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ON p-SOLVABILITY AND AVERAGE CHARACTER DEGREE IN A FINITE GROUP

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Abstract

Assume that G is a finite group, N is a nontrivial normal subgroup of G and p is an odd prime. Let $\mathrm{Irr}_p(G) = \{\chi \in \mathrm{Irr}(G) : \chi(1) = 1 \text{ or } p \mid \chi(1)\}$ and $\mathrm{Irr}_p(G|N) = \{\chi \in \mathrm{Irr}_p(G) : N \nleq \ker \chi\}$. The average character degree of irreducible characters of $\mathrm{Irr}_p(G)$ and the average character degree of irreducible characters of $\mathrm{Irr}_p(G|N)$ are denoted by $\mathrm{acd}_p(G)$ and $\mathrm{acd}_p(G|N)$, respectively. We show that if $\mathrm{Irr}_p(G|N) \neq \emptyset$ and $\mathrm{acd}_p(G|N) < \mathrm{acd}_p(\mathrm{PSL}_2(p))$, then G is p-solvable and $O^{p'}(G)$ is solvable. We find examples that make this bound best possible. Moreover, we see that if $\mathrm{Irr}_p(G|N) = \emptyset$, then N is p-solvable and $P \cap N$ and PN/N are abelian for every $P \in \mathrm{Syl}_p(G)$.

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1. Introduction

In this paper, G is a finite group and p is a prime divisor of $|G|$. Let $\mathrm{Irr}(G)$ denote the set of (complex) irreducible characters of G. For a normal subgroup N of G and $\theta \in \mathrm{Irr}(N)$, let $\mathrm{Irr}(G|N) = \{\chi \in \mathrm{Irr}(G) : N \nleq \ker \chi\}$ and $\mathrm{Irr}(\theta^G)$ denote the set of the irreducible constituents of the induced character θ^G. The average character degree of G is denoted by $\mathrm{acd}(G)$ (see [5, 8]) and it is defined by

$$\mathrm{acd}(G) = \frac{\Sigma_{\chi \in \mathrm{Irr}(G)} \chi(1)}{|\mathrm{Irr}(G)|}.$$

By $\mathrm{acd}(G|N)$, we mean the average character degree of the irreducible characters in $\mathrm{Irr}(G|N)$ (see [3]). In [1], it has been shown that if $\mathrm{acd}(G|N) < \max(\mathrm{acd}(\mathrm{PSL}_2(p)), 16/5)$, then G is p-solvable.

We write

$$\begin{aligned} \mathrm{Irr}_p(G) &= \{\chi \in \mathrm{Irr}(G) : \chi(1) = 1 \text{ or } p \mid \chi(1)\} \\ \mathrm{Irr}_p(G|N) &= \mathrm{Irr}_p(G) \cap \mathrm{Irr}(G|N) \\ \mathrm{Irr}_p(\theta^G) &= \mathrm{Irr}_p(G) \cap \mathrm{Irr}(\theta^G) \quad \text{for every } \theta \in \mathrm{Irr}(N). \end{aligned}$$

Let $\mathrm{acd}_p(G)$, $\mathrm{acd}_p(G|N)$ and $\mathrm{acd}_p(\theta^G)$ be the average degree of irreducible characters belonging to $\mathrm{Irr}_p(G)$, $\mathrm{Irr}_p(G|N)$ and $\mathrm{Irr}_p(\theta^G)$, respectively. For $\Delta \subseteq \mathrm{Irr}(G)$,

$$\mathrm{acd}_p(\Delta) = \frac{\Sigma_{\chi\in\Delta\cap\mathrm{Irr}_p(G)}\chi(1)}{|\Delta \cap \mathrm{Irr}_p(G)|}.$$

Nguyen and Tiep [7] have shown that if either $p \geq 5$ and $\mathrm{acd}_p(G) < \mathrm{acd}_p(\mathrm{PSL}_2(p))$ or $p \in \{2, 3\}$ and $\mathrm{acd}_p(G) < \mathrm{acd}_p(\mathrm{PSL}_2(5))$, then G is p-solvable and $O^{p'}(G)$ is solvable, where $O^{p'}(G)$ is the minimal normal subgroup of G whose quotient is a p'-group. Akhlaghi [2] proved that if N is a nontrivial normal subgroup of G with $\mathrm{Irr}_2(G|N) \neq \emptyset$ and $\mathrm{acd}_2(G|N) < 5/2$, then G is solvable.

We continue this investigation and show that considering the appropriate bound for $\mathrm{acd}_p(G|N)$ instead of $\mathrm{acd}_p(G)$ leads us to the p-solvability of G.

