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ENDOMORPHISM RINGS OF BUTLER GROUPS

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Abstract

This note is devoted to the question of deciding whether or not a subring of a finite-dimensional algebra over the rationals, with additive group a Butler group, is the endomorphism ring of a Butler group (a Butler group is a pure subgroup of a finite direct sum of rank-1 torsion-free abelian groups). A complete answer is given for subrings of division algebras. Several applications are included.

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A. L. S. Corner, in 1963, proved that each reduced subring of a finite-dimensional Q-algebra is isomorphic to the endomorphism ring of a finite-rank torsion-free abelian group, where Q denotes the field of rationals. M. C. R. Butler, in 1965, defined a class of finite-rank torsion-free abelian groups, subsequently called Butler groups. This class is the smallest class that contains all rank-1 torsion-free abelian groups and is closed under pure subgroups, torsion-free homomorphic images, and finite direct sums ([6]).

In 1965, S. Brenner and M. C. R. Butler showed that each finite-dimensional Q-algebra is isomorphic to the quasi-endomorphism ring of a Butler group, where the quasi-endomorphism ring of G is the tensor product of Q, over Z, with the endomorphism ring of G. Furthermore, the additive group of the endomorphism ring of a Butler group is again a Butler group ([6]). A partial converse to the latter result is:

THEOREM I. *Assume that R is a subring of $K = QR$, a finite-dimensional Q-algebra and that the additive group of R is a Butler group.*

(a) *If R is p-reduced for at least 5 primes of Z, then there is a Butler group G with endomorphism ring isomorphic to R.*

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(b) *The group G may be chosen with rank $= 2m\,rank(R)$, where $m - 1$ is the cardinality of a set of Q-algebra generators of K containing 1_K.*

A Butler group G is a B_0-group (called torsionless in [6]) if $G^*(\tau)$ is a pure subgroup of G for each type τ, where $G^*(\tau)$ is the subgroup of G generated by $\{x \in G \mid \text{type}_G(x) > \tau\}$. Among the class of almost completely decomposable groups, the B_0-groups are precisely the completely decomposable groups. Thus, the endomorphism ring of an indecomposable, almost completely decomposable Butler group of rank > 1 cannot be isomorphic to the endomorphism ring of a B_0-group in this restricted class. On the other hand, if the endomorphism ring of a Butler group is a subring of a division algebra, then the following theorem shows that the ring is also the endomorphism ring of a B_0-group.

THEOREM II. *Assume that R is a subring of $K = QR$, a finite-dimensional division Q-algebra. The following statements are equivalent*:

(a) *R is isomorphic to the endomorphism ring of a Butler group*;

(b) *R is isomorphic to the endomorphism ring of a B_0-group*;

(c) *R is a free S-module for some rank-1 torsion-free ring S; if R is p-reduced for at most 4 primes of Z, then $K = Q(\gamma)$ for some γ; and if R is p-reduced for at most 3 primes of Z, then $K = Q$.*

As a consequence of Theorem II, not every subring of a finite-dimensional Q-algebra, with additive group a Butler group, is isomorphic to the endomorphism ring of a Butler group. For example, $R = H(Z_p)$, the ring of Hamiltonian quaternions over the integers localized at a prime p fails to satisfy Theorem II (c). Applications of Theorem II are given in Examples 2 and 3.

If G is a Butler group, then typeset(G) is finite and $\langle G^*(\tau)\rangle_*/G^*(\tau)$ is finite for each type τ, where $\langle G^*(\tau)\rangle_*$ is the pure subgroup of G generated by $G^*(\tau)$ ([6]). It is conjectured in [6], that each Butler group contains a B_0-group as a subgroup of finite index. This conjecture is resolved, in the negative, by Example 4.

Notation and terminology are, unless otherwise noted, as in [1] and [2]. We write M^n for the direct sum of n copies of M and $\text{End}_R(M)$ for the ring of R-endomorphisms of an R-module M. If $S \subseteq M$, then RS denotes the R-submodule of M generated by S.

A torsion-free abelian group is *completely decomposable* (of finite rank) if it is isomorphic to a finite direct sum of subgroups of Q and *almost completely decomposable* if it contains a completely decomposable group as a subgroup of finite index.

