



# Commuting and Semi-commuting Monomial-type Toeplitz Operators on Some Weakly Pseudoconvex Domains

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*Abstract.* In this paper, we completely characterize the finite rank commutator and semi-commutator of two monomial-type Toeplitz operators on the Bergman space of certain weakly pseudoconvex domains. Somewhat surprisingly, there are not only plenty of commuting monomial-type Toeplitz operators but also non-trivial semi-commuting monomial-type Toeplitz operators. Our results are new even for the unit ball.

## 1 Introduction

The Toeplitz operators on certain pseudoconvex domains in  $\mathbb{C}^n$  have been the object of much study; see [1, 4, 10, 15] for example. In this paper, we shall consider Toeplitz operators on the weakly pseudoconvex domains

$$\Omega_m^n = \left\{ (z_1, \dots, z_n) \in \mathbb{C}^n : \sum_{i=1}^n |z_i|^{2m_i} < 1 \right\},$$

where  $m = (m_1, \dots, m_n)$  is an  $n$ -tuple of positive integers. We shall suppose that  $n > 1$  to avoid trivialities throughout the paper. Then for each  $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ , we let

$$r = \sqrt{|z_1|^{2m_1} + \dots + |z_n|^{2m_n}},$$
$$\zeta = (\zeta_1, \dots, \zeta_n) = \left( \frac{z_1}{r^{\frac{1}{m_1}}}, \dots, \frac{z_n}{r^{\frac{1}{m_n}}} \right) \in \mathbb{S}_m^n,$$

where  $\mathbb{S}_m^n$  is the boundary of  $\Omega_m^n$ . Note that these expressions define a set of coordinates  $(r, \zeta)$  for every  $z \in \mathbb{C}^n$ ; these coordinates are called  $m$ -polar coordinates (see [15]). If  $m = (1, \dots, 1)$ , then  $\Omega_m^n = \mathbb{B}^n$  is the unit ball centered at the origin.

Let  $L^2(\Omega_m^n)$  be the collection of all square integrable functions with respect to the usual Lebesgue measure  $dV$  on  $\Omega_m^n$ . The Bergman space  $A^2(\Omega_m^n)$  is the closed subspace of  $L^2(\Omega_m^n)$  consisting of holomorphic functions in  $\Omega_m^n$ . Denote by  $P: L^2(\Omega_m^n) \rightarrow A^2(\Omega_m^n)$  the orthogonal projection. Given a symbol  $u \in L^\infty(\Omega_m^n)$ , the Toeplitz operator  $T_u$  induced by  $u$  is the bounded operator defined by

$$T_u(f) = P(uf): A^2(\Omega_m^n) \longrightarrow A^2(\Omega_m^n).$$

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For two Toeplitz operators  $T_{f_1}$  and  $T_{f_2}$  on  $A^2(\Omega_m^n)$ , the commutator and semi-commutator are defined by  $[T_{f_1}, T_{f_2}] = T_{f_1}T_{f_2} - T_{f_2}T_{f_1}$  and  $(T_{f_1}, T_{f_2}) = T_{f_1}T_{f_2} - T_{f_1f_2}$ , respectively.

The problem of characterizing when two Toeplitz operators commute or semi-commute on the Bergman space over various domains has been a long-term research topic. For example, in the setting of the Bergman space over the unit disk, the Brown–Halmos type theorems were obtained in [2, 3] for Toeplitz operators with harmonic symbols, and many other types of (semi-)commuting Toeplitz operators with quasihomogeneous symbols were found in [5, 7, 13, 14]. However, the general (semi-)commuting problem remains open on the unit disk, and it becomes even more delicate and more challenging on higher-dimensional balls; see [6, 11, 12, 16, 18, 19], for example.

Just recently, the second author and Zhu [8] completely characterized when the commutators and semi-commutators of two monomial Toeplitz operators on the Bergman space of the unit ball  $A^2(\mathbb{B}^n)$  have finite rank. In this paper, we take the weakly pseudoconvex domain  $\Omega_m^n$  as our domain and consider more general symbols, namely, the monomial-type symbols. Recall that the monomial-type symbol is the function  $\varphi: \Omega_m^n \rightarrow \mathbb{C}$  given by

$$\varphi(z) = r^l \zeta^p \bar{\zeta}^q$$

for  $p, q \in \mathbb{N}^n$ ,  $l \in \mathbb{R}_+$ . (Here  $\mathbb{R}_+$  denotes the set of all nonnegative real numbers.) In this case, the corresponding Toeplitz operator  $T_\varphi$  is called a *monomial-type Toeplitz operator*.

To state our main results, we need some notations. For  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ , we write

$$|\widehat{\alpha}| = \frac{\alpha_1}{m_1} + \dots + \frac{\alpha_n}{m_n}.$$

If  $m_i = 1$  for all  $i \in \{1, \dots, n\}$ , then we will use the usual notation  $|\alpha|$  instead of  $|\widehat{\alpha}|$ . A tuple  $(x_1, x_2, y_1, y_2) \in \mathbb{R}_+^4$  is said to satisfy Condition (I) if at least one of the following conditions holds:

- (i)  $x_1 = x_2 = 0$ ,
- (ii)  $y_1 = y_2 = 0$ ,
- (iii)  $x_1 = y_1 = 0$ ,
- (iv)  $x_2 = y_2 = 0$ ,
- (v)  $x_1 = x_2$  and  $y_1 = y_2$ ,
- (vi)  $x_1 = y_1$  and  $x_2 = y_2$ .

For  $p, q, s, t \in \mathbb{N}^n$ , [8, Theorem A] shows that the operators  $T_{z^p \bar{z}^q}$  and  $T_{z^s \bar{z}^t}$  commute on  $A^2(\mathbb{B}^n)$  if and only if one of the following five conditions holds:

- (c1) One of the two operators is the identity operator;
- (c2) Both operators have analytic symbols;
- (c3) Both operators have conjugate analytic symbols;
- (c4)  $|p| = |q|$ ,  $|s| = |t|$ , and  $(p_i, q_i, s_i, t_i)$  satisfies Condition (I) for all  $i \in \{1, 2, \dots, n\}$ ;
- (c5)  $|p| = |s|$ ,  $|q| = |t|$ , and  $(p_i, q_i, s_i, t_i)$  satisfies Condition (I) for all  $i \in \{1, 2, \dots, n\}$ .

Clearly, conditions (c4) and (c5) produce lots of non-trivial commuting monomial Toeplitz operators on  $A^2(\mathbb{B}^n)$ ; see [8, Example 6]. On the setting of the Bergman

space over the weakly pseudoconvex domain  $\Omega_m^n$ , Barranco and Nungaray [15] studied the commutativity of two Toeplitz operators with special  $k$ -quasi-homogeneous symbols. Here,  $k = (k_1, \dots, k_\tau)$  is a partition of  $n$ , and then each  $z \in \mathbb{C}^n$  can be decomposed into  $\tau$  pieces. Inspired by [8, Theorem A], we obtain the following general result, which gives two non-trivial sufficient conditions for two  $k$ -quasi-homogeneous Toeplitz operators commuting on  $A^2(\Omega_m^n)$ .

