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ON THE DARBOUX PROBLEM OF NEUTRAL TYPE

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The aim of this paper is to prove uniqueness theorems for the Darboux problem of neutral type in the space L^∞ and L^1.

1. INTRODUCTION

Let $I = [0,a]$, $a > 0$. Denote by $L^\infty(I^2)$ the space of Lebesgue measurable and essentially bounded functions $z : I^2 \to \mathbb{R}$, with the norm

$$\|z\|_\infty = \operatorname{ess}\ \sup_{I^2} |z(x,y)|.$$

Furthermore, let $L^1(I^2)$ denote the space of Lebesgue measurable functions $z : I^2 \to \mathbb{R}$ such that $\int_{I^2} |z(x,y)|\,dxdy < +\infty$, with the norm

$$\|z\|_1 = \int_{I^2} |z(x,y)|\,dxdy.$$

In this paper we consider the following Darboux problem of neutral type

$$(1)\qquad \begin{aligned} &z_{xy} = f\left(x,y,z(h(x,y)),z_{xy}(H(x,y))\right), \quad (x,y) \in I^2,\\ &z(x,0) = 0, \qquad x \in I,\\ &z(0,y) = 0, \qquad y \in I. \end{aligned}$$

In Section 2 we show that under suitable assumptions on the functions f, h and H, the problem (1) has a unique solution in the space $L^\infty(I^2)$. To prove this we apply the fixed point theorem from the paper [7]. In Section 3 we apply the classical Banach contraction principle to obtain an analogous result for the problem (1) in the space $L^1(I^2)$. Similar problems (with or without translation of arguments) have been considered for example, in the papers [**2, 3, 9**] and in the monograph [**1**].

In what follows we shall need two propositions from the papers [**7**] and [**8**].

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Let $(X, \|\cdot\|, \prec, m)$ be a Banach space with a binary relation $\prec$ and a mapping $m : X \to X$. Suppose that:

(i) the relation $\prec$ is transitive,

(ii) the norm $\|\cdot\|$ is monotonic, that is, if $\theta \prec w \prec v$, then $\|w\| \leqslant \|v\|$,

(iii) $\theta \prec m(w)$ and $\|m(w)\| = \|w\|$ for all $w \in X$.

PROPOSITION 1. [7] *In the Banach space considered above, let the operators $\mathcal{A} : X \to X$, $A : X \to X$ be given with the following properties:*

(iv) *A is a linear bounded operator with spectral radius $r(A)$ less than 1,*

(v) *if $\theta \prec w \prec v$, then $Aw \prec Av$,*

(vi) *$m(\mathcal{A}w - \mathcal{A}v) \prec Am(w - v)$ for all $w, v \in X$.*

Then the equation $\mathcal{A}x = x$ has a unique solution in X.

Assume further that:

(vii) the relation $\prec$ is reflexive,

(viii) if $w \prec v$, then $w + u \prec v + u$ for $w, v, u \in X$.

PROPOSITION 2. [8] *Let $(X, \|\cdot\|, \prec)$ denote a Banach space with a binary relation $\prec$ satisfying conditions* (i), (ii), (vii) *and* (viii). *In this space, let the linear and bounded operators $A : X \to X$, $B : X \to X$ be given. Assume that the following conditions are satisfied:*

(ix) *if $\theta \prec w$, then $\theta \prec Aw$ and $\theta \prec Bw$,*

(x) *there exists an element $w_0 \in X$, $\theta \prec w_0$ such that $r(A+B) = \lim_{n\to\infty} \|(A+B)^n w_0\|^{1/n}$ and $BA^jB^kw_0 \prec A^jB^{k+1}w_0$ for $j = 1, 2, \ldots$, $k = 0, 1, \ldots$.*

Then the inequality

$$r(A+B) \leqslant r(A) + r(B)$$

holds.

2. The Darboux problem in the space $L^\infty(I^2)$.

Let $w, v \in L^\infty(I^2)$. We shall say that $w \prec v$ if and only if $w(x,y) \leqslant v(x,y)$ almost everywhere on I^2. Moreover, let $m(w)(x,y) = |w(x,y)|$ for $(x,y) \in I^2$. It is clear that the conditions (i)–(iii) and (vii)–(viii) are satisfied in this case.

