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NEW ALMOST PERIODIC TYPE FUNCTIONS AND SOLUTIONS OF DIFFERENTIAL EQUATIONS

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ABSTRACT. Let X be a Banach space and $\mathbb{J} \in \{\mathbb{R}^+, \mathbb{R}\}$. Let Π and Π_0 be two subspaces of $C(\mathbb{J}, X)$, the Banach space of bounded continuous functions from $\mathbb{J}$ to X. We seek conditions under which $\Pi + \Pi_0$ is closed in $C(\mathbb{J}, \mathbb{R})$. This led to introduce a general $\mathcal{A}\Pi_r(\mathbb{J}, X)$ space, which contains many classes of almost periodic type functions as subspaces. We prove some recent results on indefinite integral for the elements of these classes. We apply certain results on harmonic analysis to investigate solutions of differential equations. As an application we study specific $\mathcal{A}\Pi_r(\mathbb{J}, \mathbb{R})$ spaces: the spaces of asymptotic and pseudo almost automorphic functions and their solutions of some ordinary quasi-linear and a non-linear parabolic partial differential equations.

1. **Notation, Definitions and Introduction.** Throughout this paper, X denotes a Banach space and X^* denotes the dual space of X. Let $\mathbb{J} \in \{\mathbb{R}^+, \mathbb{R}\}$. Then $C(\mathbb{J}, X)$ will stand for the Banach space of bounded continuous functions φ from $\mathbb{J}$ to X with norm $\|\varphi\| = \sup_{x \in \mathbb{J}} \|\varphi(x)\|$ and $C_u(\mathbb{J}, X)$ will denote the subspace of $C(\mathbb{J}, X)$ consisting of the uniformly continuous functions. If $(\varphi_n), \varphi \subset C(\mathbb{J}, X)$, we write $\varphi_n \longrightarrow \varphi$ if and only if $\|\varphi_n - \varphi\| \longrightarrow 0$ as $n \longrightarrow \infty$. In case that $X = \mathbb{C}$, we will omit X from our notation and write, say, $C(\mathbb{J})$ for $C(\mathbb{J}, X)$. The translate and the difference of φ by $s \in \mathbb{J}$ are respectively the functions $R_s\varphi(t) := \varphi(t+s)$ for all $t \in \mathbb{J}$ and $\Delta_s\varphi := R_s\varphi - \varphi$. Let $\mathcal{F}$ be a subspace of $C(\mathbb{J}, X)$. Then $\mathcal{F}$ is said to be *translation invariant* if $R_s\mathcal{F} \subset \mathcal{F}$ for all $s \in \mathbb{J}$. If $\varphi \in C(\mathbb{R}, X)$ then $L(\varphi)$ will stand for the subspace of $C(\mathbb{R}, X)$ generated by all translates of φ. We denote by

$$\mathcal{PAP}_0(\mathbb{J}, X) = \{\varphi \in C(\mathbb{J}, X) : \lim_{t \to \infty} \frac{1}{t-c} \int_c^t \|\varphi(s)\|\, ds = 0\}, \tag{1.1}$$

and

$$\mathcal{E}(\mathbb{J}, X) = \{\varphi \in C_u(\mathbb{J}, X) : \lim_{t \to \infty} \frac{1}{t-c} \int_c^t \varphi(s+x)\, ds = M(\varphi)\}, \tag{1.2}$$

where $c = 0$ if $\mathbb{J} = \mathbb{R}^+$ and $c = -t$ if $\mathbb{J} = \mathbb{R}$, and the limit in (1.2) exists uniformly with respect to $x \in \mathbb{J}$. The set $\mathcal{PAP}_0(\mathbb{J}, X)$ is a translation invariant, closed subspace of $C(\mathbb{J}, X)$ which does not contain any constant functions except 0. $\mathcal{PAP}_0(\mathbb{J}, X)$ is introduced

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by Zhang [26–28] and its elements are called *ergodic perturbations*. We remark that the elements of $\mathcal{PAP}_0(\mathbb{J},X)$ are not necessary uniformly continuous. $\mathcal{E}(\mathbb{J},X)$ is called the *space of ergodic functions* [2, 13, 18, 22] and $M(\varphi)$ is called the *mean* of φ.

Let $\varphi \in C(\mathbb{J},X)$. The function φ is called (weakly) almost periodic if the set $\{R_s\varphi : s \in \mathbb{J}\}$ is (weakly) relatively compact in $C(\mathbb{J},X)$. Denote by $\big(\mathcal{WAP}(\mathbb{J},X)\big)$ $\mathcal{AAP}(\mathbb{J},X)$ ($\mathbb{J} = \mathbb{R}^+$) all such functions. If $\mathbb{J} = \mathbb{R}$, then the space of almost periodic functions will be denoted by $\mathcal{AP}(\mathbb{R},X)$ (see [1], [11], [17]). Recently, Ruess and Summers [22, Proposition 2.1, Theorem 2.4] and Basit [2, Theorem 2.3.4 and 2.4.7] proved that

$$\mathcal{WAP}(\mathbb{J},X) \subset \mathcal{E}(\mathbb{J},X). \tag{1.3}$$

We recall that a subset E of $\mathbb{R}$ is called relatively dense in $\mathbb{R}$ if there exists a finite subset $\{t_1,t_2,\cdots,t_m\} \subset \mathbb{R}$ such that $\mathbb{R} = \bigcup_{i=1}^m (t_i+E)$. This implies that there exists $l > 0$ such that $]x,x+l[\,\cap E \neq \emptyset$ for all $x \in \mathbb{R}$. A function $\varphi \in C_u(\mathbb{R},X)$ is said to be recurrent if, for each $\epsilon > 0$ and $N > 0$ the set

$$E(\epsilon,N,\varphi) = \{\tau \in \mathbb{R} : \|\varphi(t+\tau) - \varphi(t)\| < \epsilon, |t| \leq N\}$$

is a relatively dense subset in $\mathbb{R}$. The set of all the recurrent functions will be denoted by $r(\mathbb{R},X)$.

It is known that $r(\mathbb{R},X)$ is not linear [14, 17]. Nevertheless, $\mathcal{AP}(\mathbb{R},X)$ is a linear class of $r(\mathbb{R},X)$ and also, $L(\varphi)$ is a linear class of $r(\mathbb{R},X)$ for all $\varphi \in r(\mathbb{R},X)$.

A function $\varphi \in C_u(\mathbb{R},X)$ is said to be almost automorphic if it is recurrent and for each $\epsilon > 0$ and $N > 0$ there exist $\delta > 0$ and $M > 0$ and a relatively dense subset $B(\delta,M,\varphi)$ such that

$$B(\delta,M,\varphi) - B(\delta,M,\varphi) \subset E(\epsilon,N,\varphi).$$

The set of all almost automorphic functions will be denoted by $\mathcal{AA}(\mathbb{R},X)$.

The following needed in the sequel statement is obvious.

PROPOSITION 1.1. *Let $\varphi \in \mathcal{AA}(\mathbb{R},X)$ and $\psi \in C_u(\mathbb{R},X)$. Let for each $\epsilon > 0$ and $N > 0$ there exist $\delta > 0$ and $M > 0$ such that*

$$E(\delta,M,\varphi) \subset E(\epsilon,N,\psi).$$

Then $\psi \in \mathcal{AA}(\mathbb{R},X)$.

If the range of φ is precompact, the above definition is equivalent to the definition which S. Bochner [10] introduced for numerical almost automorphic functions. The investigation of Veech [24] and Basit [6] (see also references in [2]) showed the relationship between almost automorphic functions and almost periodic functions in the sense of Levitan. The extension to vector-valued functions and many equivalent definitions are obtained in [6, 19, 24, 25].

We shall denote by $\mathcal{WAP}_0(\mathbb{J},X) = \{\varphi \in \mathcal{WAP}(\mathbb{J},X) : M(\|\varphi(\cdot)\|) = 0\}$, and $C_0(\mathbb{J},X) = \{\varphi \in C(\mathbb{J},X) : \lim_{|t|\to\infty} \varphi(t) = 0\}$. It is obvious that $C_0(\mathbb{J},X) \subset \mathcal{WAP}_0(\mathbb{J},X)$.

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Let

(1.4) $\Pi_0(\mathbb{J},X) \in \{\mathcal{PAP}_0(\mathbb{J},X), C_0(\mathbb{J},X), \mathcal{WAP}_0(\mathbb{J},X)\}$.

$\Pi_c(\mathbb{R},X)$ denotes any class of functions satisfying:

(1.5) $\Pi_c(\mathbb{R},X)$ is a translation invariant, closed subspace of $C_u(\mathbb{R},X)$ containing all the constant functions.

(1.6) The map $m: \Pi_c(\mathbb{R},X) \longrightarrow \Pi_c(\mathbb{R}^+,X)$ defined by $m(\varphi) = \varphi|_{\mathbb{R}^+}$ is an isometry.

(1.7) $\Pi_c(\mathbb{R},X)$ is closed under multiplication by characters.