**Proposition 1.1** *Let  $p, q, s, t \in \mathbb{N}^n$ , and let  $\varphi$  and  $\psi$  be bounded  $k$ -quasi-radial functions on  $\Omega_m^n$ . Suppose one of the following statements holds:*

- (i)  $(p_i, q_i, s_i, t_i)$  satisfies Condition (I) for each  $i \in \{1, 2, \dots, n\}$ ,  $|p_{(j)}| = |q_{(j)}|$  and  $|s_{(j)}| = |t_{(j)}|$  for each  $j \in \{1, 2, \dots, \tau\}$ ;
- (ii)  $(p_i, q_i, s_i, t_i)$  satisfies Condition (I) for each  $i \in \{1, 2, \dots, n\}$ ,  $|p_{(j)}| = |s_{(j)}|$  and  $|q_{(j)}| = |t_{(j)}|$  for each  $j \in \{1, 2, \dots, \tau\}$ , and  $\varphi(r_1, \dots, r_\tau) = \psi(r_1, \dots, r_\tau)$ .

Then the operators  $T_{\zeta^p \bar{\zeta}^q \varphi}$  and  $T_{\zeta^s \bar{\zeta}^t \psi}$  commute on  $A^2(\Omega_m^n)$ .

Since the proof of this result is a direct calculation using [15, Lemma 3.6], we leave it to the interested reader. Unfortunately, so far no other non-trivial case has been obtained for two  $k$ -quasi-homogeneous Toeplitz operators commuting even on  $A^2(\mathbb{B}^n)$ . In this paper, we will give many new commuting Toeplitz operators on  $A^2(\Omega_m^n)$  induced by the monomial-type symbols.

Our first main result not only shows that there is no nonzero finite rank commutator of two monomial-type Toeplitz operators on  $A^2(\Omega_m^n)$ , but also gives a sufficient and necessary condition for such two operators to be commutative.

**Theorem 1.2** *Let  $l, k \in \mathbb{R}_+$ ,  $p, q, s, t \in \mathbb{N}^n$ . Then the following statements are equivalent.*

- (i) *The commutator  $[T_{r^l \zeta^p \bar{\zeta}^q}, T_{r^k \zeta^s \bar{\zeta}^t}]$  on  $A^2(\Omega_m^n)$  has finite rank.*
- (ii) *The operators  $T_{r^l \zeta^p \bar{\zeta}^q}$  and  $T_{r^k \zeta^s \bar{\zeta}^t}$  commute on  $A^2(\Omega_m^n)$ .*
- (iii)  *$(p_i, q_i, s_i, t_i)$  satisfies Condition (I) for all  $1 \leq i \leq n$ , and there exist real numbers  $\mu, \nu$  and  $a \geq \mu/2, b \geq \nu/2$  such that*

$$(1.1) \quad \frac{\Gamma(\eta + |\widehat{p}|)\Gamma(\eta + \nu + 1)\Gamma(\eta + \mu + |\widehat{s}|)}{\Gamma(\eta + |\widehat{s}|\Gamma(\eta + \mu + 1)\Gamma(\eta + \nu + |\widehat{p}|)} = \frac{(\eta + b)(\eta + a + \nu)}{(\eta + a)(\eta + b + \mu)}$$

for any  $\eta \in \mathbb{C}$  on some right half-plane. In this case,  $|\widehat{q}| = |\widehat{p}| - \mu, |\widehat{t}| = |\widehat{s}| - \nu, l = 2a - \mu, \text{ and } k = 2b - \nu.$

While one would wish for a more conceptual conclusion, there are too many cases for the tuple  $(|\widehat{p}|, |\widehat{s}|, \mu, \nu, a, b)$  satisfying the identity (1.1). Consequently, the commuting monomial-type Toeplitz operators on  $A^2(\Omega_m^n)$  involves not only the predictable cases but also plenty of non-trivial cases (see Corollary 3.1 and Example 3.2). It is also worth mentioning that the weakly pseudoconvex domains  $\Omega_m^n$  have many rich structural characteristics. For example, two of these domains are biholomorphically equivalent only if the corresponding  $m_i$ 's are the same up to permutation (see [17] for example). However, the operator theory on such family of domains becomes even more complicated and interesting (see Example 3.3).

As a consequence of Theorem 1.2, some interesting necessary conditions for two monomial-type Toeplitz operators commuting on  $A^2(\Omega_m^n)$  are obtained (see Corollary 3.4). Consequently, there exist a finite number of the combination of the tuple  $(|\widehat{p}|, |\widehat{s}|, \mu, \nu, a, b)$  satisfying identity (1.1). Also, we get some differences of the operator theory between on the weakly pseudoconvex domains and on the unit ball.

For the finite rank semi-commutator problem, [8, Theorem B] showed that the semi-commutator of two monomial Toeplitz operators on  $A^2(\mathbb{B}^n)$  has finite rank only in trivial cases: the first operator has conjugate analytic symbol or the second operator has analytic symbol. By contrast, our second result produces non-trivial cases for semi-commuting monomial-type Toeplitz operators on  $A^2(\Omega_m^n)$ .

**Theorem 1.3** *Let  $l, k \in \mathbb{R}_+$ ,  $p, q, s, t \in \mathbb{N}^n$ . Then the following statements are equivalent:*

- (i) *The semi-commutator  $(T_{r^l \zeta^p \bar{\zeta}^q}, T_{r^k \zeta^s \bar{\zeta}^t})$  on  $A^2(\Omega_m^n)$  has finite rank.*
- (ii)  *$T_{r^l \zeta^p \bar{\zeta}^q} T_{r^k \zeta^s \bar{\zeta}^t} = T_{r^{l+k} \zeta^{p+s} \bar{\zeta}^{q+t}}$  on  $A^2(\Omega_m^n)$ .*
- (iii) *At least one of the following conditions holds:*
  - (a)  *$l = |\widehat{q}| - |\widehat{p}|$  and  $t = 0$ ;*
  - (b)  *$k = |\widehat{s}|$  and  $t = 0$ ;*
  - (c)  *$l = |\widehat{q}|$  and  $p = 0$ ;*
  - (d)  *$k = |\widehat{s}| - |\widehat{t}|$  and  $p = 0$ .*

From Theorem 1.3 we get that the semi-commutator of two monomial Toeplitz operators  $T_{z^p \bar{z}^q}$  and  $T_{z^s \bar{z}^t}$  on  $A^2(\Omega_m^n)$  has finite rank if and only if either  $p = 0$  or  $t = 0$ , which is parallel to the result in [8, Theorem B]. In sharp contrast to the monomial case, there exist many non-trivial semi-commuting monomial-type Toeplitz operators. Furthermore, it is interesting to observe that the operator  $T_{r^{(|\widehat{q}|-|\widehat{p}|)} \zeta^p \bar{\zeta}^q}$  semi-commutes with all Toeplitz operators induced by symbol functions of the form  $r^k \zeta^s$ .

In addition, we would like to mention that the situation in the unit disk is quite different from the case of higher dimensional weakly pseudoconvex domains. In fact, the commutators or semi-commutators of two monomial-type Toeplitz operators on the Bergman space of the unit disk have finite rank only in several trivial cases; see [7, Corollaries 11 and 12].

While the main idea of our method of proofs is adapted from [8], a substantial amount of unexpected analysis is required to overcome some different nature of the weakly pseudoconvex domains and monomial-type symbols. For example, the method of characterizing (semi)-commuting  $k$ -quasi-homogeneous Toeplitz operators relies on explicit formulas for the action of the operators on the orthonormal basis of the Bergman space. This action leads to a holomorphic identity on a domain in the complex  $n$ -space; see, for example, (2.6). Then the most critical step in the proof is the analysis of such identity. To deal with this, [8] depended on the distribution of zeros of a holomorphic function, which does not work in the case of monomial-type symbols. In this paper, we devise a completely new approach, which greatly simplifies the proof even though the domain and the operators seem more general and complicated than those in [8]. Roughly speaking, we rewrite identity (2.6) as (2.7) and show that each function appeared in (2.7) must be a constant multiple of the exponent function.