Let $f(p) = \mathrm{acd}_p(\mathrm{PSL}_2(p))$ if $p \geq 5$ and otherwise, let $f(p) = \mathrm{acd}_p(\mathrm{PSL}_2(5))$. So,

$$f(p) = \begin{cases} (p+1)/2 & \text{if } p \geq 5, \\ 7/3 & \text{if } p = 3, \\ 5/2 & \text{if } p = 2. \end{cases}$$

THEOREM 1.1. *Let $1 \neq N \trianglelefteq G$ and p be an odd prime divisor of $|G|$. If G/N is not p-solvable, then $\mathrm{acd}_p(\lambda^G) \geq f(p)$ for every $\lambda \in \mathrm{Irr}(N)$ with $\mathrm{Irr}_p(\lambda^G) \neq \emptyset$.*

THEOREM 1.2. *Let p be an odd prime and $1 \neq N \trianglelefteq G$ with $\mathrm{acd}_p(G|N) < f(p)$. Then:*

(i) *either G is p-solvable and $O^{p'}(G)$ is solvable;*
(ii) *or $\mathrm{Irr}_p(G|N) = \emptyset$, N is p-solvable and for every $P \in \mathrm{Syl}_p(G)$, $P \cap N$ and PN/N are abelian.*

EXAMPLE 1.3. Let N be a cyclic group of order 2, p be an odd prime and let $G = \mathrm{PSL}_2(p) \times N$. If $p \geq 5$, then $\mathrm{acd}_p(G|N) = \mathrm{acd}_p(\mathrm{PSL}_2(p))$. Also, if $p = 5$, then $\mathrm{acd}_3(G|N) = \mathrm{acd}_3(\mathrm{PSL}_2(5))$. This example shows that the bound given in Theorem 1.2 is the best possible.

Let $\mathrm{Irr}_p(G^\sharp) = \mathrm{Irr}_p(G) - \{1_G\}$ and $\mathrm{acd}(G^\sharp) = \Sigma_{\chi\in\mathrm{Irr}_p(G^\sharp)}\chi(1)/|\mathrm{Irr}_p(G^\sharp)|$. By setting $G = N$ in Theorem 1.2, we arrive at the following corollary.

COROLLARY 1.4. *If $\mathrm{acd}_p(G^\sharp) < f(p)$, then G is p-solvable and $O^{p'}(G)$ is solvable.*

We can see that $\mathrm{acd}_3(\mathrm{Alt}_4^\sharp) = 5/3 < 7/3$ and the Sylow 3-subgroup of Alt_4 is not normal in Alt_4. This shows that the assumption $\mathrm{acd}_p(G^\sharp) < f(p)$ does not guarantee normality of the Sylow p-subgroup of G.

2. The main results

We first state some lemmas that will be used in the proof of Theorems 1.1 and 1.2. For a nonempty finite subset of real numbers X, by $\mathrm{ave}(X)$, we mean the average of X.

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Lemma 2.1 [1, Lemma 3]. *Let X be a nonempty finite subset of real numbers and $\{A_1, \ldots, A_t\}$ be a partition of X. If d is a real number such that* $\mathrm{ave}(A_i) \geq d$ *(respectively $< d$) for $1 \leq i \leq t$, then* $\mathrm{ave}(X) \geq d$ *(respectively $< d$).*

Lemma 2.2 [7, Theorem B]. *Let p be a prime divisor of $|G|$. If $\mathrm{acd}_p(G) < f(p)$, then G is p-solvable and $O^{p'}(G)$ is solvable.*

Lemma 2.3 [6, Theorem A]. *Let Z be a normal subgroup of a finite group G, $\lambda \in \mathrm{Irr}(Z)$ and let $P/Z \in \mathrm{Syl}_p(G/Z)$. If $\chi(1)/\lambda(1)$ is coprime to p for every $\chi \in \mathrm{Irr}(G)$ lying over λ, then P/Z is abelian.*