Assume that:

1^0 $h : I^2 \to I^2$ is a continuous function and $h(x,y) \leqslant (x,y)$ for every pair $(x,y) \in I^2$, where $h(x,y) = (h_1(x,y), h_2(x,y))$ and $(x_1,y_1) \leqslant (x_2,y_2)$ means that $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$;

2^0 $u \subset \mathbb{R}^2$ is an open set such that $I^2 \subset u$ and $H : u \to \mathbb{R}^2$ is a diffeomorphism "*into*" with the property $H(I^2) \subset I^2$, and $h(H(x,y)) \leqslant h(x,y)$ for $(x,y) \in I^2$;

3^0 $(x,y,u,v) \to f(x,y,u,v)$ is a real function defined on the product $I^2 \times \mathbb{R}^2$, Lebesgue measurable with respect to (x,y) for all $(u,v) \in \mathbb{R}^2$ and satisfying the Lipschitz condition

$$|f(x,y,u_1,v_1) - f(x,y,u_2,v_2)| \leqslant L_1\,|u_1 - u_2| + L_2\,|v_1 - v_2|$$

for (x,y,u_1,v_1), $(x,y,u_2,v_2) \in I^2 \times \mathbb{R}^2$, where $L_1 > 0$ and $0 < L_2 < 1$;

4^0 $|f(x,y,0,0)|$ is an essentially bounded function on I^2.

By a solution of the problem (1), defined on the set I^2, we understand a function $z : I^2 \to \mathbb{R}$ such that $z(x,y)$ is an absolutely continuous (shortly: AC) function with respect to x and y, z_x is an AC-function with respect to y for almost all $x \in I$, z_y is an AC-function with respect to x for almost all $y \in I$, $z_{xy}(x,y) = f(x,y,z(h(x,y)), z_{xy}(H(x,y)))$ almost everywhere on I^2, $z(x,0) = 0$ for $x \in I$ and $z(0,y) = 0$ for $y \in I$.

THEOREM 1. *Under the assumptions* $1^0 - 4^0$ *the problem* (1) *has a unique solution defined on* I^2.

PROOF: It is easy to verify that the problem (1) is equivalent to the following functional-integral equation

$$w(x,y) = f\Bigl(x,y,\int\limits_{D(h(x,y))} w(t,s)\,dtds, w(H(x,y))\Bigr), \quad (x,y) \in I^2, \tag{2}$$

where $D(x,y) = \{(t,s) \in I^2 :\ 0 \leqslant t \leqslant x,\ 0 \leqslant s \leqslant y\}$.

Indeed, let $z : I^2 \to \mathbb{R}$ be a solution of the problem (1) and put $z_{xy}(x,y) = w(x,y)$, $(x,y) \in I^2$. By the definition of a solution of (1) we have

$$\begin{aligned}
\int\limits_{D(h(x,y))} w(t,s)\,dtds &= \int\limits_0^{h_1(x,y)}\int\limits_0^{h_2(x,y)} z_{\xi\eta}(\xi,\eta)\,d\xi d\eta \\
&= \int\limits_0^{h_2(x,y)}\left[\int\limits_0^{h_1(x,y)} \frac{\partial}{\partial\xi}\left(\frac{\partial}{\partial\eta}z(\xi,\eta)\right)d\xi\right]d\eta \\
&= \int\limits_0^{h_2(x,y)}\left[\frac{\partial}{\partial\eta}z(h_1(x,y),\eta) - \frac{\partial}{\partial\eta}z(0,\eta)\right]d\eta \\
&= \int\limits_0^{h_2(x,y)} \frac{\partial}{\partial\eta}z(h_1(x,y),\eta)\,d\eta
\end{aligned}$$

$$= z(h_1(x,y), h_2(x,y)) - z(h_1(x,y), 0)$$
$$= z(h_1(x,y), h_2(x,y)) = z(h(x,y)).$$

