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On Higher Moments of Fourier Coefficients of Holomorphic Cusp Forms

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Abstract. Let $S_k(\Gamma)$ be the space of holomorphic cusp forms of even integral weight k for the full modular group. Let $\lambda_f(n)$ and $\lambda_g(n)$ be the n-th normalized Fourier coefficients of two holomorphic Hecke eigencuspforms $f(z), g(z) \in S_k(\Gamma)$, respectively. In this paper we are able to show the following results about higher moments of Fourier coefficients of holomorphic cusp forms.

(i) For any $\varepsilon > 0$, we have

$$\sum_{n\leq x} \lambda_f^5(n) \ll_{f,\varepsilon} x^{\frac{15}{16}+\varepsilon} \quad \text{and} \quad \sum_{n\leq x} \lambda_f^7(n) \ll_{f,\varepsilon} x^{\frac{63}{64}+\varepsilon}.$$

(ii) If $\operatorname{sym}^3 \pi_f \ncong \operatorname{sym}^3 \pi_g$, then for any $\varepsilon > 0$, we have

$$\sum_{n\leq x} \lambda_f^3(n)\lambda_g^3(n) \ll_{f,\varepsilon} x^{\frac{31}{32}+\varepsilon};$$

If $\operatorname{sym}^2 \pi_f \ncong \operatorname{sym}^2 \pi_g$, then for any $\varepsilon > 0$, we have

$$\sum_{n\leq x} \lambda_f^4(n)\lambda_g^2(n) = cx\log x + c'x + O_{f,\varepsilon}\left(x^{\frac{31}{32}+\varepsilon}\right);$$

If $\operatorname{sym}^2 \pi_f \ncong \operatorname{sym}^2 \pi_g$ and $\operatorname{sym}^4 \pi_f \ncong \operatorname{sym}^4 \pi_g$, then for any $\varepsilon > 0$, we have

$$\sum_{n\leq x} \lambda_f^4(n)\lambda_g^4(n) = xP(\log x) + O_{f,\varepsilon}\left(x^{\frac{127}{128}+\varepsilon}\right),$$

where $P(x)$ is a polynomial of degree 3.

1 Introduction and Main Results

Let $S_k(\Gamma)$ be the space of holomorphic cusp forms of even integral weight k for the full modular group $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. Suppose that $f(z)$ and $g(z)$ are two eigenfunctions of all Hecke operators belonging to $S_{2k}(\Gamma)$. Then Hecke eigencuspforms $f(z)$ and $g(z)$ have the following Fourier expansions at the cusp ∞:

$$f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi inz}, \quad g(z) = \sum_{n=1}^{\infty} b(n)e^{2\pi inz},$$

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where we normalize $f(z)$ and $g(z)$ such that $a(1) = b(1) = 1$. Instead of $a(n)$ and $b(n)$, one often considers the normalized Fourier coefficients

$$\lambda_f(n) = \frac{a(n)}{n^{\frac{k-1}{2}}}, \quad \lambda_g(n) = \frac{b(n)}{n^{\frac{k-1}{2}}}.$$

The Fourier coefficients of cusp forms are interesting objects (see [2,16]). In 1974, P. Deligne [2] proved the Ramanujan–Petersson conjecture

$$|\lambda_f(n)| \le d(n), \tag{1.1}$$

where $d(n)$ is the divisor function. As a corollary, he proved that for any $\varepsilon > 0$,

$$S(x) = \sum_{n \le x} \lambda_f(n) \ll_{f,\varepsilon} x^{\frac{1}{3}+\varepsilon}.$$

In 1989, Hafner and Ivic' [6] were able to remove the factor x^{ε} in Deligne's result, *i.e.*,

$$S(x) = \sum_{n \le x} \lambda_f(n) \ll_f x^{\frac{1}{3}}.$$

In this direction, the best known result is due to Rankin [17]

$$S(x) = \sum_{n \le x} \lambda_f(n) \ll_f x^{\frac{1}{3}} (\log x)^{-\delta},$$

where $0 < \delta < 0.06$.

