Non-steady Global Magnetic Fields
in Kinematic Theory

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Abstract: The dynamo equation for the mean field $\langle B \rangle$ contains a random forcing term of unknown magnitude, which is therefore always omitted. The influence of this term is potentially large. To evaluate its effect, we employ *ensemble averaging*. If an ensemble average is used, there is no random forcing term in the dynamo equation. The effect of fluctuations is that the ensemble members get out of phase, so that $\langle B \rangle \to 0$. The damping time of $\langle B \rangle$ can be found by requiring that the mean energy $\langle BB \rangle$ remains finite. The eigenvalues κ of the dynamo equation then all have negative real parts. Imκ determines the period, and -Reκ/Imκ the relative period stability of the dynamo. We have developed a code to solve the equation for $\langle BB \rangle$ in a spherical shell (the convection zone), assuming axisymmetry. We report our first results, which do not yet include differential rotation. Using spherically symmetric boundary conditions, we reproduce the well known α^2-dynamo, whose behaviour is known analytically. For instance, for an α^2-dynamo located in a shell with inner boundary at $R/2$, we find that $\langle BB \rangle$ remains finite for $R^2\gamma/\beta = 1.48$, where β represents turbulent diffusion and γ turbulent vorticity. Taking $\alpha = \frac{1}{4}(\beta\gamma)^{1/2}$ — a factor of four below maximum helicity — implies that we have a dynamo number $C_\alpha \equiv R\alpha/\beta = 0.30$. Using this value we find a damping time of $6 \times 10^{-2} R^2/\beta$ for $\langle B \rangle$, which is a measure for the coherence time of B in a single ensemble member. This result implies that the large-scale field of this particular α^2-dynamo reorganizes its structure completely on a time scale of only about one year (for solar values of R and β), and it shows the enormous influence of random forcing in general.

1. Introduction

The effects of turbulence in the convection zone of stars on the magnetic field B are described by the dynamo equation:

$$\partial_t \langle B \rangle = \alpha \nabla \times \langle B \rangle + \beta \nabla^2 \langle B \rangle , \qquad (1)$$

where $\langle B \rangle$ is some average of the magnetic field B, which we discuss presently. We have assumed that there is no differential rotation. The first term in (1) is the turbulent helicity term, and the second is the turbulent diffusion term. Their coefficients are defined as:

$$\alpha \approx -\langle u_1 \cdot \nabla \times u_1 \rangle \tau_c/3 ; \qquad \beta \approx \langle u_1^2 \rangle \tau_c/3 , \qquad (2)$$

which we assume to be constant; $u_1(r,t)$ is the turbulent velocity, and τ_c its correlation time.

All finite solutions of the dynamo equation are *strictly* periodic, and in our case this period is infinite, since we assume no differential rotation. *Strict* periodicity is very strange, since the underlying mechanisms are of a *stochastic* nature, so we

would expect to find at best quasi-periodic solutions. Yet, the traditional dynamo equation suggests that the dynamo has an infinite memory. The origin of this feature lies in the derivation of (1). The averaging operator $\langle \cdot \rangle$ must satisfy the Reynolds relations (e.g., Krause and Rädler, 1980):

$$\langle f + g \rangle = \langle f \rangle + \langle g \rangle \; ; \quad \langle f \langle g \rangle \rangle = \langle f \rangle \langle g \rangle \; ; \quad \langle c \rangle = c \; ; \qquad (3\mathrm{a,b,c})$$

$$\langle \cdot \rangle \text{ commutes with } \nabla, \; \partial_t \text{ and all integrals} , \qquad (3\mathrm{d})$$

where $f(\boldsymbol{r}, t)$ and $g(\boldsymbol{r}, t)$ are arbitrary functions and c is an arbitrary constant.

The average over space or time, generally used in (1), does not satisfy (3b) and (3d), which means that there is actually an extra term on the r.h.s. of (1) to account for this discrepancy. This extra term depends on the turbulent velocity $\boldsymbol{u}_1$ and has therefore the nature of a random forcing term, henceforth abbreviated as *r.f.t.* Even if $\langle \cdot \rangle$ is defined as a longitudinal average, i.e. $\langle \bullet \rangle = (2\pi)^{-1} \oint \langle \bullet \rangle d\phi$, which satisfies (3) exactly, there still is random forcing, because according to their definition α and β will now have a fluctuation component according to their definition (2).

The *r.f.t.* is usually omitted, and that explains the (unphysical) infinite phase memory of the dynamo. We wish to investigate the magnitude of this *r.f.t.*, but since it is very difficult to obtain a manageable expression for it, we adopt another approach.

