \rangle'),$$

(where $\langle\ ,\ \rangle'$ denotes the restriction of $\langle\ ,\ \rangle$ to $\mathrm{span}(\Delta'_1) \times \mathrm{span}(\Delta'_2)$) in fact constitutes a Coxeter datum in the sense of [11]. Now if we let $R' := \{r_x \mid x \in \Delta'_1\}(= \{r_y \mid y \in \Delta'_2\})$, and set $W' = \langle R'\rangle$, then it is clear that W' is a reflection subgroup of W. Furthermore, it follows from Theorem 2.8 that (W', R') forms a Coxeter system. Then upon applying Lemma 3.9 and (3.5) to $\mathscr{C}'$ and W' we may conclude that $S(W') = R'$ and consequently $\widehat{\Delta_1(W')} = \widehat{\Delta'_1}$ and $\widehat{\Delta_2(W')} = \widehat{\Delta'_2}$. Summing up, the following proposition is obtained.

PROPOSITION 3.16. *Suppose that $\Delta'_1 \subseteq \Phi_1^+$ and $\Delta'_2 \subseteq \Phi_2^+$ such that*

(A1) $\phi(\Delta'_1) = \Delta'_2$;
(A2) $\langle x, \phi(x')\rangle \leq 0$, *for all distinct* $x, x' \in \Delta'_1$;
(A3) $\langle x, \phi(x')\rangle\langle x', \phi(x)\rangle \in \{\cos^2(\pi/m) \mid m \in \mathbb{N}, m \geq 2\} \cup [1, \infty)$, *for all* $x, x' \in \Delta'_1$ *with* $r_x \neq r_{x'}$.

Then $W' = \langle\{r_x \mid x \in \Delta'_1\}\rangle$ is a reflection subgroup of W with $\widehat{\Delta'_1} = \widehat{\Delta_1(W')}$ and $\widehat{\Delta'_2} = \widehat{\Delta_2(W')}$. □

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It turns out that the converse of Proposition 3.16 is also true, namely: if W' is a reflection subgroup of W, and if $x, x' \in \Delta_1(W')$ with $r_x \neq r_{x'}$, then conditions (A2) and (A3) of Proposition 3.16 must be satisfied. Since Lemma 2.10(iii) ensures that $\langle x, \phi(x')\rangle = 0$ if and only if $\langle x', \phi(x)\rangle = 0$, it follows from this assertion and a quick argument similar to the one used immediately after (3.6) that representative elements from $\Delta_1(W')$ and $\Delta_2(W')$ can be used to form a Coxeter datum for W'. Hence this assertion and Proposition 3.16 together yield that for a reflection subgroup W', the corresponding $\Delta_i(W')$ $(i = 1, 2)$ can be characterized, up to rescalings, by a suitable Coxeter datum. We devote the rest of this section to a proof of this assertion.

Remark 3.17. Proposition 3.16 and its converse Proposition 3.24 together yield that two reflections are among the canonical generators of a reflection subgroup if and only if they are the canonical generators of a dihedral reflection subgroup. This was first proved in the classical setting in [7] and [8, 3.5 and 4.4]. By rescaling, we may obtain these results using the proofs in [8]. For the purpose of producing a more self-contained paper, we give a different, though, longer proof here.

Lemma 3.18. *Let W' be a reflection subgroup of W.*

(i) *For each $i \in \{1, 2\}$, let $x \in \Pi_i \setminus \Phi_i(W')$. Then $\Delta_i(r_x W' r_x) = r_x \Delta_i(W')$.*
(ii) *For each $i \in \{1, 2\}$, $\Phi_i(W') = W'\Delta_i(W')$.*

Proof. (i) It is readily checked that $r\Phi_i(W') = \Phi_i(rW'r)$ for all $r \in T$. Since $x \in \Pi_i \setminus \Phi_i(W')$, it follows that $r_x \in R \setminus W'$. Let $y \in \Delta_i(W')$ be arbitrary. Then