$\Pi_r(\mathbb{R},X)$ denotes any class of functions satisfying (1.5), (1.7) and

$$\Pi_r(\mathbb{R},X) \subset r(\mathbb{R},X). \tag{1.8}$$

It follows from [2, Proposition 2.1.7] that $\Pi_r(\mathbb{R},X)$ satisfies (1.6) and hence

$$\Pi_r(\mathbb{R},X) \subset \Pi_c(\mathbb{R},X) \text{ and } \Pi_r(\mathbb{R}^+,X) \subset \Pi_c(\mathbb{R}^+,X). \tag{1.9}$$

Define

$$\begin{aligned} \mathcal{A}\Pi_c(\mathbb{J},X) &= \Pi_c(\mathbb{J},X) + \Pi_0(\mathbb{J},X), \\ \mathcal{A}\Pi_r(\mathbb{J},X) &= \Pi_r(\mathbb{J},X) + \Pi_0(\mathbb{J},X). \end{aligned} \tag{1.10}$$

We have

$$\mathcal{A}\Pi_r(\mathbb{J},X) \subset \mathcal{A}\Pi_c(\mathbb{J},X). \tag{1.11}$$

If $\varphi = \psi + \xi$ with $\psi \in \Pi_c(\mathbb{J},X)$ and $\xi \in \Pi_0(\mathbb{J},X)$, then ψ will be called the almost periodic type part and ξ the ergodic perturbation. We write

$$P_0\mathcal{A}\Pi_c(\mathbb{J},X), \quad C_0\mathcal{A}\Pi_c(\mathbb{J},X), \quad W_0\mathcal{A}\Pi_c(\mathbb{J},X)$$

instead of $\mathcal{A}\Pi_c(\mathbb{J},X)$ if

$$\Pi_0(\mathbb{J},X) = \mathcal{PAP}_0(\mathbb{J},X), \quad C_0(\mathbb{J},X), \quad \mathcal{WAP}_0(\mathbb{J},X)$$

respectively.

The space $\mathcal{A}\Pi_r(\mathbb{J},X)$ is a generalization of some spaces of almost periodic type functions. Here, we mention some of them:

$\mathcal{AAP}(\mathbb{J},X)$, the space of asymptotically almost periodic functions, $\mathcal{WAP}(\mathbb{J},X)$, the space of weakly almost periodic functions (see [2, 13, 15, 19, 22]); $\mathcal{PAP}(\mathbb{J},X)$, the space of pseudo almost periodic functions (see [26–28]); $\mathcal{AAA}(\mathbb{J},X)$, the space of asymptotically almost automorphic functions (see [2]). For other classes of functions satisfying (1.5)–(1.8) see [2]. It follows easily that $\Pi_c(\mathbb{R},X)$ are Λ-classes introduced in [2, Definition 2.2.1]. We prove that $\mathcal{A}\Pi_r(\mathbb{J},X) \cap C_u(\mathbb{J},X)$ is a Λ-class.

In this paper, we mainly study the space $\mathcal{A}\Pi_r(\mathbb{J},X)$. We prove that these classes are closed subspaces of $C(\mathbb{J},X)$ in section 2. Some recent results on indefinite integral obtained in [2] and [27] are extended in section 3. In section 4, we study the harmonic analysis introduced in [2] for these classes and give some of its application to ordinary differential equations. Since the elements of these classes are not necessary uniformly continuous, the application of the results of [2] is not direct. In Sections 5, we will investigate a specific $\mathcal{A}\Pi_r(\mathbb{J},X)$ space: pseudo and asymptotically almost automorphic functions and their solutions of quasilinear ordinary differential equations. We study the solutions of non-linear parabolic partial differential equations in section 6.

2. **Properties of the Spaces $\mathcal{A}\Pi_c(\mathbb{J},X)$ and $\mathcal{A}\Pi_r(\mathbb{J},X)$.** In this section we study $\Pi_0(\mathbb{J},X)$, $\mathcal{A}\Pi_c(\mathbb{J},X)$ and $\mathcal{A}\Pi_r(\mathbb{J},X)$. We prove that every $\mathcal{A}\Pi_r(\mathbb{J},X)$ class is a closed subspace of $C(\mathbb{J},X)$ and every $C_0\mathcal{A}\Pi_c(\mathbb{J},X)$ class is a closed subspace of $C_u(\mathbb{J},X)$.

PROPOSITION 2.1. *If $\varphi \in \mathcal{A}\Pi_c(\mathbb{J},X)$ $\big(\mathcal{A}\Pi_r(\mathbb{J},X)\big)$ and $\psi \in C(\mathbb{R},X)$ such that $\psi|_{\mathbb{J}} = \varphi$, then*

(i) $R_s\psi|_{\mathbb{J}} \in \mathcal{A}\Pi_c(\mathbb{J},X)\big(\mathcal{A}\Pi_r(\mathbb{J},X)\big)$ for all $s \in \mathbb{R}$;

(ii) $\psi * f|_{\mathbb{J}} \in \mathcal{A}\Pi_c(\mathbb{J},X) \cap C_u(\mathbb{J},X)$ $\big(\mathcal{A}\Pi_r(\mathbb{J},X) \cap C_u(\mathbb{J},X)\big)$ for all $f \in L^1(\mathbb{R})$.

PROOF. First, we prove the case $\varphi \in \Pi_0(\mathbb{J},X)$.

(i) If $\mathbb{J} = \mathbb{R}$ then $\psi = \varphi$ and (i) follows from the fact that $\Pi_0(\mathbb{R},X)$ is translation invariant subspace of $C(\mathbb{R},X)$. If $\mathbb{J} = \mathbb{R}^+$, one can choose $\psi_0 \in C(\mathbb{R},X)$ such that $\psi_0(t) = \varphi(t)$ for $t \geq 0, \psi_0(t) = \varphi(0)(t+1)$ for $-1 \leq t < 0$, and $\psi_0(t) = 0$ for $t < -1$. A direct verification shows that $R_s\psi_0|_{\mathbb{J}} \in \Pi_0(\mathbb{J},X)$ for all $s \in \mathbb{R}$. Since $R_s(\psi - \psi_0)|_{\mathbb{J}}$ has compact support for all $s \in \mathbb{R}$, we conclude that $R_s(\psi - \psi_0)|_{\mathbb{J}} \in \Pi_0(\mathbb{J},X)$. This implies that $R_s\psi|_{\mathbb{J}} \in \Pi_0(\mathbb{J},X)$.

(ii) If $h > 0$ and $f = \chi_{[0,h]}$ the characteristic function of $[0,h]$, then $\psi * f(t) = \int_0^h \psi(t+x)\,dx$. Direct computation for each of the three cases of $\Pi_0(\mathbb{J},X)$ shows that $\psi * f|_{\mathbb{J}} \in \Pi_0(\mathbb{J},X) \cap C_u(\mathbb{J},X)$. If $h < 0$, then $\int_0^h \psi(t+x)\,dx = \int_{-h}^0 \psi(t+x+h)\,dx = -\int_0^{-h} R_h\psi(t+x)\,dx$. By (i), we have that $R_h\psi|_{\mathbb{J}} \in \Pi_0(\mathbb{J},X)$. Since $-h > 0$, it follows that $\int_0^{-h} R_h\psi(t+x)\,dx|_{\mathbb{J}}$ belongs to $\Pi_0(\mathbb{J},X) \cap C_u(\mathbb{J},X)$. For $d > c$, define a function $g_{c,d}$ on $\mathbb{R}$ such that $g_{c,d}(t) = 1$ for $t \in [c,d]$ and $g_{c,d}(t) = 0$ otherwise. We have that

$$\psi * g_{c,d}(t) = \int_0^d \psi(t+x)\,dx - \int_0^c \psi(t+x)\,dx \quad (t \in \mathbb{R}).$$

Hence, we get $\psi * g_{c,d}|_{\mathbb{J}} \in \Pi_0(\mathbb{J},X) \cap C_u(\mathbb{J},X)$ for all $c, d \in \mathbb{R}$. If $H(t) = \sum_{k=1}^m a_k g_{c_k,d_k}(t)$ is a step function, then, by the linearity of $\Pi_0(\mathbb{J},X)$ we get $H * \psi|_{\mathbb{J}} \in \Pi_0(\mathbb{J},X) \cap C_u(\mathbb{J},X)$. Since $f \in L^1(\mathbb{R})$, there exists a sequence of step functions $\{H_n\}$ such that $\|H_n - f\|_{L^1} \longrightarrow 0$ as $n \longrightarrow \infty$. This implies that $\|\psi * H_n - \psi * f\| \longrightarrow 0$ as $n \longrightarrow \infty$. Hence $\|\psi * H_n|_{\mathbb{J}} - \psi * f|_{\mathbb{J}}\| \longrightarrow 0$ as $n \longrightarrow \infty$. Therefore, $\psi * f|_{\mathbb{J}} \in \Pi_0(\mathbb{J},X) \cap C_u(\mathbb{J},X)$.