This means that $w : I^2 \to \mathbb{R}$ is a solution of the equation (2). On the other hand, let $w : I^2 \to \mathbb{R}$ be a solution of (2) in the space $L^\infty(I^2)$. Put $z(x,y) = \int_0^x \int_0^y w(\xi,\eta)\,d\xi d\eta$. By the Tolstov theorem [6], we have

$$z_{xy}(x,y) = \frac{\partial^2}{\partial x \partial y} \int_0^x \int_0^y w(\xi,\eta)\,d\xi d\eta = w(x,y) \quad \text{for almost all} \quad (x,y) \in I^2.$$

Thus $z_{xy}(H(x,y)) = w(H(x,y))$ for almost all $(x,y) \in I^2$ and in consequence $z : I^2 \to \mathbb{R}$ is a solution of (1).

Consider the following operator:

$$F(w)(x,y) = f\Big(x, y, \int_{D(h(x,y))} w(t,s)\,dtds, w(H(x,y))\Big),$$

where $w \in L^\infty(I^2)$, $(x,y) \in I^2$.

Since the function $(x,y) \to \int_{D(h(x,y))} w(t,s)\,dtds$ is continuous on I^2 and the function $(x,y) \to w(H(x,y))$ is Lebesgue integrable on I^2, the function $(x,y) \to f\Big(x, y, \int_{D(h(x,y))} w(t,s)\,dtds, w(H(x,y))\Big)$ is Lebesgue measurable on I^2. Moreover, in view of 3^0 we have

$$|F(w)(x,y)| \leqslant L_1 \Big| \int_{D(h(x,y))} w(t,s)\,dtds \Big| + L_2\,|w(H(x,y))| + |f(x,y,0,0)|$$
$$\leqslant L_1 a^2 \|w\|_\infty + L_2 \|w\|_\infty + |f(x,y,0,0)|$$

for $w \in L^\infty(I^2)$ and $(x,y) \in I^2$. Hence, by the above inequality and 4^0, $F(L^\infty(I^2)) \subset L^\infty(I^2)$. Again in view of 3^0, for $w, v \in L^\infty(I^2)$, $(x,y) \in I^2$ we get

$$\begin{aligned}
&|F(w)(x,y) - F(v)(x,y)| \\
&= \Big| f\Big(x, y, \int_{D(h(x,y))} w(t,s)\,dtds, w(H(x,y))\Big) \\
&\quad - f\Big(x, y, \int_{D(h(x,y))} v(t,s)\,dtds, v(H(x,y))\Big) \Big| \\
&\leqslant L_1 \int_{D(h(x,y))} |w(t,s) - v(t,s)|\,dtds + L_2\,|w(H(x,y)) - v(H(x,y))|.
\end{aligned}$$

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Thus

$$(3)\qquad |F(w)(x,y) - F(v)(x,y)| \leqslant (A_1 + A_2)(|w-v|)(x,y),$$

where $A_1(u)(x,y) = L_1 \int\limits_{D(h(x,y))} u(t,s)\,dtds$, $A_2(u)(x,y) = L_2 u(H(x,y))$, $u \in L^\infty(I^2)$.

We shall show now that the operators $A_1 + A_2$ and F satisfy the assumptions of Proposition 1. Obviously, the operator $A_1 + A_2$ is linear and $(A_1 + A_2)(L^\infty(I^2)) \subset L^\infty(I^2)$. Furthermore

$$\begin{aligned}\|(A_1 + A_2)w\|_\infty &= \operatorname{ess\,sup}_{I^2} \Big| L_1 \int\limits_{D(h(x,y))} w(t,s)\,dtds + L_2 w(H(x,y)) \Big| \\ &\leqslant L_1 \operatorname{ess\,sup}_{I^2} \int\limits_{D(h(x,y))} |w(t,s)|\,dtds + L_2 \operatorname{ess\,sup}_{I^2} |w(H(x,y))| \\ &\leqslant L_1 a^2 \|w\|_\infty + L_2 \|w\|_\infty ,\end{aligned}$$