Since all of them are bounded on the right half-plane with a certain growth at infinity, they must be the constant function.

We end this introduction by mentioning that the implication (i) to (ii) of Theorem 1.2 or Theorem 1.3 is in fact predictable: on the Bergman space, finite rank commutators and semi-commutators with bounded symbols are usually zero. For example, in the case of the Bergman space of the unit disk, Guo, Sun, and Zheng [9] showed that there is no non-zero finite rank commutator or semi-commutator of Toeplitz operators induced by bounded harmonic symbols. But the problem is very much open for other symbol classes.

## 2 Proofs of Theorems 1.2 and 1.3

We start this section with the following proposition, which will play an essential role in the proofs of our main theorems.

**Proposition 2.1** *Suppose each function  $f_i, i \in \{1, \dots, n\}$ , is analytic and has no zero on certain right half-planes  $\Pi_{a_i}^+ = \{\zeta_i \in \mathbb{C} : \text{Re } \zeta_i > a_i\}$  with  $a_i \in \mathbb{R}$ . If there exists an analytic function  $f$  such that*

$$(2.1) \quad f(\zeta_1 + \dots + \zeta_n) = \prod_{i=1}^n f_i(\zeta_i)$$

for any  $\zeta_i \in \Pi_{a_i}^+$ , then  $f_i(\zeta_i) = c_i e^{\lambda \zeta_i}, i \in \{1, \dots, n\}$ , for some complex constants  $c_i$  and  $\lambda$  (independent of  $i$ ).

**Proof** Taking the natural logarithm on both sides of (2.1), we get

$$\ln f(\zeta_1 + \dots + \zeta_n) = \ln f_1(\zeta_1) + \dots + \ln f_n(\zeta_n).$$

Fix any  $i \in \{1, \dots, n\}$  and take partial derivatives with respect to  $\zeta_i$  on both sides. The result is

$$\frac{f'(\zeta_1 + \dots + \zeta_n)}{f(\zeta_1 + \dots + \zeta_n)} = \frac{f'_i(\zeta_i)}{f_i(\zeta_i)}.$$

So for any  $i \neq j$ , we have

$$(2.2) \quad \frac{f'_i(\zeta_i)}{f_i(\zeta_i)} = \frac{f'_j(\zeta_j)}{f_j(\zeta_j)}.$$

Note that the left-hand side of (2.2) depends only on  $\zeta_i$ , and the right-hand side depends only on  $\zeta_j$ . Therefore, both sides of (2.2) should be a constant independent of  $i$ . Then we have

$$f'_i(\zeta_i) = \lambda f_i(\zeta_i)$$

for some  $\lambda \in \mathbb{C}$ , which implies that  $f_i(\zeta_i) = c_i e^{\lambda \zeta_i}$  for some  $c_i \in \mathbb{C}$ , as desired. ■

Throughout the rest of this paper, we are going to use the following notation. For two multi-indexes  $\alpha = (\alpha_1, \dots, \alpha_n)$  and  $\beta = (\beta_1, \dots, \beta_n)$  in  $\mathbb{N}^n$ , we write  $\alpha + \beta = (\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$  and  $\alpha \geq \beta$  if  $\alpha_i \geq \beta_i$  for all  $i \in \{1, \dots, n\}$ . To simplify the notation, we also write  $\alpha + 1 = (\alpha_1 + 1, \dots, \alpha_n + 1)$ . The reader should have no

problem accepting this slightly confusing notation. In addition, we will need to use the following formulas for integration on  $\Omega_m^n$  and  $\mathbb{S}_m^n$ :

$$(2.3) \quad \int_{\Omega_m^n} f(z) dV(z) = \int_0^1 r^{2(\sum_{i=1}^n \frac{1}{m_i})-1} dr \int_{\mathbb{S}_m^n} f(r, \zeta) dS(\zeta)$$

for every  $f \in L^1(\Omega_m^n)$ , and

$$(2.4) \quad \int_{\mathbb{S}_m^n} \zeta^\alpha \bar{\zeta}^\beta dS(\zeta) = \delta_{\alpha, \beta} \frac{2\pi^n \prod_{i=1}^n \Gamma(\frac{\alpha_i+1}{m_i})}{(\prod_{i=1}^n m_i) \Gamma(\sum_{i=1}^n \frac{\alpha_i+1}{m_i})},$$

where  $dS$  denotes the hypersurface measure on  $\mathbb{S}_m^n$ . For more details, we refer the reader to [15]. Then a direct calculation gives the following simple lemma.

**Lemma 2.2** *Let  $l \in \mathbb{R}_+$  and  $p, q \in \mathbb{N}^n$ . Then on  $A^2(\Omega_m^n)$ , for each  $\beta \in \mathbb{N}^n$ , we have*

$$T_{r^l \zeta^p \bar{\zeta}^q}(z^\beta) = \begin{cases} \frac{\Gamma(|\beta+1|+|\bar{p}|-|\bar{q}|+1) \prod_{i=1}^n \Gamma(\frac{\beta_i+p_i+1}{m_i})}{(|\beta+1|+\frac{l}{2}+\frac{|\bar{p}|-|\bar{q}|}{2}) \Gamma(|\beta+1|+|\bar{p}|) \prod_{i=1}^n \Gamma(\frac{\beta_i+p_i-q_i+1}{m_i})} z^{\beta+p-q}, & \beta + p \geq q, \\ 0, & \beta + p \not\geq q. \end{cases}$$

**Proof** Fix any  $\lambda \in \mathbb{N}^n$ , then by (2.3) and (2.4) we have

$$\begin{aligned} & \langle T_{r^l \zeta^p \bar{\zeta}^q}(z^\beta), z^\lambda \rangle \\ &= \int_{\Omega_m^n} r^l \zeta^p \bar{\zeta}^q z^\beta \bar{z}^\lambda dV(z) \\ &= \int_0^1 r^{2(\sum_{i=1}^n \frac{1}{m_i})+(\sum_{i=1}^n \frac{\beta_i+\lambda_i}{m_i})+l-1} dr \int_{\mathbb{S}_m^n} \zeta^{\beta+p} \bar{\zeta}^{-\lambda+q} dS(\zeta) \\ &= \begin{cases} \frac{\pi^n \delta_{\beta+p-q, \lambda} \prod_{i=1}^n \Gamma(\frac{\beta_i+p_i+1}{m_i})}{(\prod_{i=1}^n m_i) (\sum_{i=1}^n \frac{1}{m_i} (\beta_i+1+\frac{\beta_i-p_i-q_i}{2})+\frac{l}{2}) \Gamma(\sum_{i=1}^n \frac{\beta_i+p_i+1}{m_i})}, & \beta + p \geq q, \\ 0, & \beta + p \not\geq q. \end{cases} \end{aligned}$$