Secondly, let $\varphi \in \mathcal{A}\Pi_c(\mathbb{J},X)$, $\varphi = \zeta|_{\mathbb{J}} + \xi$, where $\zeta \in \Pi_c(\mathbb{R},X)$ and $\xi \in \Pi_0(\mathbb{J},X)$. The case $\varphi \in \mathcal{A}\Pi_r(\mathbb{J},X)$ can be proved exactly in the same way. Set $\eta = \psi - \zeta$. Then $\eta|_{\mathbb{J}} = \xi \in \Pi_0(\mathbb{J},X)$. By the first part (i), we get $R_s\eta|_{\mathbb{J}} \in \Pi_0(\mathbb{J},X)$ and by property (1.5) of $\Pi_c(\mathbb{R},X)$, we have $R_s\zeta \in \Pi_c(\mathbb{R},X)$. This implies that $R_s\psi|_{\mathbb{J}} = R_s\zeta|_{\mathbb{J}} + R_s\eta|_{\mathbb{J}} \in \mathcal{A}\Pi_c(\mathbb{J},X)$. By the above $\eta * f|_{\mathbb{J}} \in \Pi_0(\mathbb{J},X) \cap C_u(\mathbb{J},X)$ and from the inclusion $\zeta * f \in L(\zeta) \subset \Pi_c(\mathbb{R},X)$ for all $f \in L^1(\mathbb{R})$ by [2, Lemma 1.2.1], the proof of (ii) is complete.

COROLLARY 2.2. *Let* $\varphi \in \mathcal{A}\Pi_c(\mathbb{J},X)\big(\mathcal{A}\Pi_r(\mathbb{J},X)\big)$ *and* $P\varphi(t) = \int_0^t \varphi(x)\,dx$. *Then* $\Delta_h P\varphi \in \mathcal{A}\Pi_c(\mathbb{J},X) \cap C_u(\mathbb{J},X)\big(\mathcal{A}\Pi_r(\mathbb{J},X) \cap C_u(\mathbb{J},X)\big)$ *for all* $h \in \mathbb{J}$.

PROPOSITION 2.3. $C_0\mathcal{A}\Pi_c(\mathbb{J},X)$ $\big(C_0\mathcal{A}\Pi_r(\mathbb{J},X)\big)$ *is a closed subspace of* $C_u(\mathbb{J},X)$. *Moreover,* $C_0\mathcal{A}\Pi_c(\mathbb{J},X) = \Pi_c(\mathbb{J},X) \oplus C_0(\mathbb{J},X)$ $\big(C_0\mathcal{A}\Pi_r(\mathbb{J},X) = \Pi_r(\mathbb{J},X) \oplus C_0(\mathbb{J},X)\big)$.

PROOF. It is sufficient to prove the case $\varphi \in C_0\mathcal{A}\Pi_c(\mathbb{J},X)$. Let $\varphi = \psi|_{\mathbb{J}} + \xi$, where $\psi \in \Pi_c(\mathbb{R},X)$ and $\xi \in C_0(\mathbb{J},X)$. We show that

$$\|\psi\| \leq \|\varphi\|. \tag{2.1}$$

Indeed, $\|\varphi\| = \|\psi|_{\mathbb{J}} + \xi\| \geq \|R_s\psi|_{\mathbb{J}} + R_s\xi\| \geq \|R_s\psi|_{\mathbb{J}}\| - \|R_s\xi\|$ for $s \in \mathbb{J}$. Taking the limit when $s \longrightarrow \infty$ and using (1.6) and the fact $\lim_{s\to\infty} \|R_s\xi\| = 0$, we get (2.1).

Now, let $\{\varphi_n\}$ be Cauchy in $C_0\mathcal{A}\Pi_c(\mathbb{J},X)$. Then $\varphi_n = \psi_n|_{\mathbb{J}} + \xi_n$, where $\psi_n \in \Pi_c(\mathbb{R},X)$ and $\xi_n \in C_0(\mathbb{J},X)$. It follows from (2.1) that $\{\psi_n\}$ is Cauchy too. By property (1.5), we conclude that $\{\psi_n\}$ converges to $\psi \in \Pi_c(\mathbb{R},X)$. This implies that $\{\xi_n\}$ is a Cauchy sequence in $C_0(\mathbb{R},X)$ and hence converges to $\xi \in C_0(\mathbb{J},X)$. Hence, $\{\varphi_n\}$ converges to $\psi|_{\mathbb{J}} + \xi \in C_0\mathcal{A}\Pi_c(\mathbb{J},X)$. Therefore, $C_0\mathcal{A}\Pi_c(\mathbb{J},X)$ is a closed subspace of $C_u(\mathbb{J},X)$.

Finally, we show that $C_0\mathcal{A}\Pi_c(\mathbb{J},X) = \Pi_c(\mathbb{J},X) \oplus C_0(\mathbb{J},X)$. Assuming that for φ from $C_0\mathcal{A}\Pi_c(\mathbb{J},X)$, there are $\psi_i \in \Pi_c(\mathbb{R},X)$ and $\xi_i \in C_0(\mathbb{J},X)$, $i = 1, 2$, such that $\varphi = \psi_i|_{\mathbb{J}} + \xi_i$. Then $0 = (\psi_1 - \psi_2)|_{\mathbb{J}} + (\xi_1 - \xi_2)$. By (2.1), $\|\psi_1 - \psi_2\| \leq 0$. Therefore, $\psi_1 = \psi_2$ and $\xi_1 = \xi_2$. The proof is complete

We do not know whether or not $P_0\mathcal{A}\Pi_c(\mathbb{J},X)$ and $W_0\mathcal{A}\Pi_c(\mathbb{J},X)$ are closed subspaces of $C(\mathbb{J},X)$.

PROPOSITION 2.4. $\mathcal{A}\Pi_r(\mathbb{J},X)$ *is a closed subspace of* $C(\mathbb{J},X)$. *Moreover,* $\mathcal{A}\Pi_r(\mathbb{J},X) = \Pi_r(\mathbb{J},X) \oplus \Pi_0(\mathbb{J},X)$.

PROOF. By Proposition 2.3 the statement holds for the case that $\Pi_0(\mathbb{J},X) = C_0(\mathbb{J},X)$. Now, let $\Pi_0(\mathbb{J},X) = \mathcal{PAP}_0(\mathbb{J},X)$, let $\varphi = \psi|_{\mathbb{J}} + \xi$, where $\psi \in \Pi_r(\mathbb{R},X)$ and $\xi \in \mathcal{PAP}_0(\mathbb{J},X)$. We show that

$$\psi(\mathbb{R}) \subset \overline{\varphi(\mathbb{J})}, \tag{2.2}$$

where $\psi(\mathbb{R})$ denotes the range of ψ and $\overline{\varphi(\mathbb{J})}$ denotes the closure in X of the range of φ. Assume that (2.2) is false. This implies there exists $t_0 \in \mathbb{R}$ such that $\psi(t_0) \not\in \overline{\varphi(\mathbb{J})}$. Since ψ is recurrent, we can assume that $t_0 \in \mathbb{J}$. There exists an $\epsilon > 0$ such that $\inf_{s\in\mathbb{J}} \|\psi(t_0) - \varphi(s)\| > 2\epsilon$. Since ψ is continuous, there exists $\delta > 0$ such that

$$\inf_{s\in\mathbb{J}} \|R_{t_0}\psi(t) - R_{t_0}\varphi(s)\| > \epsilon \quad (|t| < \delta). \tag{2.3}$$

Since $R_{t_0}\psi$ is also recurrent, for $\epsilon > 0$ and $\delta > 0$ there exists $l_{\epsilon/2}$ such that each interval $(x, x + l_{\epsilon/2})$ contains a number τ with the property

$$\|R_{t_0}\psi(t+\tau) - R_{t_0}\psi(t)\| \leq \frac{\epsilon}{2} \quad (|t| < \delta). \tag{2.4}$$

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Let t and τ satisfy (2.4) and let $t+\tau \in \mathbb{J}$. It follows from (2.3) and (2.4) that

$$\begin{aligned}\|R_{t_0}\xi(t+\tau)\| &= \|R_{t_0}\varphi(t+\tau) - R_{t_0}\psi(t+\tau)\| \\ &\geq \|R_{t_0}\varphi(t+\tau) - R_{t_0}\psi(t)\| - \|R_{t_0}\psi(t) - R_{t_0}\psi(t+\tau)\| \\ &\geq \inf_{\tau}\|R_{t_0}\varphi(t+\tau) - R_{t_0}\psi(t)\| - \sup_{\tau}\|R_{t_0}\psi(t) - R_{t_0}\psi(t+\tau)\| \\ &\geq \frac{\epsilon}{2}.\end{aligned}$$

Since each interval $(x, x+l_{\epsilon/2})$ contains a number τ, it follows from inequality above that

$$\lim_{t\to\infty}\frac{1}{t-c}\int_c^t \|\xi(s)\|\,ds \geq \frac{\epsilon\delta}{2l_{\epsilon/2}}.$$

This contradicts the fact that $\xi \in \mathcal{PAP}_0(\mathbb{J},X)$. We have proved (2.2). It follows that $\|\psi\| \leq \|\varphi\|$.

Now, we can proceed exactly as in Proposition 2.3 to show that $\mathcal{A}\Pi_r(\mathbb{J},X)$ is a closed subspace of $C(\mathbb{J},X)$ and $\mathcal{A}\Pi_r(\mathbb{J},X) = \Pi_r(\mathbb{J},X) \oplus \mathcal{PAP}_0(\mathbb{J},X)$.