Rankin [16] and Selberg [19] invented the powerful Rankin–Selberg method, and then successfully showed that

$$\sum_{n \le x} \lambda_f^2(n) = c_0 x + O_f(x^{\frac{3}{5}}).$$

Later, based on the works about symmetric power L-functions, Moreno and Shahidi [15] were able to prove

$$\sum_{n \le x} \tau_0^4(n) \sim c_1 x \log x, \qquad x \to \infty,$$

where $\tau_0(n) = \tau(n)/n^{\frac{11}{2}}$ is the normalized Ramanujan tau-function. Obviously Moreno and Shahidi's result also holds true if we replace $\tau_0(n)$ by the normalized Fourier coefficient $\lambda_f(n)$. In 2001, Fomenko [3] improved Moreno and Shahidi's result by showing that

$$\sum_{n \le x} \lambda_f^4(n) = c_2 x \log x + c_3 x + O_{f,\varepsilon}\left(x^{\frac{9}{10}+\varepsilon}\right).$$

Furthermore, he proved the following results:

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(i) For any $\varepsilon > 0$, we have

$$\sum_{n\le x}\lambda_f^3(n) \ll_{f,\varepsilon} x^{\frac{5}{6}+\varepsilon}.$$

(ii) For any $\varepsilon > 0$, we have

$$\sum_{n\le x}\lambda_f^2(n)\lambda_g(n) \ll_{f,g,\varepsilon} x^{\frac{5}{6}+\varepsilon}.$$

(iii) Let F_1 be the Gelbart–Jacquet lift on GL(3) associated with f, and F_2 be the Gelbart-Jacquet lift on GL(3) associated with g. If F_1 and F_2 are distinct, then for any $\varepsilon > 0$, we have

$$\sum_{n\le x}\lambda_f^2(n)\lambda_g^2(n) = c_4x + O_{f,g,\varepsilon}(x^{\frac{9}{10}+\varepsilon}).$$

Recently, inspired by the beautiful paper of Friedlander and Iwaniec [4], I improved Fomenko's results [13]:

(i) For any $\varepsilon > 0$, we have

$$\sum_{n\le x}\lambda^4(n) = c_2x\log x + c_3x + O_{f,\varepsilon}\left(x^{\frac{7}{8}+\varepsilon}\right).$$

(ii) For any $\varepsilon > 0$, we have

$$\sum_{n\le x}\lambda_f^3(n) \ll_{f,\varepsilon} x^{\frac{3}{4}+\varepsilon}.$$

(iii) For any $\varepsilon > 0$, we have

$$\sum_{n\le x}\lambda_f^2(n)\lambda_g(n) \ll_{f,g,\varepsilon} x^{\frac{3}{4}+\varepsilon}.$$

(iv) If f and g are distinct, then for any $\varepsilon > 0$, we have

$$\sum_{n\le x}\lambda_f^2(n)\lambda_g^2(n) = c_4x + O_{f,g,\varepsilon}\left(x^{\frac{7}{8}+\varepsilon}\right).$$

More recently, in [14], I established the asymptotic formulae for the sixth and eighth moments of Fourier coefficients of cusp forms, *i.e.*,

(i) For any $\varepsilon > 0$, we have

$$\sum_{n\le x}\lambda^6(n) = xP_1(\log x) + O_{f,\varepsilon}\left(x^{\frac{31}{32}+\varepsilon}\right),$$

where $P_1(x)$ is a polynomial of degree 4.

(ii) For any $\varepsilon > 0$, we have

$$\sum_{n\leq x}\lambda^8(n) = xP_2(\log x) + O_{f,\varepsilon}\left(x^{\frac{127}{128}+\varepsilon}\right),$$

where $P_2(x)$ is a polynomial of degree 13.

In this paper we will prove higher moments of Fourier coefficients of cusp forms of the following types. To introduce our results, for $j = 1, 2, 3, 4$, let $\operatorname{sym}^j\pi_f$ be the automorphic cuspidal self-dual representation of $\mathrm{GL}_{j+1}(\mathbb{A}_\mathbb{Q})$ whose local L-factors agree with the local L-factors of the jth symmetric power L-function associated with f.