Since

$$\left\{ \sqrt{\left( \prod_{i=1}^n m_i \right) \Gamma\left( \sum_{i=1}^n \frac{\alpha_i+1}{m_i} + 1 \right)} / \pi^n \prod_{i=1}^n \Gamma\left( \frac{\alpha_i+1}{m_i} \right) z^\alpha \right\}_{\alpha \in \mathbb{N}^n}$$

is an orthonormal basis for  $A^2(\Omega_m^n)$  (see [4] for example), we have that if  $\beta + p \not\geq q$ , then  $T_{r^l \zeta^p \bar{\zeta}^q}(z^\beta) = 0$ , and if  $\beta + p \geq q$ , then

$$\begin{aligned} & T_{r^l \zeta^p \bar{\zeta}^q}(z^\beta) \\ &= \frac{\langle T_{r^l \zeta^p \bar{\zeta}^q}(z^\beta), z^{\beta+p-q} \rangle}{\langle z^{\beta+p-q}, z^{\beta+p-q} \rangle} z^{\beta+p-q} \\ &= \frac{\Gamma(\sum_{i=1}^n \frac{\beta_i+1+p_i-q_i}{m_i} + 1) \prod_{i=1}^n \Gamma(\frac{\beta_i+p_i+1}{m_i})}{(\sum_{i=1}^n \frac{1}{m_i} (\beta_i+1+\frac{\beta_i-p_i-q_i}{2})+\frac{l}{2}) \Gamma(\sum_{i=1}^n \frac{\beta_i+p_i+1}{m_i}) \prod_{i=1}^n \Gamma(\frac{\beta_i+p_i-q_i+1}{m_i})} z^{\beta+p-q}. \end{aligned}$$

This completes the proof. ■

We are now ready to prove Theorem 1.2, which characterizes all finite rank commutators for monomial-type Toeplitz operators on  $A^2(\Omega_m^n)$ .

**Proof of Theorem 1.2** It is trivial that (ii) implies (i).

To prove that (i) implies (iii), we simply write

$$\mu = |\widehat{p}| - |\widehat{q}|, \quad \nu = |\widehat{s}| - |\widehat{t}|, \quad a = \frac{l}{2} + \frac{|\widehat{p}|}{2} - \frac{|\widehat{q}|}{2}, \quad b = \frac{k}{2} + \frac{|\widehat{s}|}{2} - \frac{|\widehat{t}|}{2},$$

and define

$$(2.5) \quad H_{p,q,a}(\xi) = \frac{\Gamma(\sum_{i=1}^n \frac{\xi_i+1}{m_i} + \mu + 1) \prod_{i=1}^n \Gamma(\frac{\xi_i+p_i+1}{m_i})}{(\sum_{i=1}^n \frac{\xi_i+1}{m_i} + a) \Gamma(\sum_{i=1}^n \frac{\xi_i+1}{m_i} + |\widehat{p}|) \prod_{i=1}^n \Gamma(\frac{\xi_i+p_i-q_i+1}{m_i})}.$$

Using the same argument as in the proof of [8, Lemma 3], we can easily get that  $H_{p,q,a}(\xi)$  is holomorphic and polynomially bounded on the domain

$$\{\xi \in \mathbb{C}^n : \text{Re } \xi_i \geq \max\{0, q_i - p_i\}, 1 \leq i \leq n\}.$$

Then for each  $\beta \in \mathbb{N}^n$  with  $\beta \geq \gamma$ , where

$$\gamma_i = \max\{0, -p_i + q_i, -s_i + t_i, -p_i + q_i - s_i + t_i\}$$

for each  $i \in \{1, \dots, n\}$ , it follows from Lemma 2.2 that

$$[T_{r^l \zeta^p \bar{\zeta}^q}, T_{r^k \zeta^s \bar{\zeta}^t}](z^\beta) = [H_{s,t,b}(\beta)H_{p,q,a}(\beta + s - t) - H_{p,q,a}(\beta)H_{s,t,b}(\beta + p - q)] z^{\beta+p-q+s-t}.$$

Assume  $[T_{r^l \zeta^p \bar{\zeta}^q}, T_{r^k \zeta^s \bar{\zeta}^t}]$  has finite rank, so the set  $\{[T_{r^l \zeta^p \bar{\zeta}^q}, T_{r^k \zeta^s \bar{\zeta}^t}](z^\beta) : \beta \geq \gamma\}$  contains only finite linearly independent vectors. Thus, there exists some  $\gamma_0 \in \mathbb{N}^n$  such that

$$H_{s,t,b}(\beta)H_{p,q,a}(\beta + s - t) - H_{p,q,a}(\beta)H_{s,t,b}(\beta + p - q) = 0$$

for all  $\beta \geq \gamma_0$ . Clearly, the function

$$H_{s,t,b}(\xi)H_{p,q,a}(\xi + s - t) - H_{p,q,a}(\xi)H_{s,t,b}(\xi + p - q)$$

is holomorphic and polynomially bounded on  $\{\xi \in \mathbb{C}^n : \text{Re } \xi_i \geq \gamma_i, 1 \leq i \leq n\}$ . Then according to [11, Proposition 3.2], we obtain

$$(2.6) \quad H_{s,t,b}(\xi)H_{p,q,a}(\xi + s - t) - H_{p,q,a}(\xi)H_{s,t,b}(\xi + p - q) = 0$$

for all  $\xi \in \mathbb{C}^n$  with  $\text{Re } \xi_i \geq \gamma_i, 1 \leq i \leq n$ .

To simplify our computations later, we define

$$F_i(\eta_i) = \frac{\Gamma(\eta_i + \frac{p_i}{m_i})\Gamma(\eta_i + \frac{s_i}{m_i} - \frac{t_i}{m_i})\Gamma(\eta_i + \frac{p_i}{m_i} - \frac{q_i}{m_i} + \frac{s_i}{m_i})}{\Gamma(\eta_i + \frac{s_i}{m_i})\Gamma(\eta_i + \frac{p_i}{m_i} - \frac{q_i}{m_i})\Gamma(\eta_i + \frac{s_i}{m_i} - \frac{t_i}{m_i} + \frac{p_i}{m_i})}$$

on  $\{\eta_i \in \mathbb{C} : \text{Re } \eta_i \geq (\gamma_i + 1)/m_i\}$  for each  $i \in \{1, \dots, n\}$ , and

$$F(\eta) = \frac{(\eta + a)(\eta + b + \mu)}{(\eta + b)(\eta + a + \nu)} \frac{\Gamma(\eta + |\widehat{p}|)\Gamma(\eta + \nu + 1)\Gamma(\eta + \mu + |\widehat{s}|)}{\Gamma(\eta + |\widehat{s}|)\Gamma(\eta + \mu + 1)\Gamma(\eta + \nu + |\widehat{p}|)}$$

on  $\{\eta \in \mathbb{C} : \text{Re } \eta \geq |\widehat{\gamma} + 1|\}$ . Notice that if we replace  $\frac{\xi_i+1}{m_i}$  by  $\eta_i$  in (2.5), then the identity (2.6) becomes

$$(2.7) \quad F(\eta_1 + \dots + \eta_n) = \prod_{i=1}^n F_i(\eta_i), \quad \text{Re } \eta_i \geq (\gamma_i + 1)/m_i.$$

Then by Proposition 2.1, we see that each function  $F_i, 1 \leq i \leq n$ , should be a constant multiple of the exponential function (maybe degenerate to a constant). On the other hand, from the asymptotic expression of the logarithm of gamma function at infinity, it is known that

$$\frac{\Gamma(\eta_i + x)}{\Gamma(\eta_i + y)} = \eta_i^{x-y} \left( 1 + O\left(\frac{1}{|\eta_i|}\right) \right)$$

for large values of  $|\eta_i|$  with  $\text{Re } \eta_i > 0$  and  $x, y \geq 0$ . It is clear that each function  $F_i$  is bounded on  $\{\eta_i \in \mathbb{C} : \text{Re } \eta_i \geq (\gamma_i + 1)/m_i\}$ . Consequently, we infer that each function  $F_i$  is a constant function. Combining this with the definition of  $F_i$ , we conclude that  $F_i(\eta_i) = 1$  for each  $i \in \{1, \dots, n\}$ .

