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# THE SCHUR SUBGROUP OF A $p$-ADIC FIELD

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Let $K$ be a field. The Schur subgroup, $S(K)$, of the Brauer group, $B(K)$, consists of all classes $[\Delta]$ in $B(K)$ some representative of which is a simple component of one of the semi-simple group algebras, $KG$, where $G$ is a finite group such that char $K \nmid G$. Yamada ([**11**], p. 46) has characterized $S(K)$ for all finite extensions of the $p$-adic number field, $Q_p$. If $p$ is odd, $[\Delta] \in S(K)$ if and only if

$$\text{inv}_p \Delta \equiv z \Big/ \frac{p-1}{cs} \bmod 1,$$

where $c$ is the tame ramification index of $k/Q_p$, $k$ the maximal cyclotomic subfield of $K$, and $s = ((p-1)/c, [K:k])$. $\text{inv}_p\, \Delta$ is the Hasse invariant. Yamada showed this by proving first that $S(K)$ is the group of classes containing cyclotomic algebras and then determining the invariants of such algebras. In this paper we directly exhibit a *single* metacyclic group such that the classes of simple components of the group algebra are precisely *all* the elements of $S(K)$. This group is uniquely determined by the structure of its group algebras over $k$; furthermore, it is minimal in the sense that the simple components $M_d(\Delta)$ of the group algebra over $K$ are matrix algebras of lowest possible dimension, $d$, so that $M_e(\Delta)$ is a simple component of $KH$ for some $H$ if and only if $d|e$, provided $\Delta \neq K$.

**1. Definitions.** Let $p$ be an odd prime, $K$ a finite extension of $Q_p$, $k$ its maximal cyclotomic subfield. Let $\bar{k}$, $\bar{K}$ be the corresponding residue class fields. Let $c$ be the tame ramification index of $k/Q_p$, $m = (p-1)/c$, $q = |\bar{k}| = p^f$, $q - 1 = ln$ where $(m, n) = 1$ and all the primes dividing $l$ divide $m$, $s = (m, [K:k])$, $t = m/s$. We will assume that $t > 1$, that is, $S(K)$ is not trivial. For any integer $d$, $\zeta_d$ will denote a primitive $d$-th root of unity. And if $\Delta$ is a skew field, then $M_d(\Delta)$ denotes the full ring of $d \times d$ matrices over $\Delta$. $\Delta_{z/m}$ will denote the skew field with center $K$ and Hasse invariant $z/m$. Let $\lambda$ be an integer having order $m$ in the group $Z_p{}^*$.

We define the following cyclic algebra:

$B_z = (\delta_z, K(\zeta_p), \sigma)$ ([**8**], page 47)

where $\delta_z = \zeta_l{}^z$ and $\sigma$ generates the Galois group of $K(\zeta_p)/K$ under the map $\sigma(\zeta_p) = \zeta_p{}^\lambda$, and the following group:

$G_{q,m} = \langle X, Y : X^p = 1 = Y^{lm}, Y^{-1}XY = X^\lambda \rangle.$

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**2. Results.** Our initial results apply to the case where $K$ itself is a cyclotomic extension of $Q_p$. Thus:

PROPOSITION 1. *Let $K = k$. With the proper choice of $\zeta_l$, $B_z \simeq M_{(z,m)}(\Delta_{z/m})$.*

This proposition follows from [**11**], Theorem 4.3, p. 37. We give an independent short proof of it.

*Proof.* Note that $\delta_z \in k$, the center of $B_z$. The index of $B_z$, that is, the index of the skew field of coefficients, is the order of $\delta_z$ in $N(k^*(\zeta_p))$ where $N$ is the norm map from $k(\zeta_p)$ to $k$. ([7], 13.3, 14.19). For what values of $z$ is $\delta_z$ a norm? If $\delta_z = N(\mu)$ then $\mu$ is a unit. Since $k(\zeta_p)/k$ is totally ramified we can write $\mu = \alpha\beta$, where $\alpha \in k$ and $\beta \equiv 1 \bmod \pi'$, so $N(\mu) \equiv \alpha^m \bmod \pi$, where $\pi'$ and $\pi$ are local parameters for $k(\zeta_p)$ and $k$, respectively. Since $|\bar{k}| = ln$ and $\bar{\delta}_z = \bar{\alpha}^m$, the order of $\bar{\delta}_z$ divides $ln/m$. (The bar denotes the image in $\bar{k}$.) But $\bar{\delta}_z = \bar{\zeta}_l{}^z$ so ord $\bar{\delta}_z = l/(l, z)$. Therefore $m|z$ because $(l, n) = 1$. Conversely if $m|z$ then $\delta_z = \zeta_l{}^z = N(\zeta_l{}^{z/m})$ since $\zeta_l \in k$. So the index of $B_z$ is $m/(z, m)$. In particular, $B_1 \simeq \Delta_{w/m}$ where $(w, m) = 1$. If $\zeta_l$ is replaced by $\zeta_l' = \zeta_l{}^d$ then the corresponding algebra $B_1'$ has Hasse invariant $(dw/m) \bmod 1$ because the Brauer group is isomorphic to $Q/Z$ under the map inv. By choosing $\zeta_l$ appropriately we can assume inv $B_1 \equiv m^{-1} \bmod 1$. The result follows by a dimensionality argument since the index of $B_z$ is $m/(z, m)$.