Theorem 1.1 *For any $\varepsilon > 0$, we have*

$$\sum_{n\leq x}\lambda_f^5(n) \ll_{f,\varepsilon} x^{\frac{15}{16}+\varepsilon}.$$

Theorem 1.2 *For any $\varepsilon > 0$, we have*

$$\sum_{n\leq x}\lambda_f^7(n) \ll_{f,\varepsilon} x^{\frac{63}{64}+\varepsilon}.$$

Theorem 1.3 *If $\operatorname{sym}^3\pi_f \not\cong \operatorname{sym}^3\pi_g$, then for any $\varepsilon > 0$, we have*

$$\sum_{n\leq x}\lambda_f^3(n)\lambda_g^3(n) \ll_{f,g,\varepsilon} x^{\frac{31}{32}+\varepsilon}.$$

Theorem 1.4 *If $\operatorname{sym}^2\pi_f \not\cong \operatorname{sym}^2\pi_g$, then for any $\varepsilon > 0$, we have*

$$\sum_{n\leq x}\lambda_f^4(n)\lambda_g^2(n) = cx\log x + c'x + O_{f,g,\varepsilon}\left(x^{\frac{31}{32}+\varepsilon}\right).$$

Theorem 1.5 *If $\operatorname{sym}^2\pi_f \not\cong \operatorname{sym}^2\pi_g$, and $\operatorname{sym}^4\pi_f \not\cong \operatorname{sym}^4\pi_g$, then for any $\varepsilon > 0$, we have*

$$\sum_{n\leq x}\lambda_f^4(n)\lambda_g^4(n) = xP(\log x) + O_{f,g,\varepsilon}\left(x^{\frac{127}{128}+\varepsilon}\right),$$

where $P(x)$ is a polynomial of degree 3.

Remark 1.6 By using the same arguments, our Theorems 1.1–1.5 also hold true for the holomorphic cusp forms with respect to the congruence group of level N.

Remark 1.7 In his report, the referee introduced me to another article on the same theme by J. Wu [25]. The main difference between our works is that I insert the Rankin–Selberg L-function associated with the symmetric powers into the corresponding generating L-functions in Lemmas 2.1–2.5, and hence the generating L-functions are analytic in a much wider domain ($\operatorname{Re} s > 1/2$). This enables me to establish the asymptotic formulae with smaller error terms.

2 Some Lemmas

Let $f(z), g(z) \in S_k(\Gamma)$ be Hecke eigencuspforms of even integral weight k for the full modular group, and $\lambda_f(n)$ and $\lambda_g(n)$ denote their n-th normalized Fourier coefficients respectively. For $j = 1, 2, 3, 4$, let $L(\mathrm{sym}^j f, s)$ and $L(\mathrm{sym}^j g, s)$ be the j-th symmetric power L-functions associated with f and g respectively, and $L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s)$ the Rankin–Selberg L-function associated with $\mathrm{sym}^i f$ and $\mathrm{sym}^j g$.

Then we have the following results.

Lemma 2.1 *Define*

$$L_1(s) = \sum_{n=1}^{\infty} \frac{\lambda_f^5(n)}{n^s},$$

for $\operatorname{Re} s > 1$. *Then we have that for* $\operatorname{Re} s > 1$,

$$L_1(s) = L^4(f, s) L^3(\mathrm{sym}^3 f, s) L(\mathrm{sym}^2 f \times \mathrm{sym}^3 f, s) U_1(s),$$

where $U_1(s)$ *is a Dirichlet series, which converges uniformly and absolutely in the half plane* $\operatorname{Re} s \geq 1/2 + \varepsilon$ *for any* $\varepsilon > 0$.