Observe that  $\frac{F_i(\eta_i+1)}{F_i(\eta_i)} = 1$ . Using the formula  $\Gamma(\eta_i + 1) = \eta_i \Gamma(\eta_i)$ , we deduce that

$$\frac{(\eta_i + \frac{p_i}{m_i})(\eta_i + \frac{s_i}{m_i} - \frac{t_i}{m_i})(\eta_i + \frac{p_i}{m_i} - \frac{q_i}{m_i} + \frac{s_i}{m_i})}{(\eta_i + \frac{s_i}{m_i})(\eta_i + \frac{p_i}{m_i} - \frac{q_i}{m_i})(\eta_i + \frac{s_i}{m_i} - \frac{t_i}{m_i} + \frac{p_i}{m_i})} = 1.$$

This clearly implies that the tuple  $(p_i, q_i, s_i, t_i)$  satisfies Condition (I). Since  $F_i(\eta_i)$  is the constant function 1, it follows from (2.7) that  $F(\eta) = 1$ , and so the identity (1.1) holds on  $\{\eta \in \mathbb{C} : \text{Re } \eta \geq \lceil \gamma + 1 \rceil\}$ . This completes the proof that (i) implies (iii).

It remains to prove that (iii) implies (ii). In fact, if condition (iii) holds, then  $F_i(\eta_i) = 1$  and  $F(\eta) = 1$ , and so (2.7) holds for  $\eta_i$  with  $\text{Re } \eta_i \geq (\gamma_i + 1)/m_i, 1 \leq i \leq n$ . Thus,

$$[T_{r^l \zeta^p \bar{\zeta}^q}, T_{r^k \zeta^s \bar{\zeta}^t}](z^\beta) = 0$$

for each  $\beta \in \mathbb{N}^n$  with  $\beta \geq \gamma$ . For  $\beta \in \mathbb{N}^n$  with  $\beta \neq \gamma, \gamma_i > \beta_i \geq 0$  for some  $i \in \{1, 2, \dots, n\}$ . Using [8, Lemma 1] we have  $\gamma_i = -p_i + q_i - s_i + t_i > \beta_i$ , and hence Lemma 2.2 implies

$$[T_{r^l \zeta^p \bar{\zeta}^q}, T_{r^k \zeta^s \bar{\zeta}^t}](z^\beta) = T_{r^l \zeta^p \bar{\zeta}^q} T_{r^k \zeta^s \bar{\zeta}^t}(z^\beta) - T_{r^k \zeta^s \bar{\zeta}^t} T_{r^l \zeta^p \bar{\zeta}^q}(z^\beta) = 0.$$

Consequently,  $[T_{r^l \zeta^p \bar{\zeta}^q}, T_{r^k \zeta^s \bar{\zeta}^t}] = 0$ , as desired. This completes the proof. ■

In the rest of this section, we will prove Theorem 1.3, which completely characterizes all finite rank semi-commutators of two monomial-type Toeplitz operators on  $A^2(\Omega_m^n)$ . First, we need the following critical lemma.

**Lemma 2.3** *Let  $p, q, s, t \in \mathbb{N}^n$ . If there exist some real numbers  $a$  and  $b$  such that*

$$(2.8) \quad \frac{(\eta + a + b)}{(\eta + b)(\eta + a + |\widehat{s}| - \widehat{t})} = \frac{\Gamma(\eta + |\widehat{s}|)\Gamma(\eta + |\widehat{s}| - \widehat{t} + \widehat{p})}{\Gamma(\eta + |\widehat{s}| - \widehat{t} + 1)\Gamma(\eta + \widehat{p} + |\widehat{s}|)}$$

for any  $\eta$  over the right half-plane, then either  $\widehat{t} = 0$  or  $\widehat{p} = 0$ .

**Proof** Notice that the function on the right-hand side above is a rational function, so it has at most finitely many poles. Since the function  $\Gamma(\eta + |\widehat{s}|)$  has infinitely many poles, all but finitely many of them must cancel. Therefore, either  $\widehat{t}$  or  $\widehat{p}$  is a non-negative integer.

First, we assume that  $\widehat{t}$  is a positive integer. We need to show that  $\widehat{p} = 0$ . If  $\widehat{t} = 1$  or  $\widehat{t} = 2$ , then an elementary argument shows that either  $\widehat{p} = a = 0$  or



$|\widehat{p}| = b - |\widehat{s}| + |\widehat{t}| = 0$ , as desired. So we consider  $|\widehat{t}| \geq 3$ . Then (2.8) becomes

$$(2.9) \quad \frac{(\eta + a + b)}{(\eta + b)(\eta + a + |\widehat{s}| - |\widehat{t}|)} = \frac{(\eta + |\widehat{s}| - |\widehat{t}| + 1)(\eta + |\widehat{s}| - |\widehat{t}| + 2) \cdots (\eta + |\widehat{s}| - 1)}{(\eta + |\widehat{p}| + |\widehat{s}| - |\widehat{t}|) \cdots (\eta + |\widehat{p}| + |\widehat{s}| - 2)(\eta + |\widehat{p}| + |\widehat{s}| - 1)}.$$

Since  $|\widehat{t}| \neq 0$ , it is clear that  $|\widehat{p}| \neq 1$ . To see that  $|\widehat{p}| = 0$ , let us assume that  $|\widehat{p}| > 0$  and  $|\widehat{p}| \neq 1$ . Observe that  $|\widehat{s}| - |\widehat{t}| + 1 \neq |\widehat{p}| + |\widehat{s}| - |\widehat{t}| + i$  for any  $i \in \{0, \dots, |\widehat{t}| - 1\}$  and  $|\widehat{p}| + |\widehat{s}| - 1 > |\widehat{s}| - 1$ . It follows from (2.9) that

$$|\widehat{s}| - |\widehat{t}| + 1 = a + b$$

and

$$|\widehat{p}| + |\widehat{s}| - 1 = b \quad \text{or} \quad |\widehat{p}| + |\widehat{s}| - 1 = a + |\widehat{s}| - |\widehat{t}|.$$

Consequently, one of the following conditions holds:

- $a = -|\widehat{p}| - |\widehat{t}| + 2$  and  $b = |\widehat{p}| + |\widehat{s}| - 1$ .
- $a = |\widehat{p}| + |\widehat{t}| - 1$  and  $b = -|\widehat{p}| + |\widehat{s}| - 2|\widehat{t}| + 2$ .

In each case, (2.9) becomes

$$\frac{1}{(\eta - |\widehat{p}| + |\widehat{s}| - 2|\widehat{t}| + 2)} = \frac{(\eta + |\widehat{s}| - |\widehat{t}| + 2) \cdots (\eta + |\widehat{s}| - 1)}{(\eta + |\widehat{p}| + |\widehat{s}| - |\widehat{t}|)(\eta + |\widehat{p}| + |\widehat{s}| - |\widehat{t}| + 1) \cdots (\eta + |\widehat{p}| + |\widehat{s}| - 2)}.$$

However,

$$-|\widehat{p}| + |\widehat{s}| - 2|\widehat{t}| + 2 \neq |\widehat{p}| + |\widehat{s}| - |\widehat{t}| + j$$

for all  $j \in \{0, \dots, |\widehat{t}| - 2\}$ , since  $|\widehat{t}| \geq 3$  and  $|\widehat{p}| > 0$ , a contradiction. This shows that  $|\widehat{p}| = 0$ .