2. Ensemble average

We interpret $\langle \cdot \rangle$ in (1) as an *ensemble average*, which does obey (3) exactly. In this case α and β are constant, meaning that (1) holds as it stands: there is *no r.f.t.* if an ensemble average is used. One has to take the ensemble literally: an infinite set of identical systems, each having a different realization of $\boldsymbol{u}_1(\boldsymbol{r}, t)$. This, however, implies that $\langle \boldsymbol{B} \rangle$ no longer refers to a specific dynamo, or in other words: we don't know which of the ensemble members is *our* dynamo. Nevertheless, the ensemble averaging turns out to be a useful approach for estimating the order of magnitude of the *r.f.t.*

Our line of thought is as follows: Since each of the ensemble members has a different realization of fluctuations, the ensemble members gradually get out of phase. In other words: eventually $\langle \boldsymbol{B} \rangle \to 0$. If we can determine the damping time of this process, we know the *correlation time* of $\boldsymbol{B}$ itself in a given ensemble member.

3. Mean magnetic energy

In order to find the damping time of $\langle \boldsymbol{B} \rangle$, we analyse the *mean magnetic energy* $\langle \boldsymbol{B} \boldsymbol{B} \rangle$. On physical grounds we must require that this mean magnetic energy remains *finite*. For if $\langle \boldsymbol{B} \boldsymbol{B} \rangle \to 0$, then $B = 0$ in the end, contrary to the assumption that there is a dynamo. And if $\langle \boldsymbol{B} \boldsymbol{B} \rangle$ would increase, the energy would grow indefinitely, which is physically not acceptable either. Therefore we must require that $\langle \boldsymbol{B} \boldsymbol{B} \rangle$ remains finite. We can now find the damping time of $\langle \boldsymbol{B} \rangle$, because the requirement "$\langle \boldsymbol{B} \boldsymbol{B} \rangle$ *remains finite*" fixes a combination of parameters also present in the dynamo equation.

The equation for the tensor $T_{ij} = \langle B_i B_j \rangle$ is (Knobloch, 1978; Hoyng, 1987):

$$\partial_t T_{ij} = \alpha(\epsilon_{ikl}\nabla_k T_{lj} + \epsilon_{jkl}\nabla_k T_{li}) + \beta\nabla^2 T_{ij} + \frac{2}{5}\gamma(2T_{kk}\delta_{ij} - T_{ij}), \qquad (4)$$

where i, j represent Cartesian coordinates; recall that differential rotation is omitted. The first and second terms in (4) are turbulent helicity and diffusion, also present in the dynamo equation (1). The third term depends on the turbulent vorticity, and it represents generation of magnetic energy. The coefficient is defined as:

$$\gamma \approx \langle|\nabla \times \boldsymbol{u}_1|^2\rangle\tau_c/3 , \qquad (5)$$

which we also assume to be constant. Note that α, β and γ must fulfill Schwarz' inequality:

$$\alpha^2 \leq \beta\gamma \quad \text{or} \quad C_\alpha^2 \leq C_\gamma, \qquad (6)$$

where:

$$C_\alpha \equiv R\alpha/\beta ; \qquad C_\gamma \equiv R^2\gamma/\beta , \qquad (7)$$

with R the outer radius of the dynamo. C_α is the usual dynamo number, important in both (1) and (4); C_γ is a second "dynamo number" relevant for (4) only. The definition of these two numbers follows upon the introduction of a dimensionless time $t\beta/R^2$ in both (1) and (4).

The damping time of $\langle B \rangle$ is found as follows. We choose a value of C_α and determine, subject to condition (6), the value of C_γ for which eq. (4) has a solution that remains finite. With the adopted value of C_α we determine the eigenvalues of (1), from which we obtain the damping time of $\langle B \rangle$. This is expounded in the following sections.
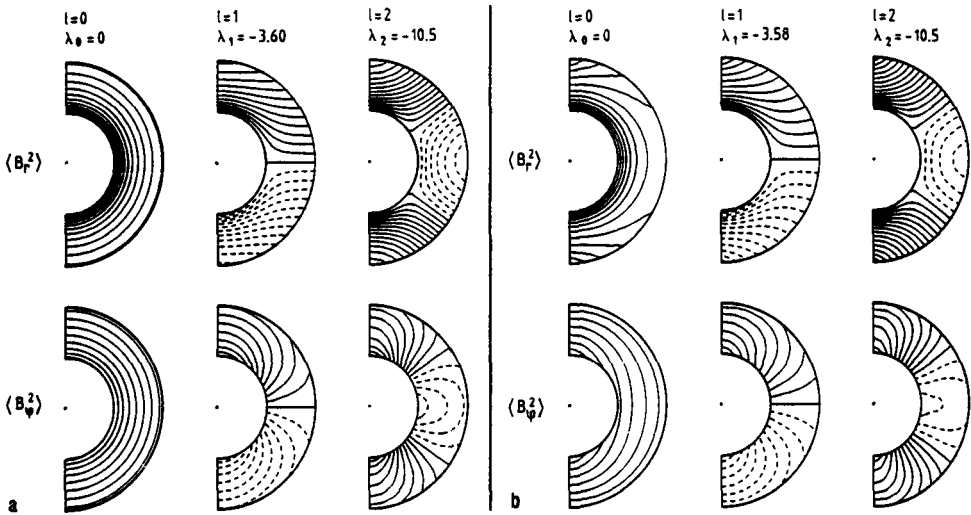
4. Results