Proof According to Deligne [2], for any prime number p there are $\alpha_f(p)$ and $\beta_f(p)$ such that

$$\lambda_f(p) = \alpha_f(p) + \beta_f(p), \quad \text{and} \quad |\alpha_f(p)| = \alpha_f(p)\beta_f(p) = 1. \tag{2.1}$$

The L-function attached to $f \in S_k(\Gamma)$ is defined by

$$L(f, s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p (1 - \alpha_f(p) p^{-s})^{-1} (1 - \beta_f(p) p^{-s})^{-1} \tag{2.2}$$

for $\operatorname{Re} s > 1$. The j-th symmetric power L-function attached to $f \in S_k(\Gamma)$ is defined by

$$L(\mathrm{sym}^j f, s) = \prod_p \prod_{m=0}^{j} (1 - \alpha_f(p)^{j-m} \beta_f(p)^m p^{-s})^{-1} := \prod_p L_p(\mathrm{sym}^j f, s) \tag{2.3}$$

for $\operatorname{Re} s > 1$. The product over primes gives a Dirichlet series representation for $L(\mathrm{sym}^j f, s)$: for $\operatorname{Re} s > 1$,

$$L(\mathrm{sym}^j f, s) = \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(n)}{n^s},$$

where $\lambda_{\mathrm{sym}^j f}(n)$ is a multiplicative function. Then we have that for $\operatorname{Re} s > 1$,

$$L(\mathrm{sym}^j f, s) = \prod_p \left(1 + \frac{\lambda_{\mathrm{sym}^j f}(p)}{p^s} + \cdots + \frac{\lambda_{\mathrm{sym}^j f}(p^k)}{p^{ks}} + \cdots \right). \tag{2.4}$$

From (2.3) and (2.4), we have

$$\lambda_{\mathrm{sym}^j f}(p) = \sum_{m=0}^{j} \alpha_f(p)^{j-m}\beta_f(p)^m. \tag{2.5}$$

From (2.1), we have

$$|\lambda_{\mathrm{sym}^j f}(n)| \le d_{j+1}(n), \tag{2.6}$$

where $d_k(n)$ is the n-th coefficient of the Dirichlet series $\zeta^k(s)$.

The Rankin–Selberg L-function associated with $\mathrm{sym}^i f$ and $\mathrm{sym}^j f$ is defined by

$$\begin{aligned} L(\mathrm{sym}^i f \times \mathrm{sym}^j f, s) &:= \\ &\prod_p \prod_{m=0}^{i} \prod_{u=0}^{j} (1 - \alpha_f(p)^{i-m}\beta_f(p)^m\alpha_f(p)^{j-u}\beta_f(p)^u p^{-s})^{-1} \end{aligned} \tag{2.7}$$

for $\operatorname{Re} s > 1$. The product over primes also gives a Dirichlet series representation for $L(\mathrm{sym}^i f \times \mathrm{sym}^j f, s)$: for $\operatorname{Re} s > 1$,

$$L(\mathrm{sym}^i f \times \mathrm{sym}^j f, s) = \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j f}(n)}{n^s},$$

where $\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j f}(n)$ is a multiplicative function. Then we have that for $\operatorname{Re} s > 1$,

$$\begin{aligned} L(\mathrm{sym}^i f \times \mathrm{sym}^j f, s) &= \\ &\prod_p \left(1 + \frac{\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j f}(p)}{p^s} + \cdots + \frac{\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j f}(p^k)}{p^{ks}} + \cdots \right). \end{aligned} \tag{2.8}$$

From (2.7) and (2.8), we have

$$\begin{aligned} \lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j f}(p) &= \sum_{m=0}^{i} \sum_{u=0}^{j} \alpha_f(p)^{i-m}\beta_f(p)^m\alpha_f(p)^{j-u}\beta_f(p)^u \\ &= \lambda_{\mathrm{sym}^i f}(p)\lambda_{\mathrm{sym}^j f}(p). \end{aligned} \tag{2.9}$$

From (2.1), we have

$$|\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j f}(n)| \le d_{(i+1)(j+1)}(n). \tag{2.10}$$

For $\operatorname{Re} s > 1$, we can write $L^4(f,s)L^3(\mathrm{sym}^3 f, s)L(\mathrm{sym}^2 f \times \mathrm{sym}^3 f, s)$ as an Euler product

$$\begin{aligned} L^4(f,s)L^3(\mathrm{sym}^3 f, s)L(\mathrm{sym}^2 f \times \mathrm{sym}^3 f, s) &:= \\ &\prod_p \left(1 + \frac{b(p)}{p^s} + \cdots + \frac{b(p^k)}{p^{ks}} + \cdots \right). \end{aligned} \tag{2.11}$$