which means that $A_1 + A_2$ is a bounded operator. Moreover, $A_1 + A_2$ is an increasing operator. Indeed, if $w, v \in L^\infty(I^2)$ and $\theta \prec w \prec v$, then for almost all $(x,y) \in I^2$ we have

$$\begin{aligned}(A_1 + A_2)(w)(x,y) &= L_1 \int\limits_{D(h(x,y))} w(t,s)\,dtds + L_2 w(H(x,y)) \\ &\leqslant L_1 \int\limits_{D(h(x,y))} v(t,s)\,dtds + L_2 v(H(x,y)) = (A_1 + A_2)(v)(x,y).\end{aligned}$$

Notice that in view of (3) the condition (vi) of Proposition 1 is satisfied. It remains to prove that $r(A_1 + A_2) < 1$. First we shall show that the operators A_1, A_2 and $A_1 + A_2$ satisfy the assumptions of Proposition 2. For $\theta \prec w$ we have $\theta \prec A_1 w$ and $\theta \prec A_2 w$. Let K denote a cone of nonnegative functions in $L^\infty(I^2)$, that is, $K = \{w \in L^\infty(I^2) : w(x,y) \geqslant 0$ almost everywhere on $I^2\}$ and let $w_0(x,y) = 1$ almost everywhere on I^2. It is easy to verify that the cone K is normal and $w_0 \in intK$. Hence $r(A_1 + A_2) = \lim\limits_{n\to\infty} \|(A_1 + A_2)w_0\|_\infty^{1/n}$ (for example, see [4, 5]). For $j = 1, 2, \ldots$, $k = 0, 1, \ldots$, we have

$$\begin{aligned}&A_2 A_1^j A_2^k (w_0)(x,y) \\ &\qquad = L_1^j L_2^{k+1} \int\limits_{D(h(H(x,y)))} \int\limits_{D(h(x_1,y_1))} \cdots \int\limits_{D(h(x_{j-1},y_{j-1}))} 1 \;\, dx_j dy_j \ldots dx_1 dy_1\end{aligned}$$

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and

$$A_1^j A_2^{k+1}(w_0)(x,y)$$
$$= L_1^j L_2^{k+1} \int\limits_{D(h(x,y))} \int\limits_{D(h(x_1,y_1))} \cdots \int\limits_{D(h(x_{j-1},y_{j-1}))} 1 \; dx_j dy_j \ldots dx_1 dy_1.$$

Hence, in view of 2^0,

$$A_2 A_1^j A_2^k w_0 \prec A_1^j A_2^{k+1} w_0$$

for $j = 1, 2, \ldots$, $k = 0, 1, \ldots$. Therefore, by Proposition 2, the inequality $r(A_1 + A_2) \leqslant r(A_1) + r(A_2)$ holds. Finally, an easy computation shows that $\|A_1^n w_0\|_\infty \leqslant (L_1^n a^{2n})/(n!)^2$ while $\|A_2^n w_0\|_\infty = L_2^n$. Thus $r(A_1) = 0$ and $r(A_2) = L_2$. Since $L_2 < 1$, this gives $r(A_1 + A_2) < 1$. It follows from Proposition 1 that the equation (2) has exactly one solution in $L^\infty(I^2)$. This completes the proof of Theorem 1. ☐

3. The Darboux problem in the space $L^1(I^2)$

Assume now that

5^0 $h : I^2 \to I^2$ is a continuous function;

6^0 $U, V \subset \mathbb{R}^2$ are any open sets such that $I^2 \subset U$, $I^2 \subset V$ and $H : U \to V$ is a diffeomorphism with the property $H(I^2) = I^2$;

7^0 $(x, y, u, v) \to f(x, y, u, v)$ is a real function defined on the product $I^2 \times \mathbb{R}^2$ which is Lebesgue measurable in (x, y) for every $(u, v) \in \mathbb{R}^2$ and satisfies the Lipschitz condition

$$|f(x, y, u_1, v_1) - f(x, y, u_2, v_2)| \leqslant L_1 |u_1 - u_2| + L_2 |v_1 - v_2|$$

for (x, y, u_1, v_1), $(x, y, u_2, v_2) \in I^2 \times \mathbb{R}^2$, where $L_1, L_2 > 0$, $L_1 a^2 + L_2 M < 1$, $M = \left(\min\limits_{(x,y) \in I^2} |H'(x, y)|\right)^{-1}$ and $|H'(x, y)|$ denotes the absolute value of the Jacobian of the mapping H;