To show the assertion for the case that $\Pi_0(\mathbb{J},X) = \mathcal{WAP}_0(\mathbb{J},X)$, let $\varphi = \psi|_{\mathbb{J}} + \xi$, where $\psi \in \Pi_r(\mathbb{R},X)$ and $\xi \in \mathcal{WAP}_0(\mathbb{J},X)$. We need only to show that $\|\psi\| \leq \|\varphi\|$. For $x^* \in X^*$, the composition function $x^*\circ\xi$ is in $\mathcal{WAP}_0(\mathbb{J})$. Since $\mathcal{WAP}_0(\mathbb{J}) \subset \mathcal{PAP}_0(\mathbb{J})$ [9, 4.3.13], $x^*\circ\xi \in \mathcal{PAP}_0(\mathbb{J})$. Note that the numerical function $x^*\circ\psi$ is recurrent and $x^*\circ\varphi = x^*\circ\psi + x^*\circ\xi$, it follows that

$$|x^*\circ\psi|_\infty \leq |x^*\circ\varphi|_\infty \leq \|x^*\||\varphi|_\infty \quad (x^*\in X^*).$$

Therefore, $\|\psi\| \leq \|\varphi\|$. The proof is complete.

We remark that the above proposition gives a new proof that $\mathcal{WAP}(\mathbb{J},X) = \mathcal{AP}(\mathbb{J},X) \oplus \mathcal{WAP}_0(\mathbb{J},X)$ [15]; and $\mathcal{AAP}(\mathbb{J},X) = \mathcal{AP}(\mathbb{J},X) \oplus C_0(\mathbb{J},X)$ [18, 19]. As a consequence, we have

COROLLARY 2.5. *The spaces $\mathcal{A}\Pi_r(\mathbb{J},X) \cap C_u(\mathbb{J},X)$, $C_0\mathcal{A}\Pi_c(\mathbb{J},X)$ are Λ-classes.*

Now, we prove

THEOREM 2.6. *Let $\psi \in C_u(\mathbb{R},X)$. Then $\psi|_{\mathbb{J}} \in \mathcal{A}\Pi_r(\mathbb{J},X)\big(C_0\mathcal{A}\Pi_c(\mathbb{J},X)\big)$ if and only if $\psi * f|_{\mathbb{J}} \in \mathcal{A}\Pi_r(\mathbb{J},X)\big(C_0\mathcal{A}\Pi_c(\mathbb{J},X)\big)$ for all $f \in L^1(\mathbb{R})$.*

PROOF. Necessity follows from Proposition 2.1. We show sufficiency. Let $h_n(t) = n/2$ for $|t| \leq 1/n$ and $h_n(t) = 0$ otherwise. Since $\psi \in C_u(\mathbb{R},X)$, we conclude that $h_n*\psi(t) = n/2\int_{-1/n}^{1/n}\psi(t+x)\,dx \longrightarrow \psi(t)$ as $n\longrightarrow\infty$ in $C_u(\mathbb{R},X)$. Hence $\|h_n*\psi|_{\mathbb{J}} - \psi|_{\mathbb{J}}\| \longrightarrow 0$ as $n\longrightarrow\infty$. By assumption, we have $h_n*\psi|_{\mathbb{J}} \in \mathcal{A}\Pi_r(\mathbb{J},X), n\in\mathbb{N}$. By Proposition 2.4, $\mathcal{A}\Pi_r(\mathbb{J},X)\big(C_0\mathcal{A}\Pi_c(\mathbb{J},X)\big)$ is closed and hence the limit $\lim_{n\to\infty} h_n*\psi|_{\mathbb{J}} = \psi|_{\mathbb{J}} \in \mathcal{A}\Pi_r(\mathbb{J},X)\big(C_0\mathcal{A}\Pi_c(\mathbb{J},X)\big)$.

3. **Indefinite Integral of Functions in $\mathcal{A}\Pi_r(\mathbb{J},X)$.** In this section we study the indefinite integral of functions in $\mathcal{A}\Pi_r(\mathbb{J},X)$ $\big(C_0\mathcal{A}\Pi_c(\mathbb{J},X)\big)$. Throughout the section, if $\varphi \in C(\mathbb{J},X)$ then $P\varphi$ will denote the definite integral $P\varphi(t) = \int_0^t \varphi(x)\,dx$. We extend some results obtained in [2, 7, 16, 27].

THEOREM 3.1. *Let $\varphi \in \Pi_0(\mathbb{J},X)$. Then $P\varphi \in \mathcal{A}\Pi_r(\mathbb{J},X)\big(C_0\mathcal{A}\Pi_c(\mathbb{J},X)\big)$ if and only if there exists a constant $k \in X$ such that $P\varphi - k \in \Pi_0(\mathbb{J},X)\big(C_0\mathcal{A}\Pi_c(\mathbb{J},X)\big)$.*

PROOF. We need only to prove the necessity. Let $P\varphi \in \mathcal{A}\Pi_r(\mathbb{J},X)$. Then $P\varphi = \psi|_{\mathbb{J}} + \xi$, where $\psi \in \Pi_r(\mathbb{R},X)$ and $\xi \in \Pi_0(\mathbb{J},X)$. We have

$$\Delta_h P\varphi(t) = P\varphi(t+h) - P\varphi(t) = \int_0^h \varphi(t+x)\,dx$$

and

$$\Delta_h P\varphi(t) = \Delta_h\psi(t)|_{\mathbb{J}} + \Delta_h\xi(t) = [\psi(t+h) - \psi(t)]|_{\mathbb{J}} + [\xi(t+h) - \xi(t)].$$

By Corollary 2.2, we get $\Delta_h P\varphi \in \Pi_0(\mathbb{J},X)$ for all $h \in \mathbb{J}$. Since $\Pi_r(\mathbb{R},X)$ and $\Pi_0(\mathbb{J},X)$ are translation invariant, $\Delta_h\psi|_{\mathbb{J}} \in \Pi_r(\mathbb{J},X)$ and $\Delta_h\xi \in \Pi_0(\mathbb{J},X)$. By the uniqueness of the decomposition of elements of $\mathcal{A}\Pi_r(\mathbb{J},X)$, we get $\Delta_h\psi = 0$ and $\Delta_h\xi(t) = \int_0^h \varphi(t+x)\,dx$. Since ψ is continuous, we conclude that $\psi = k \in X$.

The case $C_0\mathcal{A}\Pi_c(\mathbb{J},X)$ can be proved in the same way.

We remark that Theorem 3.1 gives a new proof of [27, Theorem 7] first proved by Zhang for the case that $\Pi_r(\mathbb{R},X) = \mathcal{AP}(\mathbb{R},X)$ and $\Pi_0(\mathbb{J},X) = \mathcal{PAP}_0(\mathbb{J},X)$. As a consequence of Theorem 3.1, we have

COROLLARY 3.2. *Let $\varphi = \psi|_{\mathbb{J}} + \xi \in \mathcal{A}\Pi_r(\mathbb{J},X)$, where $\psi \in \Pi_r(\mathbb{R},X)$ and $\xi \in \Pi_0(\mathbb{J},X)$. Then $P\varphi \in \mathcal{A}\Pi_r(\mathbb{J},X)$ if and only if $P\psi|_{\mathbb{J}} \in \Pi_r(\mathbb{J},X)$ and there exists $k \in X$ such that $P\xi - k \in \Pi_0(\mathbb{J},X)$.*

PROOF. We need only to prove the necessity. Let $P\varphi = \zeta|_{\mathbb{J}} + \eta$, where $\zeta \in \Pi_r(\mathbb{R},X)$ and $\eta \in \Pi_0(\mathbb{J},X)$. Then, for $h \in \mathbb{J}$,

$$\begin{aligned}\Delta_h P\varphi(t) &= \Delta_h\zeta|_{\mathbb{J}}(t) + \Delta_h\eta(t)\\ &= \Delta_h P\psi|_{\mathbb{J}}(t) + \Delta_h P\xi(t)\\ &= \int_0^h \psi(t+x)\,dx + \int_0^h \xi(t+x)\,dx.\end{aligned}$$

Similarly, we have that $\Delta_h\zeta|_{\mathbb{J}}, \int_0^h \psi(t+x)\,dx \in \Pi_r(\mathbb{J},X)$ and $\Delta_h\eta, \int_0^h \xi(t+x)\,dx \in \Pi_0(\mathbb{J},X)$. By the uniqueness of the decomposition, we get $\Delta_h(\eta - P\xi) = 0$. This implies that there exists $k \in X$ such that $P\xi = \eta + k$. Hence $P\psi|_{\mathbb{J}} = \zeta|_{\mathbb{J}} - k$ and $P\psi|_{\mathbb{J}} \in \Pi_r(\mathbb{J},X)$. Corollary 3.2 holds also for the case $\varphi \in C_0\mathcal{A}\Pi_c(\mathbb{J},X)$.

Now, we study sufficient conditions for $P\varphi$ in $\mathcal{A}\Pi_r(\mathbb{J},X)$. We have

THEOREM 3.3. *Let $\varphi \in \mathcal{A}\Pi_r(\mathbb{J},X)\big(C_0\mathcal{A}\Pi_c(\mathbb{J},X)\big)$ and $P\varphi \in \mathcal{E}(\mathbb{J},X)$. Then $P\varphi \in \mathcal{A}\Pi_r(\mathbb{J},X)\big(C_0\mathcal{A}\Pi_c(\mathbb{J},X)\big)$.*

PROOF. Let $\varphi = \psi + \xi$ with $\psi \in \Pi_r(\mathbb{J},X)$ and $\xi \in \Pi_0(\mathbb{J},X)$. By Corollary 2.2, $\Delta_h P\xi \in \Pi_0(\mathbb{J},X) \cap C_u(\mathbb{J},X)$ and by [2, Lemma 1.2.1], $\Delta_h P\psi \in L(\varphi) \subset \Pi_r(\mathbb{R},X)$ for all $h \in \mathbb{R}$.