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From (2.2), (2.4), and (2.8), we have

$$b(p) = 4\lambda_f(p) + 3\lambda_{\mathrm{sym}^3 f}(p) + \lambda_{\mathrm{sym}^2 f\times \mathrm{sym}^3 f}(p).$$

From (2.1), (2.5), and (2.9), it is easy to check that

$$\begin{aligned} b(p) &= 4(\alpha_f(p)+\beta_f(p)) + 3(\alpha_f(p)^3+\alpha_f(p)+\beta_f(p)+\beta_f(p)^3) \\ &\quad + (\alpha_f(p)^2+1+\beta_f(p)^2)(\alpha_f(p)^3+\alpha_f(p)+\beta_f(p)+\beta_f(p)^3) \\ &= (\alpha_f(p)+\beta_f(p))^5 = \lambda^5(p). \end{aligned} \tag{2.12}$$

On the other hand, from (1.1) we learn that

$$L_1(s) = \sum_{n=1}^{\infty} \frac{\lambda_f^5(n)}{n^s}$$

is absolutely convergent in the half plane $\operatorname{Re} s > 1$. On noting that $\lambda_f^5(n)$ is a multiplicative function, we have that for $\operatorname{Re} s > 1$

$$L_1(s) = \sum_{n=1}^{\infty} \frac{\lambda_f^5(n)}{n^s} = \prod_p \left(1 + \frac{\lambda_f^5(p)}{p^s} + \frac{\lambda_f^5(p^2)}{p^{2s}} + \cdots + \frac{\lambda_f^5(p^k)}{p^{ks}} + \cdots\right). \tag{2.13}$$

Therefore from (2.11), (2.12), and (2.13), we have that for $\operatorname{Re} s > 1$

$$\begin{aligned} L_1(s) &= L^4(f,s)L^3(\mathrm{sym}^3 f,s)L(\mathrm{sym}^2 f\times \mathrm{sym}^3 f,s) \\ &\quad \times \prod_p \left(1 + \frac{\lambda^5(p^2)-b(p^2)}{p^{2s}} + \cdots\right) \\ &:= L^4(f,s)L^3(\mathrm{sym}^3 f,s)L(\mathrm{sym}^2 f\times \mathrm{sym}^3 f,s)U_1(s). \end{aligned}$$

From (1.1), (2.6), and (2.10), it is obvious that $U_1(s)$ converges uniformly and absolutely in the half plane $\operatorname{Re} s \geq \frac{1}{2}+\varepsilon$ for any $\varepsilon > 0$. This completes the proof of Lemma 2.1. ■

The key point in the proof of Lemma 2.1 is the following. Let $t_f = \alpha_f(p)+\beta_f(p)$. The polynomials $S_j(f)$ for the trace of j-th symmetric power associated with f are defined by

$$S_0(f) = 1; \quad S_1(f) = \alpha_f(p)+\beta_f(p) = t_f;$$

$$S_2(f) = \alpha_f(p)^2+1+\beta_f(p)^2 = t_f^2-1;$$

$$S_3(f) = \alpha_f(p)^3+\alpha_f(p)+\beta_f(p)+\beta_f(p)^3 = t_f^3-2t_f;$$

$$S_4(f) = \alpha_f(p)^4+\alpha_f(p)^2+1+\beta_f(p)^2+\beta_f(p)^4 = t_f^4-3t_f^2+1;$$

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$$S_5(f) = \alpha_f(p)^5 + \alpha_f(p)^3 + \alpha_f(p) + 1 + \beta_f(p) + \beta_f(p)^3 + \beta_f(p)^5 = t_f^5 - 4t_f^3 + 3t_f;$$
$$S_6(f) = \alpha_f(p)^6 + \alpha_f(p)^4 + \alpha_f(p)^2 + 1 + \beta_f(p)^2 + \beta_f(p)^4 + \beta_f(p)^6 = t_f^6 - 5t_f^4 + 6t_f^2 - 1;$$
$$S_7(f) = \alpha_f(p)^7 + \alpha_f(p)^5 + \alpha_f(p)^3 + \alpha_f(p) + 1 + \beta_f(p) + \beta_f(p)^3 + \beta_f(p)^5 + \beta_f(p)^7 = t_f^7 - 6t_f^5 + 10t_f^3 - 4t_f.$$

Then $t_f^5 = 5S_1(f) + 4S_3(f) + S_5(f)$. On the other hand, we have

$$S_2(f)S_3(f) = S_1(f) + S_3(f) + S_5(f). \tag{2.14}$$

Therefore, $t_f^5 = 4S_1(f) + 3S_3(f) + S_2(f)S_3(f)$. This identity determines Lemma 2.1.