We have developed a code to determine the eigenvalues of (4) in a spherical shell, using spherical coordinates r, θ, ϕ, and assuming axisymmetry. Eq. (4) is in fact a set of six coupled differential equations, and therefore six boundary conditions are needed, both on the inner boundary $r = r_c$ and on the outer boundary $r = R$; for completeness they are summarized in the Appendix.

Let us write the eigenvalues of (4) as $\Lambda_k = \lambda_k + \mu_k\text{i}$ $(k = 0, 1, 2, \ldots)$. The real parts λ_k of the eigenvalues form a decreasing series (because (4) is a parabolic equation). The requirement that $\langle BB \rangle$ remains finite implies that the largest of the real parts must equal zero, i.e. $\lambda_0 = 0$. The eigenfunction $T_{ij,0}$ then describes the corresponding energy distribution within the dynamo; $T_{ij,0}$ depends on the value of C_α, but λ_0 does not (Van Geffen and Hoyng, 1990). Figure 1 shows two examples:

a) *Spherically symmetric* boundary conditions, which reproduces the spherically symmetric α^2-dynamo, whose behaviour is known analytically (Van Geffen and Hoyng, 1990), and can serve as a check on the computations.

b) *Axisymmetric* boundary conditions. In this case there is a preferred direction (the vertical axis).

In both figures, the top row shows the distribution of magnetic energy along the radial direction $\langle B_r^2 \rangle$, the bottom row along the ϕ direction $\langle B_\phi^2 \rangle$. Since there is no magnetic field within the core, we have $B_r = 0$ at the inner boundary, implying

that $\langle B_r^2 \rangle$ equals zero at $r = r_c$, and increases outward. On the other hand, $\langle B_\phi^2 \rangle$ is maximal on the inner boundary and decreases outward.

Fig. 1. Contour lines of the distribution of magnetic energy in a spherical shell, using a) spherically symmetric and b) axisymmetric boundary conditions. Solid lines represent positive contours, broken lines negative contours. In both cases $C_\gamma = R^2 \gamma / \beta = 1.48$, $C_\alpha = 1/4 \sqrt{C_\gamma} = 0.30$. See the text for an explanation.

The left column shows the eigenfunction $T_{ij,0}$ belonging to the eigenvalue with the largest real part $\lambda_0 = 0$. The second and third column show the eigenfunctions belonging to the states with the next two largest real parts: λ_1 and λ_2. The solutions shown in Figure 1 are non-periodic, i.e. they have $\mu_k = 0$. The states with negative energy ($k = 1, 2, \ldots$) are damped and will play only a transient role, as the solution of (4) relaxes from arbitrary initial conditions to the stationary solution. Some of the higher eigenfunctions are periodic (that is: they have $\mu_k \neq 0$) in addition to being damped; we found, however, that μ_0 always equals zero.

We can assign quantum numbers to the eigenmodes. First of all m equals zero because of axisymmetry. The quantum number l, representing the number of zeros along the θ direction, takes values 0, 1 and 2 as indicated in Figure 1. The radial quantum number n equals zero in the cases shown, but going to larger values of $|\lambda_k|$ there will also be states with $n = 1, 2, \ldots$.

5. Damping time of $\langle B \rangle$

It turns out that $\lambda_0 = 0$ is reached if $C_\gamma = 1.48$, independent of C_α (see Van Geffen and Hoyng, 1990). In the figure we have taken $C_\alpha = 1/4 \sqrt{C_\gamma} = 0.30$, so that α equals $1/4$ of its maximum value according (6). With this value we solve (1) analytically by extending the method of Krause and Rädler (1980, Ch. 14), to allow for the existence of an inner boundary (where $B_r = 0$). The eigenvalues κ_k of (1) are real, since there is no differential rotation. For $C_\alpha = 0.30$ it turns out

that $\kappa_k < 0$ for *all k*, thus confirming that *all* solutions of the dynamo equation are damped (cf. Hoyng, 1987). The longest living mode is determined by the the smallest $|\kappa_k|$, which we found to be equal to $16\,\beta/R^2$. Hence the damping time of $\langle B \rangle$ is $(16\,\beta/R^2)^{-1} \approx 6 \times 10^{-2}\,R^2/\beta$.