This implies that $\Delta_h P\varphi \in \mathcal{A}\Pi_r(\mathbb{J},X) \cap C_u(\mathbb{J},x)$. By Corollary 2.5 $\mathcal{A}\Pi_r(\mathbb{J},X) \cap C_u(\mathbb{R},X)$ is a Λ-class. Since $P\varphi \in \mathcal{E}(\mathbb{R},X), P\varphi \in \mathcal{A}\Pi_r(\mathbb{J},X) \cap C_u(\mathbb{J},X)$ by [2, Theorem 3.1.2]. By Corollary 2.5, $C_0\mathcal{A}\Pi_c(\mathbb{J},X)$ is a Λ-class, hence the statement follows directly from [2, Theorem 3.1.1].

4. **Harmonic Analysis of $\mathcal{A}\Pi_r(\mathbb{J},X)$ and Solutions of Linear Differential Equations.** In this section we introduce the spectrum $\mathrm{sp}_{\mathcal{A}\Pi_r}(\varphi)$ of the function $\varphi \in C(\mathbb{R},X)$ with respective to $\mathcal{A}\Pi_r(\mathbb{J},X)\big(C_0\mathcal{A}\Pi_c(\mathbb{J},X)\big)$. This kind of spectrum proved to be a useful tool to examine solutions of differential equations of many function classes [2, 5, 8, 17 Ch.6]. See also the recent works [4], [20], [23]. We apply the results on spectrum to the differences of functions and linear differential equations on the half line. Since the proofs for the case $C_0\mathcal{A}\Pi_c(\mathbb{J},X)$ are the same as in the case $C_0\mathcal{A}\Pi_r(\mathbb{J},X)$ we restrict ourselves to $\mathcal{A}\Pi_r(\mathbb{J},X)$.

Let $\varphi \in C(\mathbb{R},X)$. Denote by $I_{\mathcal{A}\Pi_r}(\varphi) = \{f \in L^1(\mathbb{R}) : f * \varphi|_{\mathbb{J}} \in \mathcal{A}\Pi_r(\mathbb{J},X)\}$. Since $\mathcal{A}\Pi_r(\mathbb{J},X)$ is a closed subspace of $C(\mathbb{J},X)$, $I_{\mathcal{A}\Pi_r}(\varphi)$ is a closed ideal of $L^1(\mathbb{R})$. We set

$$\mathrm{sp}_{\mathcal{A}\Pi_r}(\varphi) = \text{ hull } I_{\mathcal{A}\Pi_r}(\varphi) = \{\lambda \in \mathbb{R} : \hat{f}(\lambda) = 0, f \in I_{\mathcal{A}\Pi_r}(\varphi)\}. \tag{4.1}$$

We denote by $\mathrm{sp}(\varphi)$ the Beurling spectrum of φ [21, p138]. Similar to the corresponding properties of Beurling spectrum, the following can be proved exactly as in [2].

PROPOSITION 4.1. *Let $\varphi \in C(\mathbb{R},X)$ and $f \in L^1(\mathbb{R})$. Then*

(a) $\mathrm{sp}_{\mathcal{A}\Pi_r}(\varphi) \subset \mathrm{sp}(\varphi)$;

(b) $\mathrm{sp}_{\mathcal{A}\Pi_r}(\varphi * f) \subset \mathrm{sp}_{\mathcal{A}\Pi_r}(\varphi) \cap \mathrm{supp}\hat{f}; f \in L^1(\mathbb{R})$ *such that* $f \in I_{\mathcal{A}\Pi_r}(\varphi)$ *and* $\hat{f}(\lambda) \neq 0$.

(c) Let $\varphi \in C_u(\mathbb{R},X)$. Then $\varphi|_{\mathbb{J}} \in \mathcal{A}\Pi_r(\mathbb{J},X)$ if and only if $\mathrm{sp}_{\mathcal{A}\Pi_r}(\varphi) = \emptyset$.

THEOREM 4.2. *Let $\varphi \in C_u(\mathbb{R},X)$ and let* $\mathrm{sp}(\varphi)$ *be separated from zero.*

(a) If $\Delta_h\varphi|_{\mathbb{J}} = (R_h\varphi - \varphi)|_{\mathbb{J}} \in \mathcal{A}\Pi_r(\mathbb{J},X)$ for all $h \in \mathbb{R}$, then $\varphi|_{\mathbb{J}} \in \mathcal{A}\Pi_r(\mathbb{J},X)$.

(b) If $\varphi|_{\mathbb{J}} \in \mathcal{A}\Pi_r(\mathbb{J},X)$, then $P\varphi|_{\mathbb{J}} \in \mathcal{A}\Pi_r(\mathbb{J},X) \cap C_u(\mathbb{J},X)$.

PROOF. Corollary 2.5, $\mathcal{A}\Pi_r(\mathbb{J},X) \cap C_u(\mathbb{J},X)$ is a Λ-class. Therefore (a) follows directly from [2, Theorem 4.2.4].

(b) By [3, Corollary 4.4] $P\varphi$ is a bounded uniformly continuous function and there exists $k \in X$ such that $\mathrm{sp}(P\varphi + k)$ is isolated from zero. By Corollary 2.2, $\Delta_h P\varphi|_{\mathbb{J}} \in \mathcal{A}\Pi_r(\mathbb{J},X)$ for all $h \in \mathbb{R}$. Therefore, $P\varphi|_{\mathbb{J}} \in \mathcal{A}\Pi_r(\mathbb{J},X)$ by (a).

Now we study the linear differential equation

$$y'' + a_1 y' + a_0 y = \varphi \quad \big(\varphi \in \mathcal{A}\Pi_r(\mathbb{J},X)\big). \tag{4.2}$$

We give sufficient conditions under which the bounded solutions of (4.2) is in $\mathcal{A}\Pi_r(\mathbb{J},X)$.

THEOREM 4.3. *If $a_0, a_1 \in \mathbb{C}$ in (4.2) are such that $(i\alpha)^2 + i\alpha a_1 + a_0 \neq 0$ for all $\alpha \in \mathbb{R}$ and if y is a bounded solution of (4.2), then $y \in \mathcal{A}\Pi_r(\mathbb{J},X)$.*

PROOF. If $\mathbb{J} = \mathbb{R}^+$, we define $Y(t) = y(t)$ for $t \geq 0$ and $Y(t) = y(0)\cos(1 - \cos t) + y'(0)\sin t + y''(0)(\sin^2 t)/2$ for $t < 0$. It is easy to verify that Y is a solution of the differential equation:

$$Y'' + a_1 Y' + a_0 Y = \psi \tag{4.3}$$

where $\psi(t) = \varphi(t)$ for $t \geq 0$ and $\psi(t) = Y'' + a_1 Y' + a_0 Y$ for $t < 0$. Since Y is bounded, by Landau-Esclangon lemma we conclude that Y', Y'' are also bounded (see [17, Proposition 4, p.95]). This implies that Y is uniformly continuous. It is easy to check that $\psi \in C(\mathbb{R}, X)$. We prove that $\mathrm{sp}_{\mathcal{A}\Pi_r}(Y) = \emptyset$. Indeed, let $\alpha \in \mathbb{R}$, there exists $f \in S(\mathbb{R})$ such that $\hat{f}(\alpha) \neq 0$. We have $f*Y'' + f*a_1 Y' + f*a_0 Y = f*\psi$. Hence $Y*[f'' + a_1 f' + a_0 f] = \psi*f$. We have $(f'' + a_1 \widehat{f'} + a_0 f)(\alpha) = [(i\alpha)^2 + i\alpha a_1 + a_0]\hat{f}(\alpha)$. From the assumptions we conclude that the function $g := f'' + a_1 f' + a_0 f$ satisfies $\hat{g}(\alpha) \neq 0$. From (4.1), it follows that $\alpha \notin \mathrm{sp}_{\mathcal{A}\Pi_r}(Y)$. Since α is arbitrary, we get $\mathrm{sp}_{\mathcal{A}\Pi_r}(Y) = \emptyset$. By Proposition 4.1 (c), $y = Y|_{\mathbb{J}} \in \mathcal{A}\Pi_r(\mathbb{J}, X)$.