In addition, we have $t_f^7 = 14S_1(f) + 14S_3(f) + 6S_5(f) + S_7(f)$. On noting (2.14) and $S_3(f)S_4(f) = S_1(f) + S_3(f) + S_5(f) + S_7(f)$, we have

$$t_f^7 = 8S_1(f) + 8S_3(f) + 5S_2(f)S_3(f) + S_3(f)S_4(f). \tag{2.15}$$

If we use the similar notations $t_g = \alpha_g(p) + \beta_g(p)$, $S_j(g)$ for g, then we can prove the following identities:

$$t_f^3t_g^3 = 4S_1(f)S_1(g) + 2S_3(f)S_1(g) + 2S_1(f)S_3(g) + S_3(f)S_3(g), \tag{2.16}$$
$$t_f^4t_g^2 = 2 + 3S_2(f) + 2S_2(g) + S_4(f) + 3S_2(f)S_2(g) + S_4(f)S_2(g), \tag{2.17}$$

and

$$t_f^4t_g^4 = 4 + 6S_2(f) + 6S_2(g) + 2S_4(f) + 2S_4(g) + 9S_2(f)S_2(g) + 3S_2(f)S_4(g) + 3S_4(f)S_2(g) + S_4(f)S_4(g). \tag{2.18}$$

These identities (2.15), (2.16), (2.17), and (2.18) determine Lemmas 2.2, 2.3, 2.4, and 2.5 below respectively.

Lemma 2.2 *Define*

$$L_2(s) = \sum_{n=1}^{\infty} \frac{\lambda_f^7(n)}{n^s},$$

for $\operatorname{Re} s > 1$. *Then we have that for* $\operatorname{Re} s > 1$,

$$L_2(s) = L^8(f,s)L^8(\operatorname{sym}^3 f,s)L^5(\operatorname{sym}^2 f \times \operatorname{sym}^3 f,s)L(\operatorname{sym}^3 f \times \operatorname{sym}^4 f,s)U_2(s),$$

where $U_2(s)$ *is a Dirichlet series, which converges uniformly and absolutely in the half plane* $\operatorname{Re} s \geq 1/2 + \varepsilon$ *for any* $\varepsilon > 0$.

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Lemma 2.3 *Define*

$$L_3(s) = \sum_{n=1}^{\infty} \frac{\lambda_f^3(n)\lambda_g^3(n)}{n^s},$$

for $\operatorname{Re} s > 1$. *Then we have that for* $\operatorname{Re} s > 1$,

$$L_3(s) = L^4(f \times g, s)L^2(\operatorname{sym}^3 f \times g, s)L^2(f \times \operatorname{sym}^3 g, s)L(\operatorname{sym}^3 f \times \operatorname{sym}^3 g, s)U_3(s)$$

where $U_3(s)$ *is a Dirichlet series, which converges uniformly and absolutely in the half plane* $\operatorname{Re} s \geq 1/2 + \varepsilon$ *for any* $\varepsilon > 0$.

Lemma 2.4 *Define*

$$L_4(s) = \sum_{n=1}^{\infty} \frac{\lambda_f^4(n)\lambda_g^2(n)}{n^s},$$

for $\operatorname{Re} s > 1$. *Then we have that for* $\operatorname{Re} s > 1$,

$$\begin{aligned} L_4(s) = \zeta^2(s)L^3(\operatorname{sym}^2 f, s)L^2(\operatorname{sym}^2 g, s)L(\operatorname{sym}^4 f, s) \\ \times L^3(\operatorname{sym}^2 f \times \operatorname{sym}^2 g, s)L(\operatorname{sym}^4 f \times \operatorname{sym}^2 g, s)U_4(s), \end{aligned}$$

where $U_4(s)$ *is a Dirichlet series, which converges uniformly and absolutely in the half plane* $\operatorname{Re} s \geq 1/2 + \varepsilon$ *for any* $\varepsilon > 0$.