We are ready to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. We complete the proof by induction on $|G| + |N|$. Take $\lambda \in \mathrm{Irr}(N)$ with $\mathrm{Irr}_p(\lambda^G) \neq \emptyset$. Let E be a maximal normal subgroup of G such that $N \leq E$ and G/E is not p-solvable. Then, G/E admits the unique minimal normal subgroup M/E and it is easy to check that M/E is not p-solvable. Assume that $\{\mu_1, \ldots, \mu_t\} \subseteq \mathrm{Irr}(\lambda^E)$ such that every element of $\mathrm{Irr}(\lambda^E)$ is conjugate to exactly one of the elements in $\{\mu_1, \ldots, \mu_t\}$. If $N \neq E$, then from the hypothesis, $\mathrm{Irr}_p(\mu_i^G) = \emptyset$ or $\mathrm{acd}_p(\mu_i^G) \geq f(p)$, for $1 \leq i \leq t$. As $\mathrm{Irr}(\lambda^G) = \dot\cup_{i=1}^t \mathrm{Irr}(\mu_i^G)$ and $\mathrm{Irr}_p(\lambda^G) \neq \emptyset$, we conclude that $\mathrm{Irr}_p(\mu_j^G) \neq \emptyset$ for some j with $1 \leq j \leq t$. So, it follows from Lemma 2.1 that $\mathrm{acd}_p(\lambda^G) \geq f(p)$, as desired. Next, suppose that $N = E$. If λ is extendible to $\chi \in \mathrm{Irr}(G)$, then Gallagher's theorem [4, Corollary 6.17] implies that $\mathrm{Irr}(\lambda^G) = \{\chi\mu : \mu \in \mathrm{Irr}(G/N)\}$ and for every $\mu_1, \mu_2 \in \mathrm{Irr}(G/N)$ with $\mu_1 \neq \mu_2$, we have $\chi\mu_1 \neq \chi\mu_2$. Thus, either $p \mid \chi(1)$ and $\mathrm{acd}_p(\lambda^G) = \chi(1)\mathrm{acd}(G/N)$ or $p \nmid \chi(1)$ and $\mathrm{acd}_p(\lambda^G) = \chi(1)\mathrm{acd}_p(G/N)$. Obviously, $\mathrm{acd}(G/N) \geq 1$. So, in the former case, $\mathrm{acd}_p(\lambda^G) \geq p > f(p)$, as needed. Since G/N is not p-solvable, Lemma 2.2 yields $\mathrm{acd}_p(G/N) \geq f(p)$. Hence, if $p \nmid \chi(1)$, then $\mathrm{acd}_p(\lambda^G) = \chi(1)\mathrm{acd}_p(G/N) \geq f(p)$, as desired. Finally, suppose that λ is not extendible to G. Then, for every $\chi \in \mathrm{Irr}(\lambda^G)$, $\chi(1) > \lambda(1) \geq 1$. This means that $p \mid \chi(1)$ for every $\chi \in \mathrm{Irr}_p(\lambda^G)$. Therefore, $\mathrm{acd}_p(\lambda^G) \geq p > f(p)$. Now, the proof is complete. □

Proof of Theorem 1.2. First, assume that $\mathrm{Irr}_p(G|N) \neq \emptyset$. As $\mathrm{acd}_p(G|N) < f(p) < p$, we see that $\mathrm{Irr}_p(G|N)$ contains a linear character χ. Then, $\chi_N \neq 1_N$ and as $\chi(1) = 1$, we have $\chi_N \in \mathrm{Irr}(N)$. This implies that N admits some linear characters which are extendible to G and they are nonprincipal. Assume that $\{\mu_1, \ldots, \mu_t\}$ is the set of all linear characters of N which are extendible to G and are nonprincipal. Since the μ_is are extendible to G, none of them are G-conjugate. If $1 \leq i \neq j \leq t$ and there exists $\chi \in \mathrm{Irr}(\mu_i^G) \cap \mathrm{Irr}(\mu_j^G)$, then μ_i and μ_j are irreducible constituents of χ_N. It follows from Clifford's correspondence that μ_i and μ_j are G-conjugate, which is a contradiction with our former assumption on the μ_is. This shows that

$$\mathrm{Irr}(\mu_i^G) \cap \mathrm{Irr}(\mu_j^G) = \emptyset \quad \text{for } 1 \leq i \neq j \leq t. \tag{2.1}$$

Let $1 \leq i \leq t$. Our assumption on the μ_i guarantees the existence of a linear character $\chi_i \in \mathrm{Irr}(G)$ such that $(\chi_i)_N = \mu_i$. By Gallagher's theorem [4, Corollary 6.17], $\mathrm{Irr}(\mu_i^G) = \{\chi_i\varphi : \varphi \in \mathrm{Irr}(G/N)\}$ and for distinct characters $\varphi_1, \varphi_2 \in \mathrm{Irr}(G/N)$, $\chi_i\varphi_1 \neq \chi_i\varphi_2$.