In the same way, we have

COROLLARY 4.4. *Let $\varphi \in \mathcal{A}\Pi_r(\mathbb{J}, X)$. Let $\lambda = u + iv$ with $u \neq 0$. Then every bounded solution of the equation $y' = \lambda y + \varphi$ belongs to $\mathcal{A}\Pi_r(\mathbb{J}, X)$.*

THEOREM 4.5. *If $\lambda = u + iv \in \mathbb{C}$ with $u > 0$ and $\varphi \in \mathcal{A}\Pi_r(\mathbb{J}, X)$, then the solutions of the equation*

$$\frac{dy}{dx} = \lambda y + \varphi \tag{4.4}$$

of the form $y(x) = e^{\lambda x}[c + \int_0^x e^{-\lambda t}\varphi(t)\,dt]$ are bounded on $\mathbb{R}^+$. Furthermore, if $\mathbb{J} = \mathbb{R}^+$, then $y \in \mathcal{A}\Pi_r(\mathbb{R}^+, X)$ and $\|y\| \leq u^{-1}\|\varphi\|$. If $\mathbb{J} = \mathbb{R}$, then (4.4) has a unique bounded solution defined on $\mathbb{R}$ if and only if $y(0) = -\int_0^\infty e^{-\lambda t}\varphi(t)\,dt$. In this case the function $y_0(x) = -\int_x^\infty e^{\lambda(x-t)}\varphi(t)\,dt$ is a bounded solution from $\mathcal{A}\Pi_r(\mathbb{R}, X)$.

PROOF. Direct verification shows that

$$y(x) = e^{\lambda x}[c + \int_0^x e^{-\lambda t}\varphi(t)\,dt], \tag{4.5}$$

is a bounded solution of (4.4). If $\mathbb{J} = \mathbb{R}^+$, then $y \in \mathcal{A}\Pi_r(\mathbb{R}^+, X)$ by Corollary 4.4. Since $u > 0, |e^{\lambda x}| = e^{ux} \longrightarrow \infty$ as $x \longrightarrow \infty$. For y to be bounded on $\mathbb{R}$, we must have $c + \int_0^x e^{-\lambda t}\varphi(t)\,dt \longrightarrow 0$ as $x \longrightarrow \infty$. This means that we must take in (4.5) $c = -\int_0^\infty e^{-\lambda t}\varphi(t)\,dt$. We note that the improper integral in (4.5) is convergent since $|e^{-\lambda t}\varphi(t)| \leq \|\varphi\| e^{-ut}$ for $t \geq 0$. Thus, the unique bounded solution of equation (4.4) can only be $y_0(x) = -\int_x^\infty e^{\lambda(x-t)}\varphi(t)\,dt$, and we do have $\|y_0(x)\| \leq \|\varphi\| e^{ux}\int_x^\infty e^{-ut}\,dt = \|\varphi\|/u$, so that y_0 is bounded. By Corollary 4.4, $y_0 \in \mathcal{A}\Pi_r(\mathbb{R}, X)$.

When $u < 0$, we have that $y_0(x) = \int_{-\infty}^x e^{\lambda(t-x)}\varphi(t)\,dt$ is the unique bounded solution of (4.3) and $y_0 \in \mathcal{A}\Pi_r(\mathbb{R}, X)$. At the same time we have the similar estimate $\|y_0(x)\| \leq \|\varphi\|/|u|$.

THEOREM 4.6. *Let $\psi \in C(\mathbb{R}, X)$ and $\psi|_{\mathbb{J}} \in \mathcal{A}\Pi_r(\mathbb{J}, X)$. Let $\varphi(t) = e^{-ivt}\psi(t)$ with $v \in \mathbb{R}$ and let the Beurling spectrum of φ be separated from zero. Then all solutions of equation (4.3) belong to $\mathcal{A}\Pi_r(\mathbb{J}, X) \cap C_u(\mathbb{J}, X)$.*

PROOF. (4.4) has general solution $y(x) = e^{ivx}[c + \int_0^x e^{-ivt}\psi(t)\,dt]$, where c is an arbitrary element of X. By Theorem 4.2 (b), we conclude $\int_0^x e^{-ivt}\psi(t)\,dt]|_{\mathbb{J}}$ is in $\mathcal{A}\Pi_r(\mathbb{J}, X) \cap C_u(\mathbb{J}, X)$. Since $\mathcal{A}\Pi_r(\mathbb{J}, X)$ contains all constant functions and invariant under multiplication by characters, $y \in \mathcal{A}\Pi_r(\mathbb{J}, X) \cap C_u(\mathbb{J}, X)$.

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5. **Asymptotically and Pseudo Almost Automorphic Functions.** In this section we will concentrate two new, specific $\mathcal{A}\Pi_r(\mathbb{J},X)$ spaces. That is the space of asymptotically almost automorphic introduced in [2] and the space of pseudo almost automorphic functions defined below.

DEFINITION 5.1. A function $\psi \in C(\mathbb{J},X)$ is called *pseudo (asymptotically) almost automorphic* if $\psi = \varphi|_{\mathbb{J}} + \xi$, where $\varphi \in \mathcal{AA}(\mathbb{R},X)$ and $\xi \in \mathcal{PAP}_0(\mathbb{J},X)\big(C_0(\mathbb{J},X)\big)$. The functions φ and ξ are called the almost automorphic component and the ergodic perturbation respectively of the function ψ. Denote by $\mathcal{PAA}(\mathbb{J},X)\big(\mathcal{AAA}(\mathbb{J},X)\big)$ the set of all such functions ψ.

By Theorem 2.1.1 and Proposition 2.1.2 in [2], $\mathcal{AA}(\mathbb{R},X)$ satisfies (1.5)–(1.8). Therefore, $\mathcal{AA}(\mathbb{R},X)$ is a $\Pi_r(\mathbb{R},X)$ space and $\mathcal{PAA}(\mathbb{J},X)$ and $\mathcal{AAA}(\mathbb{J},X)$ are $\mathcal{A}\Pi_r(\mathbb{J},X)$ spaces. It follows from Proposition 2.4 that

$$\mathcal{PAA}(\mathbb{J},X) = \mathcal{AA}(\mathbb{R},X)|_{\mathbb{J}} \oplus \mathcal{PAP}_0(\mathbb{J},X),$$
$$\mathcal{AAA}(\mathbb{J},X) = \mathcal{AA}(\mathbb{R},X)|_{\mathbb{J}} \oplus C_0(\mathbb{J},X)$$

and

$$\varphi(\mathbb{R}) \subset \overline{\psi(\mathbb{J})}, \quad \|\varphi\| \le \|\psi\|. \tag{5.1}$$

THEOREM 5.2. *The following statements hold.*

(1) A function $\xi \in C\mathbb{R}$ is in $\mathcal{PAP}_0(\mathbb{R})\big(C_0(\mathbb{R})\big)$ if and only if ξ^2 is.

(2) $\Xi \in C(\mathbb{R})^n$ is in $\mathcal{PAP}_0(\mathbb{R})^n\big(C_0(\mathbb{R})^n\big)$ if and only if the norm function $|\Xi(\cdot)|$ is in $\mathcal{PAP}_0\mathbb{R}\big(C_0(\mathbb{R})^n\big)$.

PROOF. We show the theorem only for the case of $\mathcal{PAP}_0(\mathbb{R})$. The case of $C_0(\mathbb{R})$ is similar.

(1) The sufficiency follows since

$$\frac{1}{2t}\int_{-t}^{t} |\xi(x)|/\,dx \le \frac{1}{2t}\Big[\int_{-t}^{t} |\xi(x)|^2\,dx\Big]^{1/2}\Big[\int_{-t}^{t} 1\,dx\Big]^{1/2} = \Big[\frac{1}{2t}\int_{-t}^{t} |\xi(x)|^2\,dx\Big]^{1/2}.$$

The necessity follows from the fact that $\mathcal{PAP}_0(\mathbb{R})$ is an ideal of $C(\mathbb{R})$.

(2) By (1), $\Xi = (\xi_1,\xi_2,\cdots,\xi_n) \in \mathcal{PAP}_0(\mathbb{R})^n$ if and only if $\xi_i\overline{\xi_i} \in \mathcal{PAP}_0(\mathbb{R}), i = 1,2,\cdots,n$. The latter is equivalent to that $|\Xi(\cdot)|^2 = \sum_{i=1}^n |\xi_i(\cdot)|^2 \in \mathcal{PAP}_0(\mathbb{R})$, which, again by (1), is equivalent to that $|\Xi(\cdot)| \in \mathcal{PAP}_0(\mathbb{R})$.

Let $\Omega \subset \mathbb{C}^n$ be compact and define

$$\mathcal{PAP}_0(\Omega\times\mathbb{R}) = \mathcal{PAP}_0\big(\mathbb{R}, C(\Omega)\big), C_0(\Omega\times\mathbb{R}) = C_0\big(\mathbb{R}, C(\Omega)\big)$$

and

$$\mathcal{AA}(\Omega\times\mathbb{R}) = \mathcal{AA}\big(\mathbb{R}, C(\Omega)\big).$$

A function $\varphi \in \mathcal{AA}(\Omega\times\mathbb{R})$ is called almost automorphic in $t \in \mathbb{R}$ uniformly in $Z \in \Omega$. $\mathcal{PAA}(\Omega\times\mathbb{R})\big(\mathcal{AAA}(\Omega\times\mathbb{R})\big)$ is defined to consist of functions ψ such that

$$\psi = \varphi + \xi \quad \Big(\varphi \in \mathcal{AA}(\Omega\times\mathbb{R}), \xi \in \mathcal{PAP}_0(\Omega\times\mathbb{R})\big(C_0(\Omega\times\mathbb{R})\big)\Big).$$

The following lemma is straight forward.