Let G be a torsion-free abelian group of finite rank. Then G is *p-reduced* for a prime p of Z if G contains no elements of infinite p-height. If τ is a type, then $G(\tau) = \{x \in G \mid \text{type}_G(x) \geqslant \tau\}$ is a pure fully invariant subgroup of G. Thus, $G^*(\tau)$ is the subgroup of G generated by $\{G(\sigma) \mid \sigma > \tau\}$. Define *typeset*$(G) = \{\text{type}_G(x) \mid 0 \neq x \in G\}$. The group G is *homogeneous* if typeset(G) has cardinality 1.

LEMMA 1 (BRENNER [4]). *Let K be a finite-dimensional Q-algebra and let $m - 1$ be the cardinality of a set of Q-algebra generators for K that contains 1_K.*

(a) *There are five left K-submodules of K^{2m} such that K is isomorphic to the algebra of Q-endomorphisms of K^{2m} that leave each of the submodules invariant.*

(b) *If $m - 1 = 2$, then there are four left K-submodules of K^2 such that K is isomorphic to the algebra of Q-endomorphisms of K^2 that leave each of the submodules invariant.*

PROOF. An explicit construction of these submodules is given for later reference.

(a) Define $m \times 2m$ matrices as follows: $M_1 = (IO)$, $M_2 = (OI)$, $M_3 = (II)$, $M_4 = (IJ)$, and $M_5 = (IM)$, where I is an $m \times m$ identity matrix, O is an $m \times m$ zero matrix, J is an $m \times m$ Jordan matrix with ones on the superdiagonal and zeros elsewhere, M is an $m \times m$ matrix of the form

$$M = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \gamma_2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \gamma_3 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & \gamma_{m-1} & 1 & 0 \end{pmatrix}$$

and $\Gamma = \{\gamma_1 = 1, \gamma_2, \ldots, \gamma_{m-1}\}$ is a set of Q-algebra generators of K. Define K_i to be the K-submodule of K^{2m} generated by the rows of M_i.

Note that left multiplication induces a well-defined algebra injection from K to

$$\left\{ f \in \text{End}_Q(K^{2m}) \mid f(K_i) \subseteq K_i, 1 \leqslant i \leqslant 5 \right\}.$$

Next, let $f \in \text{End}_Q(K^{2m})$ with $f(K_i) \subseteq K_i$ for each i. Represent f as a $2m \times 2m$ matrix with entries in $\text{End}_Q(K)$, say,

$$f = \begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix}$$

$N_1 = N_4$, since $f(K_3) \subseteq K_3$; N_1 is a lower triangular matrix with equal diagonal elements and equal subdiagonal elements, since $f(K_4) \subseteq K_4$; N_1 is a diagonal matrix with equal diagonal elements, since f sends the first row of M_5 into K_5; and there is $k \in K$ with f acting as left multiplication by k, since $f(K_5) \subseteq K_5$ and Γ is a set of Q-algebra generators for K. For the latter statement, observe that if $f = \alpha I_{2m \times 2m}$ for some $\alpha \in \mathrm{End}_Q(K)$, then $\alpha(k\gamma_i) = \alpha(k)\gamma_i$ for each $k \in K$, $2 \leqslant i \leqslant m-1$, since $f(K_5) \subseteq K_5$. Therefore, f is left multiplication by $\alpha(1)$, as desired.

(b) Let $\Gamma = \{1, \gamma\}$, $x_1 = (1,0)$, $x_2 = (0,1)$, $x_3 = (1,1)$, and $x_4 = (1,\gamma)$. Define K_i to be the $K-$ submodule of K^2 generated by x_i. Then, as in the proof of (a), left multiplication induces an isomorphism $K \to \{f \in \mathrm{End}_Q(K^2) \mid f(K_i) \subseteq K_i, 1 \leqslant i \leqslant 4\}$.