THEOREM 2. *$kG_{q,m} \simeq k\langle Y\rangle \oplus (lc/m) \sum_{z=1}^{m} M_{(z,m)}(\Delta_{z/m})$ where $\langle Y\rangle$ is the cyclic subgroup of $G$ generated by $Y$.*

*Proof.* For simplicity write $G = G_{q,m}$. $k\langle Y\rangle$ occurs as a direct summand of $kG$ since $\langle x\rangle$ is normal in $G$. Since $lm + (lc/m).\ m^3 = |G|$ the result will follow from comparing dimensions if it can be shown that each of the simple algebras $B_z$ occurs as a direct summand of $kG$ at least $lc/m$ times. Consider the $m \times m$ matrices $X(t) = [\delta(i, j)\zeta_p{}^{t\lambda^i}]$, $Y(z) = [\delta(i-1, j)\zeta_{lm}{}^z]$, where $\delta(i, j)$ is the Kronecker delta function. The $k$-subalgebra of $M_m(\tilde{k})$ generated by these two matrices is $k$-isomorphic to $B_z$ ($\tilde{k}$ denotes the algebraic closure of $k$). If $t \not\equiv 0 \bmod p$ the map $X \to X(t)$, $Y \to Y(z)$ extends to a homomorphism $\phi(t, z)$ of $kG$ onto $B_z$. Let $\mu$ have multiplicative order $p - 1 \bmod p$. Let $t_h = \mu^h$, $z_j = \zeta_{lm}{}^{z+(j-1)m}$, where $0 \leqq z < m$. Then each of the $lc/m$ representations $\phi(t_h, z_j)$; $h = 1 \ldots c$; $j = 1 \ldots l/m$ takes $kG$ onto $B_z$. It will follow that these representations are inequivalent if it can be shown that their corresponding characters $\chi(t_h, z_j)$ are different. For $j$ fixed the $c$ elements $\chi(t_h, z_j)(X) = \sum_{i=1} \zeta_p{}^{\mu^h\lambda^i}$, $h = 1 \ldots c$ are pairwise unequal because $1, \zeta_p, \ldots \zeta_p{}^{p-2}$ are linearly independent over $Q$. Similarly for $h$ fixed the $l/m$ elements $\chi(t_h, z_j)(Y^m) = m\zeta_l{}^{z+(j-1)m}$, $j = 1 \ldots l/m$ are pairwise unequal. Thus the $lc/m$ characters $\chi(t_h, z_j)$; $h = 1 \ldots c$; $j = 1 \ldots l/m$ are pairwise unequal. But each of the inequivalent representations corresponds to one copy of $B_z$ in the direct sum decomposition of $kG$.

We remark that the structure of $k\langle Y\rangle$ as a direct sum of cyclotomic extensions of $k$ can be easily described by a theorem of Perlis and Walker [**5**].

COROLLARY 3. *$S(Q_p)$ is a cyclic group of order $p - 1$.*

*Proof.* This follows from Theorem 2 and [**11**], Proposition 6.2, p. 89.

The group $G_{q,m}$ is uniquely determined by the following:

THEOREM 4. *If $kG_{q,m} \simeq kG'$ then $G_{q,m} \simeq G'$.*

*Proof.* Since $k\langle Y\rangle$ is the largest abelian direct summand of $kG_{q,m}$ both $G_{q,m}$ and $G'$ have precisely $lm$ linear characters. So $G'$ has an element $X'$ of order $p$ which generates the commutator subgroup. If $m > 1$ then $\zeta_{lm} \notin k$. But $k(\zeta_{lm}) \subset k\langle Y\rangle$, so by the Perlis-Walker theorem there must be an element $Y'$ in $G'$ such that ord $Y' = lm$. Now suppose $(Y')^{-1}(X')(Y') = (X')^{\mu}$. Let $v =$ ord $\mu$ in $Z_p{}^*$. Consider one of the mappings $\phi$ of $kG'$ onto $\Delta_{1/m}$. Under this map $k\langle X'\rangle$ is taken onto a field which can be identified with $k(\zeta_p)$. Since $(Y')^v$ commutes with $X'$, $\phi(Y')^v$ is in the centralizer of $k(\zeta_p)$, that is $\alpha = \phi(Y')^v \in k(\zeta_p)$. $\sigma(\zeta_p) = \zeta_p{}^{\mu}$ defines an automorphism of $k(\zeta_p)$ with fixed field $F \supset k$. So $\phi$ maps $k(G')$ onto the cyclic algebra $(\alpha, F, \sigma)$, which, being isomorphic to $\Delta_{1/m}$, means $v = m$. Finally, the isomorphism between $G_{q,m}$ and $G'$ is obtained by mapping $X$ onto $X'$ and $Y$ onto $(Y')^{\nu}$ where $\mu^{\nu} \equiv \lambda \bmod p$.