$$\begin{aligned}
N_i(r_{(r_x y)}) \cap \Phi_i(\widehat{r_x W' r_x}) &= N_i(r_x r_y r_x) \cap \Phi_i(\widehat{r_x W' r_x}) && \text{(by (3.3) and (3.4))}\\
&= (r_x N_i(r_x r_y) \dotplus N_i(r_x)) \cap \Phi_i(\widehat{r_x W' r_x}) && \text{(by Lemma 3.9 (iii))}\\
&= (r_x r_y N_i(r_x) \dotplus r_x N_i(r_y) \dotplus N_i(r_x)) \cap \Phi_i(\widehat{r_x W' r_x}) && \text{(by Lemma 3.9 (iii))}\\
&= r_x((r_y N_i(r_x) \dotplus N_i(r_y) \dotplus N_i(r_x)) \cap \widehat{\Phi_i(W')})\\
&= r_x((r_y\{\widehat{x}\} \dotplus N_i(r_y) \dotplus \{\widehat{x}\}) \cap \widehat{\Phi_i(W')}) && \text{(by Lemma 3.9 (i))}\\
&= r_x(N_i(r_y) \cap \widehat{\Phi_i(W')}) && \text{(since } \widehat{x}, r_y\widehat{x} \notin \widehat{\Phi_i(W')})\\
&= \{\widehat{r_x y}\} && \text{(since } y \in \Delta_i(W')),
\end{aligned}$$

and therefore $r_x y \in \Delta_i(r_x W' r_x)$. This proves that $r_x \Delta_i(W') \subseteq \Delta_i(r_x W' r_x)$. But $x \in \Pi_i \setminus r_x \Phi_i(W')$, so the above yields that $r_x \Delta_i(r_x W' r_x) \subseteq \Delta_i(W')$ proving the desired result.

(ii) Since $\Delta_i(W') \subseteq \Phi_i(W')$ for each $i \in \{1, 2\}$, it follows by Lemma 3.2 that $W'\Delta_i(W') \subseteq \Phi_i(W')$.

Conversely if $x \in \Phi_i(W')$ then $r_x \in W' \cap T$. By Lemma 3.15 (i), there are $x_0, x_1, \cdots, x_m \in \Delta_i(W')$ $(m \in \mathbb{N})$ such that

$$r_x = r_{x_m} \cdots r_{x_1} r_{x_0} r_{x_1} \cdots r_{x_m}.$$

Calculations similar to (3.3) and (3.4) give $\lambda x = r_{x_m} \cdots r_{x_1} x_0 \in W'\Phi_i(W')$ for some (nonzero) scalar λ. Now since $(1/\lambda)x_0 = (r_{x_m} \cdots r_{x_1})^{-1} x \in \Phi_i$, it follows that $(1/\lambda)x_0 \in \Delta_i(W')$ and hence $x = r_{x_m} \cdots r_{x_1}((1/\lambda)x_0) \in W'\Delta_i(W')$ as required. □

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Definition 3.19. Let W' be a reflection subgroup of W, and let $\ell_{W'} : W' \to \mathbb{N}$ be the length function on $(W', S(W'))$ defined by

$$\ell_{W'}(w) = \min\{n \in \mathbb{N} \mid w = r_1 \cdots r_n, \text{ where } r_i \in S(W')\}.$$

If $w = r_1 \cdots r_n \in W'$ $(r_i \in S(W'))$ and $n = \ell_{W'}(w)$ then $r_1 \cdots r_n$ is called a *reduced expression* for w (with respect to $S(W')$).

Lemma 3.20. *Let W' be a reflection subgroup. For each $i \in \{1, 2\}$,*

(i) $N_i(r_x) \cap \widehat{\Phi_i(W')} = \{\widehat{x}\}$ *for all* $x \in \Delta_i(W')$;

(ii) *for all* $w_1 \in W$ *and* $w_2 \in W'$

$$N_i(w_1 w_2) \cap \widehat{\Phi_i(W')} = w_2^{-1}(N_i(w_1) \cap \widehat{\Phi_i(W')}) \dotplus (N_i(w_2) \cap \widehat{\Phi_i(W')}).$$

Proof. (i) is just the definition of $\Delta_i(W')$.

(ii) Lemma 3.9 (iii) yields that $N_i(w_1 w_2) = w_2^{-1} N_i(w_1) \dotplus N_i(w_2)$, and hence

$$N_i(w_1 w_2) \cap \widehat{\Phi_i(W')} = (w_2^{-1} N_i(w_1) \cap \widehat{\Phi_i(W')}) \dotplus (N_i(w_2) \cap \widehat{\Phi_i(W')}).$$

Since $w_2 \in W'$ it follows from Lemma 3.2 that $w_2^{-1}\widehat{\Phi_i(W')} = \widehat{\Phi_i(W')}$. Thus, $w_2^{-1} N_i(w_1) \cap \widehat{\Phi_i(W')} = w_2^{-1}(N_i(w_1) \cap \widehat{\Phi_i(W')})$, giving us

$$N_i(w_1 w_2) \cap \widehat{\Phi_i(W')} = w_2^{-1}(N_i(w_1) \cap \widehat{\Phi_i(W')}) \dotplus (N_i(w_2) \cap \widehat{\Phi_i(W')}). \qquad \square$$

Lemma 3.21. *Let W' be a reflection subgroup. For each $i \in \{1, 2\}$ and all $w \in W'$, the following hold.*

(i) $|N_i(w) \cap \widehat{\Phi_i(W')}| = \ell_{W'}(w)$.