Proof of Theorem I. Choose distinct primes p_1, p_2, p_3, p_4, p_5 such that R is p_i-reduced. Let X_j be the subgroup of $Q1_K \subseteq K$ generated by $\{1_K/p_i^k \mid 1 \leqslant i \neq j \leqslant 5, k = 1,2,\dots\}$. Define $B = (X_1R)^m \oplus \cdots \oplus (X_5R)^m \subseteq K^{5m}$. Then B is a Butler group, since R is a Butler group, noting that $X_1R \simeq X_1 \otimes_Z R$ and X_1 is a flat Z-module.

Next, define $G = \mathrm{Image}\,\Phi \cap B$, where Φ: $K^{2m} \to K^{5m}$ is given by $\Phi(x,y) = (y, x, y-x, y-x\alpha, y-x\gamma)$, $y \in K^m$, $x = (x_1, \dots, x_m) \in K^m$, $x\alpha = (0, x_1, x_2, \dots, x_{m-1})$, $x\gamma = (x_2 + x_3\gamma_2, x_3 + x_4\gamma_3, \dots, x_m, 0)$, and $\Gamma = \{1 = \gamma_1, \gamma_2, \dots, \gamma_{m-1}\} \subseteq R$ is a set of Q-algebra generators for $K = QR$. Then G is a Butler group, being a pure subgroup of B. Moreover, $\mathrm{rank}(G) = \dim_Q(\mathrm{Image}\,\Phi) = 2m \dim_Q(K) = 2m\,\mathrm{rank}(R)$, since $QB = K^{5m}$, Φ is an injection, and $QR = K$.

It is sufficient to prove that left multiplication induces an isomorphism μ: $R \to \mathrm{End}(G)$. Note that $RB = B$ and μ is a well-defined ring injection. The strategy is to first prove that if $f \in \mathrm{End}(G)$, then f is left multiplication by some $k \in K$ and then to prove that $k \in R$.

Define $\tau_j = \mathrm{type}(X_j^*S)$ for each $1 \leqslant j \leqslant 5$, where S is the pure subgroup of R generated by $1_R = 1_K$, and $X_j^* = \bigcap\{X_i \mid 1 \leqslant i \neq j \leqslant 5\} = Z[1/p_j] \cdot 1_K$. Since G is a pure subgroup of B, then $G(\tau_j) = G \cap B(\tau_j)$, is a fully invariant subgroup of G. In fact, $B(\tau_j) = \oplus\{(X_iR)^m \mid 1 \leqslant i \neq j \leqslant 5\}$, since if $i \neq j$ then $\tau_j \leqslant \mathrm{type}(x)$ for each $x \in X_iR$ while $\tau_j \nleqslant \mathrm{type}(x)$ for each non-zero x in X_jR. Consequently, $QG(\tau_j) = \Phi(K_j)$, where K_j is defined as in Lemma 1.a, since $\Phi(K_j) = \{\Phi(x,y) \mid (x,y) \in K^{2m}$ and the jth coordinate of $\Phi(x,y) = 0\}$.

Now let $f \in \mathrm{End}(G)$. Then $f(G(\tau_j)) \subseteq G(\tau_j)$, whence $f(QG(\tau_j)) \subseteq QG(\tau_j)$, for each $1 \leqslant j \leqslant 5$. Therefore, by Lemma 1 and the fact that $\Phi(K_j) = QG(\tau_j)$, f is left multiplication by some $k \in K$.

To prove that $k \in R$, let $x = y = (1, 0, \dots, 0) \in K^m$. Then $\Phi(0,y) \in G(\tau_2)$, $f(\Phi(0,y)) = k\Phi(0,y) = (ky, 0, ky, ky, ky) \in G(\tau_2) \subseteq B$, and $k \in X_1R \cap X_3R \cap X_4R \cap X_5R$. Similarly, $k\Phi(x,0) \in G(\tau_1)$ implies that $k \in X_2R$. Thus, $k \in$

$\bigcap\{X_iR \mid 1 \leqslant i \leqslant 5\} = R$, since $Z \cdot 1_K = \bigcap\{X_i \mid 1 \leqslant i \leqslant 5\}$. Consequently, $k \in R$, as desired.

PROOF OF THEOREM II. (b) $\Rightarrow$ (a) is clear.