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LEMMA 5.3. *Let* $\Phi = (\varphi_1, \varphi_2, \cdots, \varphi_n) \in C(\mathbb{R})^n$. *Then* $\Phi \in \mathcal{AA}(\mathbb{R}, \mathbb{C}^n)$ *if and only if* $\Phi \in \mathcal{AA}(\mathbb{R})^n$.

For $H = (h_1, h_2, \cdots, h_n) \in C(\mathbb{R})^n$, suppose that $H(t) \in \Omega$ for all $t \in \mathbb{R}$. Define

$$H \times \iota : \mathbb{R} \to \Omega \times \mathbb{R} \text{ by } H \times \iota(t) = \big(h_1(t), h_2(t), \cdots, h_n(t), t\big) \quad (t \in \mathbb{R}).$$

For $\Psi = (\psi_1, \psi_2, \cdots, \psi_n) \in \mathcal{PAA}(\mathbb{R})^n\big(\mathcal{AAA}(\mathbb{R})^n\big)$, let $\Phi = (\varphi_1, \varphi_2, \cdots, \varphi_n)$ and $\Xi = (\xi_1, \xi_2, \cdots, \xi_n)$, where φ_i and ξ_i are the almost automorphic component and the ergodic perturbation respectively of ψ_i, $i = 1, 2, \cdots, n$.

The following theorem generalizes [26, Theorem 1.5]

THEOREM 5.4. *Let* $\psi \in \mathcal{PAA}\big(\mathbb{J}, C(\Omega)\big)\Big(\mathcal{AAA}\big(\mathbb{J}, C(\Omega)\big)\Big)$. *If* $\Psi \in \mathcal{PAA}(\mathbb{J})^n\big(\mathcal{AAA}(\mathbb{J})^n\big)$ *and* $\Psi(\mathbb{J}) \subset \Omega$, *then* $\psi \circ (\Psi \times \iota) \in \mathcal{PAA}(\mathbb{J})\big(\mathcal{AAA}(\mathbb{J})\big)$.

PROOF. It is sufficient to prove the case $\mathbb{J} = \mathbb{R}$, $\psi \in \mathcal{PAA}\big(\mathbb{J}, C(\Omega)\big)$, and $\Psi \in \mathcal{PAA}(\mathbb{J})^n$.

Let $\psi = \varphi + \xi$ and $\Psi = \Phi + \Xi$ with $\Phi = (\varphi_1, \varphi_2, \cdots, \varphi_n) \in \mathcal{AA}(\mathbb{R})^n$, as above. Since $\Psi(t) \in \Omega$ when $t \in \mathbb{R}$, it follows from (5.1) that $\Phi(t) \in \Omega$ for $t \in \mathbb{R}$. Note that

$$\psi \circ (\Psi \times \iota) = \varphi \circ (\Psi \times \iota) + \xi \circ (\Psi \times \iota) = \varphi \circ (\Phi \times \iota) + [\varphi \circ (\Psi \times \iota) - \varphi \circ (\Phi \times \iota) + \xi \circ (\Psi \times \iota)].$$

We show that $\varphi \circ (\Phi \times \iota) \in \mathcal{AA}(\mathbb{R})$. Let $\epsilon > 0$. Since $\varphi \in C_u(\Omega \times \mathbb{R})$, there is a $\delta, \epsilon/2 > \delta > 0$ such that

$$|\varphi(Z_1, t) - \varphi(Z_2, t)| < \frac{\epsilon}{2} \quad (Z_1, Z_2 \in \Omega, |Z_1 - Z_2| < \delta; t \in \mathbb{R}). \tag{5.2}$$

Note Lemma 5.3 and the hypothesis that $\Phi \in \mathcal{AA}(\mathbb{R})^n$ and $\varphi \in \mathcal{AA}\big(\mathbb{R}, C(\Omega)\big)$, we conclude $(\varphi, \Phi) \in \mathcal{AA}\big(\mathbb{R}, C(\Omega) \times \mathbb{R}^n\big)$. From the identity

$$\begin{aligned} \varphi\big(\Phi(t+\tau), t+\tau\big) - \varphi\big(\Phi(t), t\big) &= \varphi\big(\Phi(t+\tau), t+\tau\big) \\ &- \varphi\big(\Phi(t), t+\tau\big) + \varphi\big(\Phi(t), t+\tau\big) - \varphi\big(\Phi(t), t\big) \end{aligned} \tag{5.3}$$

we conclude $E\big(\delta, N, (\varphi, \Phi)\big) \subset E\big(\delta, N, (\varphi \circ \Phi \times \iota)\big)$. This implies $\varphi \circ (\Phi \times \iota) \in \mathcal{AA}(\mathbb{R})$ by Proposition 1.1.

To finish the proof, we need to show that the function $h = \varphi \circ (\Psi \times \iota) - \varphi \circ (\Phi \times \iota) + \xi \circ (\Psi \times \iota)$ is in $\mathcal{PAP}_0(\mathbb{R})$. First we show that $\varphi \circ (\Psi \times \iota) - \varphi \circ (\Phi \times \iota) \in \mathcal{PAP}_0(\mathbb{R})$. It is trivial in the case that $\varphi = 0$. So we assume that $\varphi \neq 0$. Let ϵ and δ be as in (5.2). Set

$$C_\delta = \{t \in \mathbb{R} : |\Psi(t) - \Phi(t)| = |\Xi(t)| \geq \delta\}. \tag{5.4}$$

It follows from [27, Definition 3 and Proposition 4] that there is a $T > 0$ such that when $t \geq T$

$$\frac{m([-t, t] \cap C_\delta)}{2t} < \frac{\epsilon}{4\|\varphi\|}, \tag{5.5}$$

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where m stands for the Lebesgue measure on $\mathbb{R}$. It follows from (5.2), (5.4) and (5.5) that

$$\begin{aligned}\frac{1}{2t}\int_{-t}^{t}|\varphi\big(\Psi(s),s\big)-\varphi\big(\Phi(s),s\big)|\,ds \\ &= \frac{1}{2t}\left\{\int_{[-t,t]\setminus C_\delta}+\int_{[-t,t]\cap C_\delta}|\varphi\big(\Psi(s),s\big)-\varphi\big(\Phi(s),s\big)|\,ds\right\} \\ &\leq \frac{\epsilon}{2}+2\|\varphi\|\frac{m([-t,t]\cap C_\delta)}{2t}<\epsilon.\end{aligned}$$

Therefore, $\varphi\circ(\Psi\times\iota)-\varphi\circ(\Phi\times\iota)\in\mathcal{PAP}_0(\mathbb{R})$.

A little modification of the proof for [26, Theorem 1.5] shows that $\xi\circ(\Psi\times\iota)\in\mathcal{PAP}_0(\mathbb{R})$. The proof is complete.

THEOREM 5.5. *Consider systems of the form*

$$\frac{dY}{dx}=AY+\Psi, \tag{5.6}$$

where $A=(a_{ij})$ *is a complex* $n\times n$ *matrix and* $\Psi=(\psi_1,\psi_2,\cdots,\psi_n)'\in\mathcal{PAA}(\mathbb{R})^n$ $\big(\mathcal{AAA}(\mathbb{R})^n\big)$. *If the matrix* $A=(a_{ij})$ *has no eigenvalues with real part zero, then system (5.6) admits a unique solution* $Y=(y_1,y_2,\cdots,y_n)'\in\mathcal{PAA}(\mathbb{R})^n\big(\mathcal{AAA}(\mathbb{R})^n\big)$. *Moreover*

$$\|Y\|\leq K\|\Psi\|, \tag{5.7}$$

where $K>0$ *depends only on the matrix* A.

PROOF. By a discussion in [11, Theorem 4.2] the matrix A can be considered triangular. Therefore the theorem follows by applying Theorem 4.5 n times.

Now, consider a system of the form

$$\frac{dY}{dx}=AY+\Psi+\mu G\circ(Y\times\iota), \tag{5.8}$$

where $\mu\in\mathbb{C}\setminus\{0\}$, A is a complex $n\times n$ matrix, $\Psi\in\mathcal{PAA}(\mathbb{R})^n\big(\mathcal{AAA}(\mathbb{R})^n\big)$, and $G\in\mathcal{PAA}(\Omega\times\mathbb{R})^n\ \big(\mathcal{AAA}(\Omega\times\mathbb{R})^n\big)$. Such a system is called quasi-linear. We get the generating system of (5.8) by putting $\mu=0$.

As in the proof for [26, Theorem 2.3], by using Theorems 5.4 and 5.5 one shows the following theorem.

THEOREM 5.6. *Let* Ψ *and* A *be as in Theorem 5.5. Let* $Y^{(0)}$ *be the unique solution in* $\mathcal{PAA}(\mathbb{R})^n\big(\mathcal{AAA}(\mathbb{R})^n\big)$ *of the generating system of (5.8), let* $a>0$ *and let* $\Omega=\bigcup\{Z\in\mathbb{C}^n:|Z-Y^{(0)}(t)|\leq a,t\in\mathbb{R}\}$. *Assume that*

(1) $G\in\mathcal{PAA}(\Omega\times\mathbb{R})^n\ \big(\mathcal{AAA}(\Omega\times\mathbb{R})^n\big)$ *such that*

$$|G(Z',t)-G(Z'',t)|\leq L\Sigma_{i=1}^n|z_i'-z_i''|,\quad (Z',Z''\in\Omega,t\in\mathbb{R}), \tag{5.9}$$

where $L>0$;

(2) $0<|\mu|<\min\{1/LK,a/K\|G\|\}$, *where* $K>0$ *in (5.7) depends only on the matrix* A.