Moreover, if $w = r_{x_1} \cdots r_{x_n}$ (where $x_1, \cdots, x_n \in \Delta_i(W')$) is reduced with respect to $(W', S(W'))$ then $N_i(w) \cap \widehat{\Phi_i(W')} = \{\widehat{y_1}, \cdots \widehat{y_n}\}$ where the y_j are given by $y_j = (r_{x_n} \cdots r_{x_{j+1}}) x_j$ for all $j = 1, \cdots, n$.

(ii) $N_i(w) \cap \widehat{\Phi_i(W')} = \{\widehat{x} \in \widehat{\Phi_i(W')} \mid \ell_{W'}(w r_x) < \ell_{W'}(w)\}$.

Proof. (i) For each $j \in \{1, \cdots, n\}$, set $t_j = r_{x_n} \cdots r_{x_{j+1}} r_{x_j} r_{x_{j+1}} \cdots r_{x_n}$, that is, $t_j = r_{y_j}$. If $t_j = t_k$ where $j > k$ then

$$\begin{aligned} w &= r_{x_1} \cdots r_{x_{k-1}} r_{x_{k+1}} \cdots r_{x_n} t_k \\ &= r_{x_1} \cdots r_{x_{k-1}} r_{x_{k+1}} \cdots r_{x_n} t_j \\ &= r_{x_1} \cdots r_{x_{k-1}} r_{x_{k+1}} \cdots r_{x_{j-1}} r_{x_{j+1}} \cdots r_{x_n} \end{aligned}$$

contradicting $\ell_{W'}(w) = n$. Hence the t_j's are all distinct and consequently all the $\widehat{y_j}$'s are all distinct. Now by repeated application of Lemma 3.20(ii), for each $i \in \{1, 2\}$,

$$\begin{aligned} &N_i(w) \cap \widehat{\Phi_i(W')} \\ &= (N_i(r_{x_n} \cap \widehat{\Phi_i(W')}) \dotplus r_{x_n}(N_i(r_{n-1}) \cap \widehat{\Phi_i(W')}) \dotplus \cdots \\ &\qquad\qquad \dotplus r_{x_n} \cdots r_{x_2}(N_i(r_{x_1}) \cap \widehat{\Phi_i(W')}) \\ &= \{\widehat{y_n}\} \dotplus \{\widehat{y_{n-1}}\} \dotplus \cdots \dotplus \{\widehat{y_1}\} \\ &= \{\widehat{y_1}, \cdots, \widehat{y_n}\} \end{aligned}$$

and consequently $|N_i(w) \cap \widehat{\Phi_i(W')}| = \ell_{W'}(w)$.

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(ii) Let $w = r_{x_1} \cdots r_{x_n}$ be a reduced expression for $w \in W'$ with respect to $S(W')$ $(x_1, \ldots, x_n \in \Delta_i(W'))$. Then for each $i \in \{1, 2\}$, Part (i) above yields that

$$N_i(w) \cap \widehat{\Phi_i(W')} = \{\widehat{y_1}, \cdots, \widehat{y_n}\}$$

where $y_j = (r_{x_n} \cdots r_{x_{j+1}})x_j$, for all $j \in \{1, \ldots, n\}$. Now for each such j,

$$wr_{y_j} = wr_{x_n} \cdots r_{x_{j+1}} r_{x_j} r_{x_{j+1}} \cdots r_{x_n} = r_{x_1} \cdots r_{x_{j-1}} r_{x_{j+1}} \cdots r_{x_n}$$

and so $\ell_{W'}(wr_{y_j}) \leq n - 1 < \ell_{W'}(w)$. Hence, $\widehat{x} \in N_i(w) \cap \widehat{\Phi_i(W')}$ implies that $\ell_{W'}(wr_x) < \ell_{W'}(w)$.