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Since $\chi_i(1) = 1$,

$$\mathrm{Irr}_p(\mu_i^G) = \{\chi_i\varphi : \varphi \in \mathrm{Irr}_p(G/N)\}. \tag{2.2}$$

As $\mu_i \neq 1_N$, $\chi_i \in \mathrm{Irr}(G|N)$. Therefore,

$$\mathrm{Irr}_p(\mu_i^G) \subseteq \mathrm{Irr}_p(G|N).$$

In view of (2.1), $\bigcup_{i=1}^t \mathrm{Irr}(\mu_i^G)$ is disjoint. Take

$$\mathfrak{A} = \mathrm{Irr}_p(G|N) - \dot\bigcup_{i=1}^t \mathrm{Irr}(\mu_i^G).$$

If $\chi \in \mathrm{Irr}(G|N)$ is linear, then $\chi_N \neq 1_N$ and $\chi_N(1) = \chi(1) = 1$. Thus, $\chi_N \in \mathrm{Irr}(N)$ is nonprincipal. It follows from our assumption on the μ_i that $\chi_N \in \{\mu_1, \ldots, \mu_t\}$. Therefore, $\chi \in \mathrm{Irr}(\mu_j^G)$ for some $1 \leq j \leq t$. This implies that $\chi(1) \geq p$ for every $\chi \in \mathfrak{A}$. Therefore,

$$\mathrm{acd}_p(\mathfrak{A}) \geq p > f(p). \tag{2.3}$$

By (2.1) and (2.2), $|\dot\bigcup_{i=1}^t \mathrm{Irr}_p(\mu_i^G)| = t|\mathrm{Irr}_p(G/N)|$ and

$$\begin{aligned}\mathrm{acd}_p(\dot\bigcup_{i=1}^t \mathrm{Irr}(\mu_i^G)) &= \frac{\sum_{i=1}^t \sum_{\chi \in \mathrm{Irr}_p(\mu_i^G)} \chi(1)}{|\dot\bigcup_{i=1}^t \mathrm{Irr}_p(\mu_i^G)|}\\ &= \frac{\sum_{i=1}^t \sum_{\varphi \in \mathrm{Irr}_p(G/N)} (\chi_i\varphi)(1)}{t|\mathrm{Irr}_p(G/N)|}\\ &= \frac{t\sum_{\varphi \in \mathrm{Irr}_p(G/N)} \varphi(1)}{t|\mathrm{Irr}_p(G/N)|} = \mathrm{acd}_p(G/N).\end{aligned}$$

If $\mathrm{acd}_p(G/N) \geq f(p)$, then

$$\mathrm{acd}_p(\dot\bigcup_{i=1}^t \mathrm{Irr}(\mu_i^G)) \geq f(p). \tag{2.4}$$

Note that $\mathrm{Irr}_p(G|N) = (\dot\bigcup_{i=1}^t \mathrm{Irr}_p(\mu_i^G)) \dot\cup \mathfrak{A}$. It follows from (2.3), (2.4) and Lemma 2.1 that $\mathrm{acd}_p(G|N) \geq f(p)$, which is a contradiction. This implies that $\mathrm{acd}_p(G/N) < f(p)$. As $\mathrm{acd}_p(G|N) < f(p)$ and $\mathrm{Irr}_p(G) = \mathrm{Irr}_p(G|N) \dot\cup \mathrm{Irr}_p(G/N)$, we deduce from Lemma 2.1 that $\mathrm{acd}_p(G) < f(p)$. Hence, Lemma 2.2 implies that G is p-solvable and $O^{p'}(G)$ is solvable, as desired.

Now, assume that $\mathrm{Irr}_p(G|N) = \emptyset$. Working towards a contradiction, suppose that there exists $\theta \in \mathrm{Irr}(N)$ such that $p \mid \theta(1)$. We have $\theta(1) \mid \chi(1)$ for every $\chi \in \mathrm{Irr}(\theta^G)$. Thus, $p \mid \chi(1)$ for every $\chi \in \mathrm{Irr}(\theta^G)$. Clearly, $\theta \neq 1_N$. So, $\chi \in \mathrm{Irr}_p(\theta^G) \subseteq \mathrm{Irr}_p(G|N)$. This means that $\mathrm{Irr}_p(G|N) \neq \emptyset$, which is a contradiction. This implies that $p \nmid \theta(1)$ for every $\theta \in \mathrm{Irr}(N)$. It follows from the Ito–Michler theorem [4, Corollary 12.34] that N has a normal and abelian Sylow p-subgroup. Thus, N is p-solvable. Now, assume that $1_N \neq \theta \in \mathrm{Irr}(N)$ and $\chi \in \mathrm{Irr}(\theta^G)$. Hence, $\chi \in \mathrm{Irr}(G|N)$. As $\mathrm{Irr}_p(G|N) = \emptyset$, we deduce that $p \nmid \chi(1)$. Thus, $p \nmid \chi(1)/\theta(1)$. It follows from Lemma 2.3 that G/N has an abelian Sylow p-subgroup. This completes the proof. □

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