(a) $\Rightarrow$ (c). Assume that $R = \operatorname{End}(G)$ for some Butler group G. Then R is a free S-module, where S is the pure subgroup of R generated by 1_R, by [2], Corollary 5.2. Note that if p is a prime, then R is p-reduced if and only if R is not p-divisible, whence R is p-reduced if and only if G is not p-divisible.

It is now sufficient to prove that if R is divisible by all but four, respectively three, primes, then there are four, respectively three, QR-submodules of QG such that QR is the algebra of Q-endomorphisms of QG that leave each of these submodules invariant. In this case, $QR = Q(\gamma)$, respectively $QR = Q$, as a consequence of [4], Proposition 5 and Section 7.

The case that G is divisible by all but four primes, say p_1, p_2, p_3, p_4, is proved and the other case, having an analogous proof, is left to the reader.

The four desired submodules of QG are $L_i = QG(\tau_i)$ for $1 \leqslant i \leqslant 4$, where $\tau_j = \inf\{\sigma_i \mid 1 \leqslant i \neq j \leqslant 4\}$, $\sigma_i = \operatorname{type}(Z_{p_i})$, and Z_{p_i} is the localization of Z at p_i. To see this is indeed the case, first note that $\operatorname{typeset}(G) \subseteq T$, the finite sublattice of the lattice of all types generated by σ_1, σ_2, σ_3, σ_4. Then τ_1, τ_2, τ_3, τ_4 are minimal elements of $T \setminus \{\tau_0\}$, where $\tau_0 = \inf\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$, and σ_1, σ_2, σ_3, σ_4 are maximal elements of $T \setminus \{\operatorname{type}(Q)\}$.

Since $\operatorname{typeset}(G) \subseteq T$, G is a pure subgroup of $C = C_1 \oplus \cdots \oplus C_n$, with $C_i \subseteq Q$ and $\operatorname{type}(C_i) \in T$ ([6]). For each $1 \leqslant i \leqslant 4$, $\operatorname{typeset}(C(\sigma_i)) \subseteq \{\sigma_i, \operatorname{type}(Q)\}$ so that $G(\sigma_i)$ is a summand of $C(\sigma_i)$ ([1], Exercise 5.7), hence of G. Therefore, $G(\sigma_i) = 0$, since $Q \operatorname{End}(G) \simeq K$, a division algebra, implies that G is indecomposable. It now follows that $\operatorname{typeset}(G) \subseteq \{\tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \tau^{ij} \mid 1 \leqslant i \neq j \leqslant 4\}$, where $\tau^{ij} = \sup\{\tau_i, \tau_j\} = \inf\{\sigma_k, \sigma_l\}$, and $\{i, j, k, l\} = \{1, 2, 3, 4\}$.

Finally, assume that $f \in \operatorname{End}_Q(QG)$, with $f(L_i) \subseteq L_i$ for each $1 \leqslant i \leqslant 4$. Then $f(QG(\tau^{ij})) \subseteq QG(\tau^{ij})$, for each $1 \leqslant i \neq j \leqslant 4$, noting that $QG(\tau^{ij}) = QG(\tau_i) \cap QG(\tau_j)$ for each $\tau \in \operatorname{typeset}(G)$. Therefore, $f \in Q \operatorname{End}(G)$ by [2], Theorem 1.5. Consequently, left multiplication induces an isomorphism $QR \to \{f \in \operatorname{End}_Q(QG) \mid f(L_i) \subseteq L_i, 1 \leqslant i \leqslant 4\}$, as desired.

(c) $\Rightarrow$ (b). If R is p-reduced for at most 3 primes, then $K = Q$, by hypothesis. In this case, the additive group of R is a B_0-group with endomorphism ring R.

The next case considered is that R is p-reduced for at least 5 primes. The construction of Theorem I yields a Butler group $G = \operatorname{Image}\Phi \cap B$ with $R \simeq \operatorname{End}(G)$, where $B = (X_1R)^m \oplus \cdots \oplus (X_5R)^m \subseteq K^{5m}$ and $\Gamma = \{1 = \gamma_1, \ldots, \gamma_{m-1}\}$ is a set of Q-algebra generators of K, which may be asssumed to be a subset of R since $K = QR$. We prove that G is a B_0-group.