Conversely, suppose that $x \in \Phi_i(W') \cap \Phi_i^+$ and $\widehat{x} \notin N_i(w)$. We are finished if we could show that then $\ell_{W'}(wr_x) > \ell_{W'}(w)$. Observe that the given choice of x implies that $\widehat{x} \in N_i(r_x) \cap \widehat{\Phi_i(W')}$, furthermore, $\widehat{x} \notin r_x(N_i(w) \cap \widehat{\Phi_i(W')})$. Therefore

$$\widehat{x} \in r_x(N_i(w) \cap \widehat{\Phi_i(W')}) \dotplus (N_i(r_x) \cap \widehat{\Phi_i(W')}) = N_i(wr_x) \cap \widehat{\Phi_i(W')},$$

and by what has just been proved, this implies that

$$\ell_{W'}(w) = \ell_{W'}((wr_x)r_x) < \ell_{W'}(wr_x),$$

as desired. □

The following is a mild generalization of [7, Lemma 3.2]:

Lemma 3.22. *Let W' be a reflection subgroup. For each $i \in \{1, 2\}$, let $x, y \in \Delta_i(W')$ such that $r_x \neq r_y$. Let $n = \operatorname{ord}(r_x r_y)$. Then for $0 \leq m < n$*

$$\underbrace{\cdots r_y r_x r_y}_{m\ factors}\, x \in \Phi_i^+ \quad \textit{and} \quad \underbrace{\cdots r_x r_y r_x}_{m\ factors}\, y \in \Phi_i^+.$$

Proof. It is easily checked that when $0 \leq m < n$,

$$\ell_{W'}((\underbrace{\cdots r_y r_x r_y}_{m \text{ factors}})r_x) = m + 1 > m = \ell_{W'}(\underbrace{\cdots r_y r_x r_y}_{m \text{ factors}}),$$

as well as

$$\ell_{W'}((\underbrace{\cdots r_x r_y r_x}_{m \text{ factors}})r_y) = m + 1 > m = \ell_{W'}(\underbrace{\cdots r_x r_y r_x}_{m \text{ factors}}).$$

Hence, the desired result follows immediately from Lemma 3.21. □

In fact we can refine Lemma 3.22 with the following generalization of [7, Lemma 3.3].

Lemma 3.23. *Let W' be a reflection subgroup. For each $i \in \{1, 2\}$, let $x, y \in \Delta_i(W')$ with $r_x \neq r_y$. Let $n = \operatorname{ord}(r_x r_y)$, and let c_m, d_m, c'_m, and d'_m be constants such that*

$$(\underbrace{\cdots r_y r_x r_y}_{m\ factors})\, x = c_m x + d_m y \quad \textit{and} \quad (\underbrace{\cdots r_x r_y r_x}_{m\ factors})\, y = c'_m x + d'_m y.$$

Then $c_m \geq 0$, $d_m \geq 0$, $c'_m \geq 0$, and $d'_m \geq 0$ whenever $m < n$.

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Proof. By symmetry, it will suffice to prove that $d_m \geq 0$ and $d'_m \geq 0$. The proof of this will be based on an induction on $\ell(r_x)$.

Suppose first that $\ell(r_x) = 1$. Then $\lambda x \in \Pi_i$ for some $\lambda > 0$. Observe that by replacing x by λx, one may assume that $x \in \Pi_i$. Write $y = \sum_{z \in \Pi_i} \lambda_z z$ where $\lambda_z \geq 0$ for all $z \in \Pi_i$. In fact, $\lambda_{z_0} > 0$ for some $z_0 \in \Pi_i \setminus \{x\}$, since otherwise we would have $y \in \mathbb{R}x$ and so $r_x = r_y$. Now for $0 \leq m < n$, Lemma 3.22 yields that

$$(\underbrace{\cdots r_y r_x r_y}_{m \text{ factors}})x = c_m x + \sum_{z \in \Pi_i} d_m \lambda_z z \in \Phi_i^+.$$

That is

$$c_m x + d_m\Big(\sum_{z \in \Pi_i} \lambda_z z\Big) = \sum_{z \in \Pi_i} \mu_z z, \text{ where } \mu_z \geq 0, \text{ for all } z \in \Pi_i.$$

Now if $d_m < 0$ then the above yields that

$$(c_m - \mu_x + d_m \lambda_x)x = (\mu_{z_0} - d_m \lambda_{z_0})z_0 + \sum_{z \in \Pi_i \setminus \{x, z_0\}} (\mu_z - d_m \lambda_z)z. \tag{3.7}$$

If $c_m - \mu_x + d_m \lambda_x > 0$, then (3.7) allows $x \in \Pi_i$ to be expressed as a positive linear combination of elements in $\Pi_i \setminus \{x\}$; else if $c_m - \mu_x + d_m \lambda_x \leq 0$, upon rearranging (3.7),

$$0 = -(c_m - \mu_x + d_m \lambda_x)x + (\mu_{z_0} - d_m \lambda_{z_0})z_0 + \sum_{z \in \Pi_i \setminus \{x, z_0\}} (\mu_z - d_m \lambda_z)z \in \mathrm{PLC}(\Pi_i),$$

both contradicting (D2). Therefore, $d_m \geq 0$ as required. Similarly $d'_m \geq 0$.