Next, we suppose that  $|\widehat{p}|$  is a positive integer. Then, using the same method as before, we get that  $|\widehat{t}| = 0$ . This completes the proof. ■

**Proof of Theorem 1.3** It is trivial that (ii) implies (i).

To prove that (i) implies (iii), we consider each  $\beta \in \mathbb{N}^n$  with  $\beta \geq \delta$ , where

$$\delta_i = \max\{0, -s_i + t_i, -p_i + q_i - s_i + t_i\}.$$

Then we deduce from Lemma 2.2 and the notation of (2.5) that

$$(T_{r^l \zeta^p \bar{\zeta}^q}, T_{r^k \zeta^s \bar{\zeta}^t})(z^\beta) = [H_{s,t,b}(\beta)H_{p,q,a}(\beta + s - t) - H_{p+s,q+t,a+b}(\beta)](z^{\beta+p-q+s-t}).$$

Since  $(T_{r^l \zeta^p \bar{\zeta}^q}, T_{r^k \zeta^s \bar{\zeta}^t})$  has finite rank, we can proceed as in the proof of Theorem 1.2 to obtain

$$(2.10) \quad H_{s,t,b}(\xi)H_{p,q,a}(\xi + s - t) - H_{p+s,q+t,a+b}(\xi) = 0$$

for all  $\xi \in \mathbb{C}^n$  with  $\text{Re } \xi_i \geq \delta_i, 1 \leq i \leq n$ . If we define

$$G_i(\eta_i) = \frac{\Gamma(\eta_i + \frac{s_i}{m_i} - \frac{t_i}{m_i})\Gamma(\eta_i + \frac{p_i}{m_i} + \frac{s_i}{m_i})}{\Gamma(\eta_i + \frac{s_i}{m_i})\Gamma(\eta_i + \frac{s_i}{m_i} - \frac{t_i}{m_i} + \frac{p_i}{m_i})}$$

on  $\{\eta_i \in \mathbb{C} : \operatorname{Re} \eta_i \geq (\delta_i + 1)/m_i\}$  for each  $i \in \{1, \dots, n\}$  and

$$G(\eta) = \frac{(\eta + a + b)\Gamma(\eta + \nu + 1)\Gamma(\eta + |\widehat{p}| + |\widehat{s}|)}{(\eta + b)(\eta + a + \nu)\Gamma(\eta + |\widehat{s}|)\Gamma(\eta + \nu + |\widehat{p}|)}$$

on  $\{\eta \in \mathbb{C} : \operatorname{Re} \eta \geq |\widehat{\delta} + 1|\}$ , then (2.10) is equivalent to

$$G(\eta_1 + \dots + \eta_n) = \prod_{i=1}^n G(\eta_i)$$

for all  $\eta_i \in \mathbb{C}$  with  $\operatorname{Re} \eta_i \geq (\delta_i + 1)/m_i, 1 \leq i \leq n$ . Using the same argument as in the proof of Theorem 1.2, we have that  $G(\eta) = 1$ . Then by Lemma 2.3, we get either  $|\widehat{t}| = 0$  or  $|\widehat{p}| = 0$ . If  $|\widehat{t}| = 0$ , then it follows from  $G(\eta) = 1$  that

$$(\eta + a + b)(\eta + |\widehat{s}|) = (\eta + b)(\eta + a + |\widehat{s}|),$$

which implies that either  $a = 0$  or  $b = |\widehat{s}|$ . Similarly, if  $|\widehat{p}| = 0$ , then we have either  $a = 0$  or  $b = \nu$ . This completes the proof that (i) implies (iii).

To prove that (iii) implies (ii), we first observe that condition (iii) implies that either  $p = 0$  or  $t = 0$ , and consequently,  $G_i(\eta_i) = 1$  on  $\{\eta_i \in \mathbb{C} : \operatorname{Re} \eta_i \geq (\delta_i + 1)/m_i\}$  for each  $i \in \{1, \dots, n\}$ . It is also easy to check that  $G(\eta) = 1$ , and so (2.10) holds for all  $\zeta \in \mathbb{C}^n$  with  $\operatorname{Re} \zeta_i \geq \delta_i, 1 \leq i \leq n$ . Thus,

$$(T_{r^l \zeta^p \bar{\zeta}^q}, T_{r^k \zeta^s \bar{\zeta}^t}](z^\beta) = 0$$

for each  $\beta \in \mathbb{N}^n$  with  $\beta \geq \delta$ . If  $\beta \in \mathbb{N}^n$  with  $\beta \neq \delta$ , then  $\delta_{i_0} > \beta_{i_0} \geq 0$  for some  $i_0 \in \{1, 2, \dots, n\}$ . As in the proof of Theorem 1.2, condition (ii) will follow if we can show that  $\delta_{i_0} = -p_{i_0} + q_{i_0} - s_{i_0} + t_{i_0}$ . To this end, first observe that either  $p_{i_0} = 0$  or  $t_{i_0} = 0$ . If  $p_{i_0} = 0$ , then it is obvious that

$$-s_{i_0} + t_{i_0} \leq -p_{i_0} + q_{i_0} - s_{i_0} + t_{i_0}.$$

Since  $\delta_{i_0} > 0$ , the desired result then follows from the definition of  $\delta_{i_0}$ . If  $t_{i_0} = 0$ , then  $\delta_{i_0} = \max\{0, -s_{i_0}, -p_{i_0} + q_{i_0} - s_{i_0}\}$ . Since  $\delta_{i_0} > 0$ , the desired result is then obvious. This completes the proof. ■

### 3 Corollaries and Examples

In this section we observe some interesting consequences of Theorem 1.2. More specifically, we will give some sufficient conditions and necessary conditions for two monomial-type Toeplitz operators  $T_{r^l \zeta^p \bar{\zeta}^q}$  and  $T_{r^k \zeta^s \bar{\zeta}^t}$  commuting on  $A^2(\Omega_m^n)$ . For the convenience of our proofs, we rewrite (1.1) as

$$(3.1) \quad \frac{\Gamma(\eta + |\widehat{p}|)\Gamma(\eta + |\widehat{s}| - |\widehat{t}| + 1)\Gamma(\eta + |\widehat{p}| - |\widehat{q}| + |\widehat{s}|)}{\Gamma(\eta + |\widehat{s}|)\Gamma(\eta + |\widehat{p}| - |\widehat{q}| + 1)\Gamma(\eta + |\widehat{s}| - |\widehat{t}| + |\widehat{p}|)} = \frac{(\eta + \frac{k}{2} + \frac{|\widehat{s}|}{2} - \frac{|\widehat{t}|}{2})(\eta + \frac{l}{2} + |\widehat{s}| - |\widehat{t}| + \frac{|\widehat{p}|}{2} - \frac{|\widehat{q}|}{2})}{(\eta + \frac{l}{2} + \frac{|\widehat{p}|}{2} - \frac{|\widehat{q}|}{2})(\eta + \frac{k}{2} + |\widehat{p}| - |\widehat{q}| + \frac{|\widehat{s}|}{2} - \frac{|\widehat{t}|}{2})}.$$

As a direct consequence of Theorem 1.2, we first present five cases for two monomial-type Toeplitz operators commuting on  $A^2(\Omega_m^n)$ , which correspond to [8, Theorem A].

