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FULL SATISFACTION CLASSES AND RECURSIVE SATURATION

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ABSTRACT. It is shown that a nonstandard model of Peano arithmetic which has a full satisfaction class is necessarily recursively saturated.

The purpose of this note is to complement the paper [1] which immediately precedes this one by proving:

THEOREM. *If $\mathcal{M}$ is a nonstandard model of Peano arithmetic having a full satisfaction class then $\mathcal{M}$ is recursively saturated.*

We shall use the notation and terminology of [1]. Fix a nonstandard model $\mathcal{M}$ of PA. Since a finite number of elements of $\mathcal{M}$ may be coded by a single element it is sufficient to show that $\mathcal{M}$ satisfies

$$\forall y\Big(\bigwedge_{n<\omega}\Big(\exists x\bigwedge_{i<n}\varphi_i(x,y)\Big)\rightarrow\exists x\bigwedge_{i<\omega}\varphi_i(x,y)\Big) \tag{1}$$

for any recursive sequence $\langle\varphi_i(x,y):i<\omega\rangle$ of formulas of L having at most x, y free. Fix such a recursive sequence $\langle\varphi_i(x,y):i<\omega\rangle$ then without loss of generality we may assume that $\mathcal{M}$ satisfies

$$\forall x\forall y(\varphi_{i+1}(x,y)\rightarrow\varphi_i(x,y)). \tag{2}$$

It is convenient to let $\delta_0(x,y)$ denote the formula $\neg\varphi_0(x,y)$ and $\delta_{i+1}(x,y)$ denote $\varphi_i(x,y)\wedge\neg\varphi_{i+1}(x,y)$. Let $a\in\mathcal{M}$ be nonstandard.

The key to our proof is the construction of a certain sequence of nonstandard formulas. First note that inside $\mathcal{M}$ there is an $\mathcal{M}$-infinite sequence of formulas $\langle\varphi_i(x,y):i\le a\rangle$ having at most x, y free such that its standard part is the recursive sequence fixed above. The δ-sequence is extended in the obvious way to an $\mathcal{M}$-finite sequence $\langle\delta_i(x,y):i\le a\rangle$. Now inside $\mathcal{M}$ we define by simultaneous induction two sequences of L-formulas $\langle\psi_i(x,y):i\le a\rangle$ and $\langle\theta_{i,j}(x,y):j\le i<a\rangle$ by letting $\psi_0(x,y)$, $\theta_{i,0}(x,y)$ be $x=x$ for $i<a$, $\theta_{i,j+1}(x,y)$ be

$$(\exists x(\psi_i(x,y)\wedge\delta_{i-(j+1)}(x,y))\wedge\varphi_{i-(j+1)}(x,y)) \\ \vee(\neg\exists x(\psi_i(x,y)\wedge\delta_{i-(j+1)}(x,y))\wedge\theta_{ij}(x,y)) \tag{3}$$

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for $j < i < a$, and $\psi_{i+1}(x, y)$ be

$$(x = x \wedge \neg \exists x \psi_i(x, y)) \vee (\exists x \psi_i(x, y) \wedge \theta_{i,i}(x, y)). \tag{4}$$

The sequence $\langle \psi_i(x, y) : i \leq a \rangle$ is the one we need.

For proof by contradiction fix $b \in \mathcal{M}$ such that

$$M \models \bigwedge_{n<\omega} \Big(\exists x \bigwedge_{i>n} \varphi_i(x, b) \Big) \wedge \neg \exists x \bigwedge_{i<\omega} \varphi_i(x, b), \tag{5}$$

otherwise there is nothing to prove. Let Σ be a full satisfaction class for $\mathcal{M}$ and

$$S_i = \{c \in M : \psi_i(c, b) \in \Sigma\} \qquad (i \leq a).$$

Here it is worth recalling that a full satisfaction class is a notion of truth in $\mathcal{M}$ for the sentences of $^*L(\mathcal{M})$, which agrees with the usual notion on standaıd sentences. Call two formulas $\pi_0(x)$, $\pi_1(x)$ of $^*L(\mathcal{M})$ *equivalent* if $\forall x(\pi_0(x) \leftrightarrow \pi_1(x))$ is in Σ.

From (5) and the definition of $\delta_i(x, y)$, if $i \leq a$ and $S_i \neq \emptyset$ there is a least number $n_i < \omega$ such that

$$\exists x(\psi_i(x, b) \wedge \delta_{n_i}(x, b)) \in \Sigma.$$

Suppose $S_i \neq \emptyset$ and $n_i < i$. From (3) by descending induction on j we see that $\theta_{i,j}(x, b)$ is equivalent to $\theta_{i,i}(x, b)$ for $i - n_i \leq j \leq i$. Further, putting $j + 1 = i - n_i$ in (3) we see that $\theta_{i,i-n_i}(x, b)$ is equivalent to $\varphi_{n_i}(x, b)$. Hence $\theta_{i,i}(x, b)$ is equivalent to $\varphi_{n_i}(x, b)$, and from (4) for $i < a$ we have

$$[S_i \neq \emptyset \text{ and } i > n_i] \Rightarrow [\psi_{i+1}(x, b) \text{ is equivalent to } \varphi_{n_i}(x, b)]. \tag{6}$$

From (2), (5), and the right hand side of (6) we can deduce that n_{i+1} is defined and $n_{i+1} > n_i$. Thus for $i < a$

$$[S_i \neq \emptyset \text{ and } i > n_i] \Rightarrow [S_{i+1} \neq \emptyset \text{ and } n_i < n_{i+1}]. \tag{7}$$

The last observation we need is that for $i < a$

$$S_i = \emptyset \Rightarrow S_{i+1} \neq \emptyset \tag{8}$$

which is immediate from (4).

From (7) and (8) it is clear that $S_a, S_{a-1}, S_{a-2}, \ldots$ are all nonempty and that $n_a, n_{a-1}, n_{a-2}, \ldots$ is a strictly descending sequence of natural numbers. This contradiction completes the proof.

Kotlarski has supplied the following example which shows that in general having a full satisfaction class does not imply resplendence. Let $\mathcal{N}$ be the standard model of PA and Σ be its truth set, i.e. the set of Godel numbers of sentences true in $\mathcal{N}$. Using a theorem of McDowell and Specker [3] we obtain an elementary end extension $\langle ^*\mathcal{N}, ^*\Sigma \rangle$ of $\langle \mathcal{N}, \Sigma \rangle$ which is ω_1-like. Then $^*\Sigma$ is a full satisfaction class for $^*\mathcal{N}$ which being a two-cardinal model is not resplendent.

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