Then there exists a unique solution $Y=(y_1,y_2,\cdots,y_n)'\in\mathcal{PAA}(\mathbb{R})^n\ \big(\mathcal{AAA}(\mathbb{R})^n\big)$ *of the system (5.8) such that* $Y(x)\in\Omega$ *for all* $x\in\mathbb{R}$. *Furthermore,* $\|Y-Y^{(0)}\|\longrightarrow 0$ *as* $\mu\longrightarrow 0$.

6. **The Solutions of non-linear Parabolic Partial Differential Equations.** For $T > 0$ and $A > 0$, let $\triangle = \mathbb{R} \times [0, T]$ and $\triangle_1 = \triangle \times [-A, A]$. Consider the non-linear parabolic partial differential equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + \psi(x, t, u) \quad \big((x, t) \in \triangle\big), \tag{6.1}$$

where $\psi \in C(\triangle_1)$ satisfies the Lipschitz condition

$$|\psi(x, t, v') - \psi(x, t, v'')| \leq l|v' - v''| \quad \big((x, t) \in \triangle, v', v'' \in [-A, A]). \tag{6.2}$$

In this section, we shall use some knowledge from ordinary differential equations to establish some properties of solutions of (6.1). For this purpose, we first consider the following ordinary differential equation

$$\frac{d^2 y}{dx^2} - \alpha^2 y = r, \tag{6.3}$$

where $\alpha > 0$ and $r \in \mathcal{C}(\mathbb{R})$. Note that (6.3) admits a unique bounded solution given by

$$y_0(x) = -\frac{1}{2\alpha}\left\{e^{\alpha x}\int_x^{\infty} e^{-\alpha t} r(t)\, dt + e^{-\alpha x}\int_{-\infty}^{x} e^{\alpha t} r(t)\, dt\right\}, \tag{6.4}$$

for which

$$\|y_0\| \leq \frac{1}{\alpha^2}\|r\|. \tag{6.5}$$

It follows from Theorem 4.3 that y_0 in (6.4) is in $\mathcal{PAA}(\mathbb{R})\big(\mathcal{AAA}(\mathbb{R})\big)$ if r is.

Let us now consider the system of ordinary differential equations

$$\frac{d^2 u_k}{dx^2} = h^{-1}[u_k - u_{k-1}] + \psi(x, t_k, u_{k-1}), \quad k = 1, 2, \cdots, n, \tag{6.6}$$

where $h = T/n \in \mathbb{R}$, $t_k = kh$ and $u_0(x) = u(x, 0)$, where u is a solution of (6.1). We need to consider (6.6) for variable n; note that the partition $\{t_k\}_{k=0}^{n}$ of $[0, T]$ depends on n, as do all the functions in a solution $(u_1, u_2, \cdots, u_n)$ of (6.6).

As in the proof for [26, Lemma 3.1], one shows the following lemma.

LEMMA 6.1. *Let $\psi \in \mathcal{PAA}(\triangle_1)(\mathcal{AAA}(\triangle_1))$ satisfy the Lipschitz condition (6.2). Suppose u is a solution of equation (6.1) such that $\|u\| < A$, u and $\partial u/\partial t$ are uniformly continuous, and $u_0 = u(\cdot, 0) \in \mathcal{PAA}(\mathbb{R})$ $\big(\mathcal{AAA}(\mathbb{R})\big)$. Then there exist $n_0 \in \mathbb{N}$ and $W \geq 0$ such that (6.6) has a unique solution $(u_1, u_2, \cdots, u_n) \in \mathcal{PAA}(\mathbb{R})^n$ $\big(\mathcal{AAA}(\mathbb{R})^n\big)$ for $n \geq n_0$; it satisfies*

(1) $\|u_k\| \leq A$, $1 \leq k \leq n$, and

(2) the functions $\varepsilon_0 = 0$, $\varepsilon_k = u(\cdot, t_k) - u_k$ are in $\mathcal{C}(\mathbb{R})$ with

$$\|\varepsilon_k\| \leq W\omega(h), \tag{6.7}$$

where $h = T/n, t_k = kh$, and ω is the modulus of uniform continuity of u and $\partial u/\partial t$.

THEOREM 6.2. *Let ψ and u satisfy the conditions of Lemma 6.1. Then u is in $\mathcal{PAA}(\triangle)\big(\mathcal{AAA}(\triangle)\big)$.*

PROOF. We only show the assertion that $u \in \mathcal{PAA}(\triangle)$. Similarly, one shows the assertion that $u \in \mathcal{AAA}(\triangle)$. Let n_0 be as in Lemma 6.1, and fixed $n \geq n_0$ and $t \in [0, T]$. Then there is a $k_0 \leq n$ such that $|t - t_{k_0}| < h$. Recall that $h = T/n$ and $t_k = kh$. If $B_n = \{u_k : k = 1, 2, \cdots, n\} \subset \mathcal{PAA}(\mathbb{R})$ is the solution of (6.6) given by Lemma 6.1, the uniform continuity of u gives

$$\begin{aligned} |u(x,t) - u_{k_0}(x)| &\leq |u(x,t) - u(x,t_{k_0})| + |u(x,t_{k_0}) - u_{k_0}(x)| \\ &< (1+W)\omega(h) \quad (x \in \mathbb{R}). \end{aligned} \tag{6.8}$$

It follows that the function $u(\cdot, t)$ is in the norm closure of $\bigcup_{n=n_0}^{\infty} B_n$; hence $u(\cdot, t) \in \mathcal{PAA}(\mathbb{R})$ (Proposition 2.4).

Let $\varphi(\cdot, t)$ and $\xi(\cdot, t)$, φ_k and ξ_k be the almost automorphic components and the ergodic perturbations respectively of $u(\cdot, t), u_k, k = 1, 2, \cdots, n$. To show that $u \in \mathcal{PAA}(\triangle)$, we need to prove that $\varphi \in \mathcal{AA}(\triangle)$ and $\xi \in \mathcal{PAP}_0(\triangle)$.

Since $u(\cdot, t'), u(\cdot, t'') \in \mathcal{PAA}(\mathbb{R})$ for any $t', t'' \in [0, T]$, so is $u(\cdot, t') - u(\cdot, t'')$ and also $u(\cdot, t) - u_k$ for $t \in [0, T]$ and $k = 1, 2, \cdots, n$. It follows from (5.1) that

$$\|\varphi(\cdot, t') - \varphi(\cdot, t'')\| \leq \|u(\cdot, t') - u(\cdot, t'')\| \tag{6.9}$$

and

$$\|\varphi(\cdot, t) - \varphi_k\| \leq \|u(\cdot, t) - u_k\| \quad k = 1, 2, \cdots, n. \tag{6.10}$$

We show that $\varphi \in \mathcal{AA}(\triangle)$. Let $\epsilon > 0$. Choose $n \geq n_0$ such that $h = T/n$ implies

$$4(W+1)\omega(h) < \epsilon. \tag{6.11}$$

Let $\Phi = \{\varphi_1, \varphi_2, \cdots, \varphi_n\}$. By Lemma 5.3, $\Phi \in \mathcal{AA}(\mathbb{R}, \mathbb{C}^n)$. Let $N > 0$. It follows from (6.7), (6.9), (6.10) and (6.11) that, for $|x| < N, \tau \in E(\epsilon/2, N, \Phi)$ and $t \in [0, T]$, there exists a t_{k_0} such that $|t - t_{k_0}| < h$ and

$$\begin{aligned} |\varphi(x+\tau, t) - \varphi(x,t)| &\leq |\varphi(x+\tau,t) - \varphi(x+\tau, t_{k_0})| + |\varphi(x+\tau, t_{k_0}) - \varphi_{k_0}(x+\tau)| \\ &\quad + |\varphi_{k_0}(x+\tau) - \varphi_{k_0}(x)| + |\varphi_{k_0}(x) - \varphi(x, t_{k_0})| \\ &\quad + |\varphi(x, t_{k_0}) - \varphi(x,t)| \\ &\leq \|u(\cdot,t) - u(\cdot, t_{k_0})\| + \|u(\cdot, t_{k_0}) - u_{k_0}\| \\ &\quad + |\varphi_{k_0}(x+\tau) - \varphi_{k_0}(x)| + \|u(\cdot, t_{k_0}) - u_{k_0}\| \\ &\quad + \|u(\cdot, t_{k_0}) - u(\cdot, t)\| \\ &\leq 2(W+1)\omega(h) + |\varphi_{k_0}(x+\tau) - \varphi_{k_0}(x)| \\ &< \frac{\epsilon}{2} + |\Phi(x+\tau) - \Phi(x)| < \epsilon. \end{aligned}$$

It follows that $E(\epsilon/2, N, \Phi) \subset E(\epsilon, N, \varphi)$. This implies that φ is almost automorphic function by Proposition 1.1.

As in the proof for Theorem 3.3 in [26], one shows that $\xi \in \mathcal{PAP}_0(\triangle)$. The proof is complete.

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