**Corollary 3.1** *If one of the following conditions holds, then the operators  $T_{r^l \zeta^p \bar{\zeta}^q}$  and  $T_{r^k \zeta^s \bar{\zeta}^t}$  commute on  $A^2(\Omega_m^n)$ :*

- (c1)  $l = |\widehat{p}| = |\widehat{q}| = 0$  or  $k = |\widehat{s}| = |\widehat{t}| = 0$ , which means that one of the two operators is the identity operator.
- (c2)  $|\widehat{q}| = |\widehat{t}| = 0$ ,  $l = |\widehat{p}|$  and  $k = |\widehat{t}|$ , which means that both operators have analytic symbols.
- (c3)  $|\widehat{p}| = |\widehat{s}| = 0$ ,  $l = |\widehat{q}|$  and  $k = |\widehat{s}|$ , which means that both operators have conjugate analytic symbols.
- (c4)  $(p_i, q_i, s_i, t_i)$  satisfies Condition (I) for each  $i \in \{1, 2, \dots, n\}$ ,  $|\widehat{p}| = |\widehat{q}|$  and  $|\widehat{s}| = |\widehat{t}|$ .
- (c5)  $(p_i, q_i, s_i, t_i)$  satisfies Condition (I) for each  $i \in \{1, 2, \dots, n\}$ ,  $|\widehat{p}| = |\widehat{s}|$ ,  $|\widehat{q}| = |\widehat{t}|$ , and  $l = k$ .

It is also worth to mention that except for the above predictable sufficient conditions for two monomial-type Toeplitz operators commuting on  $A^2(\Omega_m^n)$ , there are exactly many other cases. Next, we present some non-trivial examples of commuting monomial-type Toeplitz operators on  $A^2(\Omega_m^n)$ .

**Example 3.2** Let  $(p_i, q_i, s_i, t_i)$  satisfy Condition (I) for each  $i \in \{1, 2, \dots, n\}$ , and let

$$\begin{cases} |\widehat{p}| = 2, \\ |\widehat{q}| = 1, \\ |\widehat{s}| = 2|\widehat{t}|, \\ l = 2|\widehat{t}| - 1, \\ k = |\widehat{t}|, \end{cases} \quad \text{or} \quad \begin{cases} |\widehat{p}| = 2, \\ |\widehat{q}| = 1, \\ |\widehat{s}| = 2|\widehat{t}|, \\ l = 2|\widehat{t}| + 1, \\ k = 3|\widehat{t}|, \end{cases} \quad \text{or} \quad \begin{cases} |\widehat{q}| = 2, \\ |\widehat{t}| = 1, \\ |\widehat{p}| = 2|\widehat{s}|, \\ l = 4|\widehat{s}| - 2, \\ k = 3|\widehat{s}| - 1. \end{cases}$$

Then it is easy to check that  $T_{r^l \zeta^p \bar{\zeta}^q}$  commutes with  $T_{r^k \zeta^s \bar{\zeta}^t}$  on  $A^2(\Omega_m^n)$  in each of the cases above. More specifically, if we define  $p, q, s, t \in \mathbb{N}^3$  by

$$p = (m_1, m_2, 0), \quad q = (m_1, 0, 0), \quad s = (2m_1, 2m_2, 4m_3), \quad t = (2m_1, 0, 2m_3),$$

then  $(|\widehat{p}|, |\widehat{q}|, |\widehat{s}|, |\widehat{t}|) = (2, 1, 8, 4)$ , which does not satisfy Corollary 3.1. Obviously,  $T_{r^7 \zeta^p \bar{\zeta}^q}$  commutes with  $T_{r^4 \zeta^s \bar{\zeta}^t}$ , and  $T_{r^9 \zeta^p \bar{\zeta}^q}$  commutes with  $T_{r^{12} \zeta^s \bar{\zeta}^t}$  on  $A^2(\Omega_m^3)$ .

If the weakly pseudoconvex domain is not the unit ball, we can construct the following more interesting example.

**Example 3.3** Suppose  $m = (4, \dots, 4) \in \mathbb{N}^6$  and

$$\begin{aligned} p &= (0, 2, 0, 1, 1, 4), & q &= (0, 1, 1, 0, 1, 1), \\ s &= (2, 0, 0, 8, 2, 4), & t &= (3, 0, 2, 0, 2, 1). \end{aligned}$$

Then  $(|\widehat{p}|, |\widehat{q}|, |\widehat{s}|, |\widehat{t}|) = (2, 1, 4, 2)$ , and the tuple  $(p_i, q_i, s_i, t_i)$  satisfies Condition (I) for all  $i \in \{1, \dots, 6\}$ . As a consequence of Example 3.2, we see that the operators  $T_{z^p \bar{z}^q}$  and  $T_{z^s \bar{z}^t}$  commute on  $A^2(\Omega_m^6)$ .

Next, we give some interesting necessary conditions for two monomial-type Toeplitz operators commuting on  $A^2(\Omega_m^n)$ .

**Corollary 3.4** Let  $l, k \in \mathbb{R}_+$ ,  $p, q, s, t \in \mathbb{N}^n$ . If the operators  $T_{r^l \zeta^p \bar{\zeta}^q}$  and  $T_{r^k \zeta^s \bar{\zeta}^t}$  commute on  $A^2(\Omega_m^n)$ , then the following statements hold:

- (i) At least one of the tuples  $(|\widehat{p}|, |\widehat{q}|)$ ,  $(|\widehat{s}|, |\widehat{t}|)$ ,  $(|\widehat{p}|, |\widehat{s}|)$ ,  $(|\widehat{q}|, |\widehat{t}|)$ ,  $(|\widehat{p}| - |\widehat{q}|, |\widehat{s}| - |\widehat{t}|)$ , and  $(|\widehat{p}| - |\widehat{s}|, |\widehat{q}| - |\widehat{t}|)$  belongs to  $\mathbb{Z}^2$ .
- (ii) The tuple  $(l, k, |\widehat{p}|, |\widehat{q}|, |\widehat{s}|, |\widehat{t}|)$  must satisfy

$$(3.2) \quad \frac{(\eta + |\widehat{p}|)(\eta + |\widehat{s}| - |\widehat{t}| + 1)(\eta + |\widehat{p}| - |\widehat{q}| + |\widehat{s}|)}{(\eta + |\widehat{s}|)(\eta + |\widehat{p}| - |\widehat{q}| + 1)(\eta + |\widehat{s}| - |\widehat{t}| + |\widehat{p}|)} = \frac{(\eta + b + 1)(\eta + a + |\widehat{s}| - |\widehat{t}| + 1)(\eta + a)(\eta + b + |\widehat{p}| - |\widehat{q}|)}{(\eta + b)(\eta + a + |\widehat{s}| - |\widehat{t}|)(\eta + a + 1)(\eta + b + |\widehat{p}| - |\widehat{q}| + 1)},$$

where  $a = (l + |\widehat{p}| - |\widehat{q}|)/2$  and  $b = (k + |\widehat{s}| - |\widehat{t}|)/2$ .