Since R is a free S-module, $X_i R$ is homogeneous completely decomposable with type $= \sigma_i = \text{type}(X_i S)$. Recall, from the proof of Theorem I, that $X_j^* = \bigcap\{X_i \mid 1 \leqslant i \neq j \leqslant 5\}$, $\tau_j = \text{type}(X_j^* S)$, and $\Phi(K_j) = QG(\tau_j)$, where K_j is as defined in Lemma 1.a.

A routine calculation shows that if $1 \leqslant i \neq j \leqslant 5$, then $K_i \cap K_j = 0$, unless $\{i, j\} = \{1,4\}$ or $\{1,5\}$. Consequently, typeset$(G) \subseteq \{\tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau^{14}, \tau^{15}\}$, where type$(S) = \tau_0 = \inf\{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5\}$ and $\tau^{ij} = \sup\{\tau^i, \tau^j\}$.

It is now sufficient to prove that $G^*(\tau_0\} = G$ and that $G^*(\tau_1)$ is pure in G, in which case G must be a B_0-group. In fact, $G^*(\tau_1) = G(\tau^{14}) + G(\tau^{15})$ is pure in G, since $K_1 \cap K_4 = \Phi^{-1}QG(\tau^{14}) = Q(0,0,\ldots,0,1,0,0,\ldots,0)$ and $K_1 \cap K_5 = \Phi^{-1}QG(\tau^{15}) = Q(1,0,\ldots,0,0,0,\ldots,0)$.

Note that $G^*(\tau_0) = G(\tau_1) + G(\tau_2) + G(\tau_3) + G(\tau_4) + G(\tau_5)$. To see that $G^*(\tau_0) = G$, it is sufficient to prove that if p is a prime, then $Z_p \otimes_Z G = G_p \subseteq G(\tau_1)_p + \cdots + G(\tau_5)_p$, in which case, $G = \bigcap_p G_p \subseteq G^*(\tau_0) = \bigcap_p G^*(\tau_0)_p$. Let $a = (y, x, y - x, y - x\alpha, y - x\gamma) \in G_p \subseteq (X_1 R)_p^m \oplus \cdots \oplus (X_5 R)_p^m$. If $p \notin \{p_3, p_4, p_5\}$, then $y \in (X_1 R)_p^m = (X_2^* R)_p^m$ and $x \in (X_2 R)_p^m = (X_1^* R)_p^m$. Hence, $a = \Phi(x, y) = \Phi(0, y) + \Phi(x, 0) \in G(\tau_2)_p + G(\tau_1)_p$. Similarly, if $p = p_3$, then $a = \Phi(0, y - x) + \Phi(x, x) \in G(\tau_2)_p + G(\tau_3)_p$; if $p = p_4$, then $a = \Phi(0, y - x\alpha) + \Phi(x, x\alpha) \in G(\tau_2)_p + G(\tau_4)_p$; and if $p = p_5$, then $a = \Phi(0, y - x\gamma) + \Phi(x, x\gamma) \in G(\tau_2)_p + G(\tau_5)_p$. Consequently, $G^*(\tau_0) = G$, as desired.

The final case is that R is p-reduced for exactly four primes. The proof, using Lemma 1.b in place of Lemma 1.a, is similar, but easier, and thus is omitted.

EXAMPLE 2. There is a B_0-group A with $\text{End}(A) \simeq Z \oplus 2Zi \subseteq Q(i)$, where $i^2 = -1$. In particular, End(A) is not integrally closed in its quotient field $Q(i)$.

Example 2 and Theorem II are a partial resolution of Problem 6.5, [2]. The following example is a counterexample to Conjecture 6.2, [2].

EXAMPLE 3. There are B_0-groups A and B such that A and B are nearly isomorphic but not isomorphic.

PROOF. Let $S = Z[\sqrt{-5}\,]$, a Dedekind domain that is not a principal ideal domain. By Theorem II, there is a B_0-group A with $\text{End}(A) \simeq S$. Let I be a non-principal ideal of S. Then $B = IA$ is a subgroup of finite index in A, A and B are nearly isomorphic, but A and B are not isomorphic (as in [1], Example 12.11).