Our interpretation of this result is as follows. The dynamo equation (1) with $\langle \cdot \rangle$ is a spatial or time average contains a *r.f.t.* and may be written as follows:

$$\partial_t \langle B \rangle = \alpha \nabla \times \langle B \rangle + \beta \nabla^2 \langle B \rangle + \underbrace{\cdots \langle B \rangle}_{r.f.t} . \tag{8}$$

The diffusion term is of the order of β/R^2, so we must conclude that in this case the *r.f.t.* is about 16 times *larger* than the diffusion term. Although this is only a rough estimate, it shows that the *r.f.t.* is certainly not small, as has been assumed so far (e.g., Krause and Rädler, 1980). We have analyzed values of C_α ranging from 1 to $^1/_{128}$ times the maximum value $\sqrt{C_\gamma}$, and we found the damping time to be of the same order of magnitude as in the above example.

6. Concluding remarks

We have shown above that by using ensemble averaging and requiring that $\langle BB \rangle$ remains finite, the damping time of $\langle B \rangle$ can be found, and the magnitude of the random forcing term in the dynamo equation can be estimated. We have also shown that this random forcing dominates in (8), rather than being negligible. This result raises several issues:

- First of all the role of non-linearities. Our main result is that random forcing is an essential ingredient in the dynamo equation (1), if an average over space or time is used. The use of an ensemble average to estimate the magnitude of the *r.f.t.* is merely a technical matter. Apparently the *r.f.t.* is very large and this makes the dynamo unsteady and erratic. At present it is not clear whether non-linearities (which are of course important) will further destabilize the dynamo. The traditional approach, however, where non-linear effects are included in (1) — for example by making α a function of $|\langle B \rangle|$ — while random forcing is ignored, seems rather pointless in the light of the present results.

- We have assumed isotropic, homogeneous turbulence and infinite resistivity. These assumptions are not very realistic and have to be removed, but we believe that this will not drastically change the outcome of our study.

- What happens if we adopt a larger value of C_α? We found the dipole solution of (1) to be marginally stable only at $C_\alpha = 6.57$. We have verified that for this large value of C_α, eq. (4) always has growing solutions (i.e. $\lambda_0 > 0$). A full discussion will appear later.

- The aim of our work is to study the effect of random forcing in the solar dynamo, in order to find out whether the observed period stability of the 22-year cycle of the sun ($\delta P/P \approx 0.1$) can be explained within linear theory.

Appendix: Boundary conditions

We consider a dynamo in a spherical shell: $0 < r_c \leq r \leq R$, and assume axisymmetry. The first column shows the boundary conditions and quantities specified on the inner boundary, and the second column those on the outer boundary. The first two lines follow directly from physics, where $\epsilon \equiv 2gem$ is the mean magnetic energy density, and where σ, τ are related to the distribution of magnetic energy over spatial scales; these are discussed in more detail in Van Geffen and Hoyng (1990). The bottom three lines represent cross-correlation functions at the boundaries. On the axis: $T_{r\theta} = T_{r\phi} = T_{\theta\phi} = 0$, $T_{\theta\theta} = T_{\phi\phi}$ because of axisymmetry.

$r = r_c$	$r = R$
$\partial\epsilon/\partial r = 0$	$\partial\epsilon/\partial r + \epsilon/\varrho = 0$
$T_{rr} = T_{r\theta} = T_{r\phi} = 0$	
	$\sigma = (T_{\theta\theta} + T_{\phi\phi})/T_{rr}$
$\tau^0 = T_{\theta\theta}/T_{\phi\phi}$	$\tau = T_{\theta\theta}/T_{\phi\phi}$
	$C_{r\theta} = T_{r\theta}/\sqrt{T_{rr}T_{\theta\theta}}$
	$C_{r\phi} = T_{r\phi}/\sqrt{T_{rr}T_{\phi\phi}}$
$C^0_{\theta\phi} = T_{\theta\phi}/\sqrt{T_{\theta\theta}T_{\phi\phi}}$	$C_{\theta\phi} = T_{\theta\phi}/\sqrt{T_{\theta\theta}T_{\phi\phi}}$

The distribution and correlation functions chosen for the graphs in Figure 1 are:

	spherically sym.	axisymmetric
σ	1/2	$1/2+\sin\theta$
τ^0, τ	1	$1+\sin\theta$
$C^0_{\theta\phi}, C_{\theta\phi}$	0	$\sin\theta$
$C_{r\theta}, C_{r\phi}$	0	0
R/ϱ	1	1

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