8^0 there exists a function $m_0 : I^2 \to \mathbb{R}_+$ which is integrable in the Lebesgue sense and such that

$$|f(x, y, 0, 0)| \leqslant m_0(x, y) \quad \text{for} \quad (x, y) \in I^2.$$

In the situation described above, we define a solution of (1) on I^2 analogously as in the previous section.

Now we can prove the following

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Theorem 2. *Under the above assumptions the problem* (1) *has a unique solution defined on* I^2.

Proof: The same arguments as in the proof of Theorem 1 show that (1) and (2) are equivalent. Define the operator

$$F(w)(x,y) = f\Big(x, y, \int\limits_{D(h(x,y))} w(t,s)\,dtds, w(H(x,y))\Big),$$

where $w \in L^1(I^2)$, $(x,y) \in I^2$.

Since

$$|F(w)(x,y)| \leqslant L_1 \Big| \int\limits_{D(h(x,y))} w(t,s)\,dtds \Big| + L_2\,|w(H(x,y))| + |f(x,y,0,0)|,$$

$F(L^1(I^2)) \subset L^1(I^2)$.

Further, we have

$$\begin{aligned}|F(w)(x,y) - F(v)(x,y)| \leqslant L_1 &\int\limits_{D(h(x,y))} |w(t,s) - v(t,s)|\,dtds \\ &+ L_2\,|w(H(x,y)) - v(H(x,y))|, \quad w,v \in L^1(I^2), \quad (x,y) \in I^2.\end{aligned}$$

Thus

$$\begin{aligned}&\|F(w) - F(v)\|_1 \\ &\leqslant L_1 \int\limits_{I^2} \Big(\int\limits_{D(h(x,y))} |w(t,s) - v(t,s)|\,dtds \Big) dxdy + L_2 \int\limits_{I^2} |w(H(x,y)) - v(H(x,y))|\,dxdy \\ &\leqslant L_1 a^2 \|w - v\|_1 + L_2 \int\limits_{I^2} |w(H(x,y)) - v(H(x,y))|\,|H'(x,y)| \frac{1}{|H'(x,y)|}\,dxdy \\ &\leqslant (L_1 a^2 + L_2 M)\,\|w - v\|_1 .\end{aligned}$$

In view of Banach contraction principle the mapping F has a unique fixed point. Hence the proof of Theorem 2 is completed. ∎

4. Remarks

It is clear that in the of proof Theorem 2 one can apply Proposition 1 (under additional assumption on the functions h and H). Consider the following linear operator

$$A(w)(x,y) = L_1 \int\limits_{D(h(x,y))} w(t,s)\,dtds + L_2 w(H(x,y)),$$

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$w \in L^1(I^2)$, $(x,y) \in I^2$.

One can easy verify that if h and H satisfy 1^0 and 2^0 in addition then $\|A\|_1 \leqslant L_1 a^2 + L_2 M$, $\|A^2\|_1 \leqslant L_1 a^4/4 + L_1 L_2 a^2(M+1) + L_2^2 M$.

Further, it is well known that in an arbitrary Banach space

$$r(A) \leqslant \sqrt[n]{\|A^n\|} \quad \text{for every} \quad n \in \mathbb{N}.$$

Hence, if for example, $M \geqslant 1$ then

$$L_1 a^2 + L_2 M \geqslant \sqrt{L_1^2 \frac{a^4}{4} + L_1 L_2 a^2(M+1) + L_2^2 M^2}$$

and the assumption $r(A) < 1$ is better than $L_1 a^2 + L_2 M < 1$. But we can not find an estimate of the spectral radius of the operator A in terms of some constants and, therefore, we choose the Banach theorem to prove Theorem 2.

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