Now let $K$ be an arbitrary finite extension of $Q_p$; $k$ its maximal cyclotomic subfield.

THEOREM 5. $KG_{q,m} \simeq K\langle Y\rangle \oplus (lc/t) \sum M_{(z,t)s}(\Delta_{z/t})$.

*Proof.* This follows directly from Theorem 2 and the formula

$$\text{inv}\,(A \oplus_k K) \equiv s(\text{inv}\,A)$$

for any central simple algebra $A$ over $k$. (See, for example, Chapter 7 of Deuring [**3**].)

Finally, for an arbitary group, a result concerning the order of the matrices in which the division algebras occur:

THEOREM 6. *$M_d(\Delta_{z/t})$ occurs as a direct summand of $KH$ for some finite group $H$ if and only if $(z, t)s|d$.*

*Proof.* $[Q]_d$ is a simple component of $QS_{d+1}$, where $S_{d+1}$ is the symmetric group of degree $d + 1$. So sufficiency follows by letting $H = G_{q,m} \times S_{r+1}$ where $r = d/(z, t)s$. On the other hand, suppose $\psi$ is a homomorphism of $KH$ onto $[\Delta_{z/t}]_d$ for some $H$. We can assume ker $\psi|H = 1$ by taking the factor group if necessary. $p||H|$ because $KH$ contains an algebra with index greater than 1. (This follows, for example, from [**3**], Corollary p. 150.)

Let $P$ be a subgroup of $H$ or order $p$, generated by $a$. Because $KP$ is isomorphic to the direct sum of $K$ and copies of $K(\zeta_p)$, we can write $\psi(KP) = \sum_1^{\alpha+\beta} A_i$ with $A_i \subset M_d(\Delta_{z/t})$, $\alpha$ and $\beta$ non-negative integers, $A_i \simeq K$ for $i = 1, \ldots, \alpha$, $A_i \simeq K(\zeta_p)$ for $i = \alpha + 1, \ldots, \alpha + \beta$ and the sum of the algebras in the expression for $\psi(KP)$ is a direct sum.

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We may assume $\zeta_p \notin K$, for otherwise the result is immediate. Let $1_i$ be the identity of $A_i$ for $i = 1, \ldots, \alpha + \beta$. The identity in $\psi(KH)$ is $\psi(1 \cdot e) = \sum_1^{\alpha+\beta} 1_i$. $(1 \cdot a)^p = 1 \cdot e$, and thus $\psi(1 \cdot a) \notin \sum_1^{\alpha} A_i$ because $\zeta_p \notin K$ and $\operatorname{Ker} \psi|P = 1$. So $\beta \neq 0$. Let $\epsilon_j$ be a primitive $p$-th root of unity in $A_j$ for $j = \alpha + 1, \ldots, \alpha + \beta$. Let $\rho = \sum_1^{\alpha} 1_i + \sum_{\alpha+1}^{\alpha+\beta} \epsilon_j$. $\rho^p = \psi(1 \cdot e)$. The $K$ algebra generated by $\rho$ is a subalgebra of $\psi(KH)$ isomorphic to $K(\zeta_p)$ and having the same identity as $\psi(KH)$. So $K(\zeta_p) \subset \psi(KH)$. Since

$$[K(\zeta_p) : K] = m, \ m| \deg M_d(\Delta_{z/t}) = td/(z, t). \text{ But } m = st. \text{ So } s(z, t)|d.$$

**3. Related Problems.** 1. If $\Delta_{1/r}$ is the division algebra of a simple component in $kG$ for a finite $G$, must $rmp$ divide $|G|$? This relation is true for several choices of $m$.

2. Is $G_{q,m}$ uniquely determined by its order and the fact that the entire Schur subgroup of $k$ appears in $kG_{q,m}$?

3. Is $G_{q,m}$ uniquely determined by its order and the fact that $\Delta_{1/m}$ appears as a direct summand of $kG_{q,m}$?

*Note.* We have subsequently been able to show that each of these questions has an affirmative answer.

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