Suppose inductively that the result is true for reflection subgroups W'' of W and x', $y' \in \Delta_i(W'')$ with $r_{x'} \neq r_{y'}$ and $\ell(r_{x'}) < \ell(r_x)$ where $\ell(r_x) \geq 3$. It is well-known that there exists $z \in \Pi_i$ such that $\ell(r_z r_x r_z) = \ell(r_x) - 2$. Then $\ell(r_x r_z) < \ell(r_x)$, and thus $z \in N_i(r_x)$. But since $x \in \Delta_i(W')$ and $x \neq z$ (since $\ell(r_x) \geq 3$), it follows that $r_z \notin W'$. Let $W'' = r_z W' r_z$. Lemma 3.18(i) yields that $\Delta_i(W'') = r_z \Delta_i(W')$ and therefore $r_z x, r_z y \in \Delta_i(W'')$. Now

$$r_{(r_z x)} = r_z r_x r_z \quad \text{and} \quad r_{(r_z y)} = r_z r_y r_z \tag{3.8}$$

and hence $\mathrm{ord}(r_{(r_z x)} r_{(r_z y)}) = \mathrm{ord}(r_x r_y) = n$. Since $\ell(r_{(r_z x)}) = \ell(r_x) - 2$, the inductive hypothesis gives

$$(\underbrace{\cdots r_{(r_z y)} r_{(r_z x)} r_{(r_z y)}}_{m \text{ factors}})(r_z x) = c_m(r_z x) + d_m(r_z y)$$

and

$$(\underbrace{\cdots r_{(r_z x)} r_{(r_z y)} r_{(r_z x)}}_{m \text{ factors}})(r_z y) = c'_m(r_z x) + d'_m(r_z y)$$

where $d_m, d'_m \geq 0$ for $0 \leq m < n$. Finally, by (3.8), the desired result follows on applying r_z to both sides of the last two equations. $\square$

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Proposition 3.24. *Let W' be a reflection subgroup of W. Suppose that $x, y \in \Delta_1(W')$ with $r_x \neq r_y$. Let $n = \operatorname{ord}(r_x r_y) \in \{\infty\} \cup \mathbb{N}$. Then*

$$\langle x, \phi(y)\rangle \leq 0$$

and

$$\begin{cases}\langle x, \phi(y)\rangle\langle y, \phi(x)\rangle = \cos^2 \frac{\pi}{n} & (n \in \mathbb{N}, n \geq 2)\\ \langle x, \phi(y)\rangle\langle y, \phi(x)\rangle \in [1, \infty) & (n = \infty).\end{cases}$$

Proof. Observe that since $r_{\phi(x)} = r_x \neq r_y = r_{\phi(y)}$, it follows that $\{x, y\}$ and $\{\phi(x), \phi(y)\}$ are both linearly independent, and hence conditions (D1) and (D2) are satisfied. Now set $R_1'' = R_2'' := \{r_x, r_y\}$ and $W_1'' = W_2'' := \langle\{r_x, r_y\}\rangle$, and furthermore, $\Phi_1'' := W_1''\{x, y\}$. Observe that Φ_1'' consists of elements of the form $\pm(\underbrace{\cdots r_y r_x r_y}\, m \text{ factors})x$ and $\pm(\underbrace{\cdots r_x r_y r_x}\, m \text{ factors})y$ (where $0 \leq m < \operatorname{ord}(r_x r_y)$). Lemma 3.23 then yields that

$$\Phi_1'' = \Phi_1''^{+} \uplus \Phi_1''^{-},$$

and consequently Theorem 2.4 yields that

$$\begin{cases}\langle x, \phi(y)\rangle\langle y, \phi(x)\rangle = \cos^2 \frac{\pi}{n} & (n \in \mathbb{N}, n \geq 2)\\ \langle x, \phi(y)\rangle\langle y, \phi(x)\rangle \in [1, \infty) & (n = \infty).\end{cases}$$ □

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