EXAMPLE 4. There is a rank-4 Butler group that does not contain a B_0-group as a subgroup of finite index.

Proof. Let $V = Qv_1 \oplus Qv_2 \oplus Qv_3 \oplus Qv_4$ and let $\{p_1, \ldots, p_9\}$ be a set of distinct primes. Denote by $Z[1/p_i]$ the subring of Q generated by $1/p_i$, and define $Z^{ij} = Z[1/p_i] + Z[1/p_j]$ whenever $1 < i \neq j \leqslant 9$. Let G be the subgroup of V generated by $\{A_1, \ldots, A_6\}$, where $A_1 = Z^{15}v_1$, $A_2 = Z^{25}v_2$, $A_3 = Z^{36}v_3$, $A_4 = Z^{46}v_4$, $A_5 = Z^{79}(v_1 + v_2 + p_9 v_3)$, and $A_6 = Z^{89}(v_3 + v_4 + p_9 v_1)$. Then each A_i is a pure rank-1 subgroup of G and $\text{typeset}(G) = \{\tau_1, \ldots, \tau_6, \tau, \tau_{12}, \tau_{34}, \tau_{56}\}$, where $\tau_i = \text{type}(A_i)$, $\tau = \text{type}(Z)$ and $\tau_{ij} = \inf\{\tau_i, \tau_j\}$ (see [2], Theorems 0.1 and 1.7).

Note that $G(\tau_i) = A_i$, $G^*(\tau_{12}) = A_1 + A_2$, and $G^*(\tau_{34}) = A_3 + A_4$. Since $G(\tau_{12}) \cap G(\tau_{34}) = 0$, $G(\tau_{12}) = (Qv_1 \oplus Qv_2) \cap G = A_1 + A_2 + Z(v_1 + v_2)/p$ and $G(\tau_{34}) = A_3 + A_4 + Z(v_3 + v_4)/p$, where $p = p_9$. Similarly, $G^*(\tau_{56}) = G(\tau_{56})$.

Suppose that H is a subgroup of finite index in G and that H is a B_0-group. Then $\text{typeset}(H) = \text{typeset}(G)$, so there exist non-zero integers m_i, such that $H = m_1 A_1 + \cdots + m_6 A_6$ and $m_i A_i = H(\tau_i)$, since $H(\tau_i) = H \cap G(\tau_i) = H \cap A_i$ ([2], Theorem 2.2).

Furthermore, $m_1 A_1 + m_2 A_2 = H^*(\tau_{12}) = H(\tau_{12}) = H \cap G(\tau_{12})$ and, similarly, $m_3 A_3 + m_4 A_4 = H \cap G(\tau_{34})$. In particular, $(m_3 A_3 + m_4 A_4 + m_5 A_5 + m_6 A_6) \cap G(\tau_{12}) \subseteq m_1 A_1 + m_2 A_2$ and $(m_1 A_1 + m_2 A_2 + m_5 A_5 + m_6 A_6) \cap G(\tau_{34}) \subseteq m_3 A_3 + m_4 A_4$. Localizing the first containment at $p = p_9$ yields $(p^{e(3)} Z_p v_3 + p^{e(4)} Z_p v_4 + Qv_5 + Qv_6) \cap (Z_p v_1 + Z_p v_2 + Z_p(v_1 + v_2)/p) \subseteq p^{e(1)} Z_p v_1 + p^{e(2)} Z_p v_2$, where $e(i) = p\text{-height}(m_i)$ in Z, $v_5 = (v_1 + v_2 + pv_3)/p$, and $v_6 = (v_3 + v_4 + pv_1)/p$.

Note that $v_5 - v_3 = (v_1 + v_2)/p$. Thus, $-p^{e(3)} v_3 + p^{e(3)} v_5 = p^{e(3)-1}(v_1 + v_2) \in p^{e(1)} Z_p v_1 + p^{e(2)} Z_p v_2$. Hence, $e(3) - 1 \geqslant e(1)$. However, by a similar argument on the localization of the second containment and the equation $v_6 - v_1 = (v_3 + v_4)/p$, $e(1) - 1 \geqslant e(3)$. This contradiction completes the proof.

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