**Proof** To prove part (i), let us assume the contrary. Then the analytic function on the left-hand side of (3.1) has no zero on the complex plane. However, if the right-hand side of (3.1) has no zero, then it follows that

$$\left(\eta + \frac{k}{2} + \frac{|\widehat{s}|}{2} - \frac{|\widehat{t}|}{2}\right) \left(\eta + \frac{l}{2} + |\widehat{s}| - |\widehat{t}| + \frac{|\widehat{p}|}{2} - \frac{|\widehat{q}|}{2}\right) = \left(\eta + \frac{l}{2} + \frac{|\widehat{p}|}{2} - \frac{|\widehat{q}|}{2}\right) \left(\eta + \frac{k}{2} + |\widehat{p}| - |\widehat{q}| + \frac{|\widehat{s}|}{2} - \frac{|\widehat{t}|}{2}\right).$$

From the assumption  $(|\widehat{p}| - |\widehat{q}|, |\widehat{s}| - |\widehat{t}|) \notin \mathbb{Z}^2$ , we then deduce that  $k = l$  and

$$(3.3) \quad |\widehat{p}| - |\widehat{q}| = |\widehat{s}| - |\widehat{t}|$$

Then identity (3.1) becomes

$$\Gamma(\eta + |\widehat{p}|)\Gamma(\eta + |\widehat{p}| - |\widehat{q}| + |\widehat{s}|) = \Gamma(\eta + |\widehat{s}|)\Gamma(\eta + |\widehat{s}| - |\widehat{t}| + |\widehat{p}|).$$

This together with (3.3) shows that either  $|\widehat{p}| - |\widehat{q}| = |\widehat{s}| - |\widehat{t}| = 0$  or  $|\widehat{p}| - |\widehat{s}| = |\widehat{q}| - |\widehat{t}| = 0$ , which contradicts the assumption and completes the proof of condition (i).

To prove part (ii), we can replace  $\eta$  by  $\eta + 1$  in (3.1) and apply the formula  $\Gamma(\eta + 1) = \eta\Gamma(\eta)$  to obtain

$$\frac{\Gamma(\eta + |\widehat{p}|)\Gamma(\eta + |\widehat{s}| - |\widehat{t}| + 1)\Gamma(\eta + |\widehat{p}| - |\widehat{q}| + |\widehat{s}|)}{\Gamma(\eta + |\widehat{s}|)\Gamma(\eta + |\widehat{p}| - |\widehat{q}| + 1)\Gamma(\eta + |\widehat{s}| - |\widehat{t}| + |\widehat{p}|)} = \frac{(\eta + b + 1)(\eta + a + |\widehat{s}| - |\widehat{t}| + 1)(\eta + |\widehat{s}|)(\eta + |\widehat{p}| - |\widehat{q}| + 1)(\eta + |\widehat{s}| - |\widehat{t}| + |\widehat{p}|)}{(\eta + a + 1)(\eta + b + |\widehat{p}| - |\widehat{q}| + 1)(\eta + |\widehat{p}|)(\eta + |\widehat{s}| - |\widehat{t}| + 1)(\eta + |\widehat{p}| - |\widehat{q}| + |\widehat{s}|)}.$$

The desired result then follows from (3.1) and the above identity. ■

Of course, one can expect that there should exist some differences in operator theory on the Bergman spaces between on the weakly pseudoconvex domain  $\Omega_m^n$  and on the unit ball  $\mathbb{B}^n$ . Note that  $|\widehat{p}|, |\widehat{q}|, |\widehat{s}|$  and  $|\widehat{t}|$  will always satisfy Corollary 3.4(i) for the case  $m = (1, \dots, 1)$ . In many cases, however, this is no longer true.

It is also interesting to note that the collection of tuples  $(l, k, |\widehat{p}|, |\widehat{q}|, |\widehat{s}|, |\widehat{t}|)$  satisfying (3.2) is finite. Consequently, there exist a finite number of the combination of the tuple  $(l, k, |\widehat{p}|, |\widehat{q}|, |\widehat{s}|, |\widehat{t}|)$  such that the operators  $T_{r^l \zeta^p \bar{\zeta}^q}$  and  $T_{r^k \zeta^s \bar{\zeta}^t}$  commute on  $A^2(\Omega_m^n)$ . Moreover, with the help of (3.2), we can easily obtain the next two

corollaries, which give the specific sufficient and necessary condition for some special monomial-type Toeplitz operators to be commutative on  $A^2(\Omega_m^n)$ . The first corollary characterizes commuting monomial Toeplitz operators on the Bergman space of the weakly pseudoconvex domains, which is the same as the case on the unit ball.

**Corollary 3.5** *Let  $p, q, s, t \in \mathbb{N}^n$ . Then the operators  $T_{z^p \bar{z}^q}$  and  $T_{z^s \bar{z}^t}$  commute on  $A^2(\Omega_m^n)$  if and only if  $(|\widehat{p}|, |\widehat{q}|, |\widehat{s}|, |\widehat{t}|)$  and  $(p_i, q_i, s_i, t_i)$  satisfy Condition (I) for all  $i \in \{1, 2, \dots, n\}$ .*

**Proof** Denote  $l = |\widehat{p}| + |\widehat{q}|$  and  $k = |\widehat{s}| + |\widehat{t}|$ . Then identity (3.2) becomes

$$(\eta + |\widehat{s}| - |\widehat{t}| + 1)(\eta + |\widehat{p}| + 1)(\eta + |\widehat{s}| + |\widehat{p}| - |\widehat{q}| + 1) = (\eta + |\widehat{p}| - |\widehat{q}| + 1)(\eta + |\widehat{s}| + 1)(\eta + |\widehat{p}| + |\widehat{s}| - |\widehat{t}| + 1),$$

which implies that  $(|\widehat{p}|, |\widehat{q}|, |\widehat{s}|, |\widehat{t}|)$  satisfies Condition (I). The desired result then follows from Theorem 1.2. ■

Our next corollary shows that a Toeplitz operator with a holomorphic monomial symbol can only commute with another monomial-type Toeplitz operator with a holomorphic symbol.

**Corollary 3.6** *Let  $k \in \mathbb{R}_+$ ,  $p, s, t \in \mathbb{N}^n$  with  $p \neq 0$ . Then the operators  $T_{z^p}$  and  $T_{r^k \zeta^s \bar{\zeta}^t}$  commute on  $A^2(\Omega_m^n)$  if and only if the operator  $T_{r^k \zeta^s \bar{\zeta}^t}$  also has holomorphic symbol.*

**Proof** First assume that  $T_{z^p}$  and  $T_{r^k \zeta^s \bar{\zeta}^t}$  commute on  $A^2(\Omega_m^n)$ . Then by identity (3.2), we have

$$\frac{(\eta + |\widehat{s}| - |\widehat{t}| + 1)(\eta + \frac{k}{2} + \frac{|\widehat{s}|}{2} - \frac{|\widehat{t}|}{2})}{(\eta + |\widehat{s}|)(\eta + \frac{k}{2} + \frac{|\widehat{s}|}{2} - \frac{|\widehat{t}|}{2} + 1)} = \frac{(\eta + |\widehat{p}| + |\widehat{s}| - |\widehat{t}| + 1)(\eta + \frac{k}{2} + \frac{|\widehat{s}|}{2} - \frac{|\widehat{t}|}{2} + |\widehat{p}|)}{(\eta + |\widehat{p}| + |\widehat{s}|)(\eta + \frac{k}{2} + \frac{|\widehat{s}|}{2} - \frac{|\widehat{t}|}{2} + |\widehat{p}| + 1)}.$$

Thus, the function on the left-hand side of the above identity is a bounded periodic analytic function with a period  $|\widehat{p}|$  in the right half-plane, and consequently, it must be the identity function. So we have

$$(\eta + |\widehat{s}| - |\widehat{t}| + 1)\left(\eta + \frac{k}{2} + \frac{|\widehat{s}|}{2} - \frac{|\widehat{t}|}{2}\right) = (\eta + |\widehat{s}|)\left(\eta + \frac{k}{2} + \frac{|\widehat{s}|}{2} - \frac{|\widehat{t}|}{2} + 1\right),$$

which implies that  $|\widehat{t}| = 0$  and  $k = |\widehat{s}|$ , as desired.

By Corollary 3.1(c2), the converse implication is clear. This completes the proof. ■

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