Lemma 2.5 *Define*

$$L_5(s) = \sum_{n=1}^{\infty} \frac{\lambda_f^4(n)\lambda_g^4(n)}{n^s},$$

for $\operatorname{Re} s > 1$. *Then we have that for* $\operatorname{Re} s > 1$,

$$\begin{aligned} L_5(s) = \zeta^4(s)&L^6(\operatorname{sym}^2 f, s)L^6(\operatorname{sym}^2 g, s)L^2(\operatorname{sym}^4 f, s)L^2(\operatorname{sym}^4 g, s) \\ &\times L^9(\operatorname{sym}^2 f \times \operatorname{sym}^2 g, s)L^3(\operatorname{sym}^2 f \times \operatorname{sym}^4 g, s)L^3(\operatorname{sym}^4 f \times \operatorname{sym}^2 g, s) \\ &\times L(\operatorname{sym}^4 f \times \operatorname{sym}^4 g, s)U_5(s), \end{aligned}$$

where $U_5(s)$ *is a Dirichlet series, which converges uniformly and absolutely in the half plane* $\operatorname{Re} s \geq 1/2 + \varepsilon$ *for any* $\varepsilon > 0$.

As a part of the far-reaching Langlands program, the analytic properties of symmetric power L-functions $L(\operatorname{sym}^j f, s)$ are important topics in contemporary mathematics and have a significant impact on modern number theory. The analytic continuation of the symmetric power L-functions $L(\operatorname{sym}^j f, s)$ with $j = 2, 3, 4$ over the whole complex plane and the predicted functional equations have been established by Gelbart and Jacquet [5], Kim and Shahidi [11, 12], and Kim [10] respectively.

Lemma 2.6 *Let $f(z) \in S_k(\Gamma)$ be a Hecke eigencuspform of even integral weight k. The jth symmetric power L-function $L(\mathrm{sym}^j f, s)$ is defined in* (2.3).

For $j = 1, 2, 3, 4$, there exists an automorphic cuspidal self-dual representation, denoted by

$$\mathrm{sym}^j \pi_f = \bigotimes{}' \mathrm{sym}^j \pi_{f,\nu} \text{ of } \mathrm{GL}_{j+1}(\mathbb{A}_{\mathbb{Q}})$$

whose local L-factors $L(\mathrm{sym}^j \pi_{f,p}, s)$ agree with the local L-factors $L_p(\mathrm{sym}^j f, s)$ in (2.3). *Therefore for $j = 1, 2, 3, 4$, $L(\mathrm{sym}^j f, s)$ have analytic continuations to the whole complex plane $\mathbb{C}$, and satisfy certain functional equations.*

More precisely, for $j = 1, 2, 3, 4$ the archimedean local factor of $L(\mathrm{sym}^j f, s)$ is

$$L_\infty(\mathrm{sym}^j f, s) = \begin{cases} \prod_{\nu=0}^{n} \Gamma_{\mathbb{C}}(s + (\nu + \frac{1}{2})(k-1)), & \text{if } j = 2n+1, \\ \Gamma_{\mathbb{R}}(s + \delta_{2\nmid n}) \prod_{\nu=1}^{n} \Gamma_{\mathbb{C}}(s + \nu(k-1)), & \text{if } j = 2n, \end{cases}$$

where $\Gamma_{\mathbb{R}} = \pi^{-s/2}\Gamma(s/2)$, $\Gamma_{\mathbb{C}} = 2(2\pi)^{-s}\Gamma(s)$, and

$$\delta_{2\nmid n} = \begin{cases} 1, & \text{if } 2 \nmid n, \\ 0, & \text{otherwise.} \end{cases}$$

For $1 \le j \le 4$, it is known that the complete L-function

$$\Lambda(\mathrm{sym}^j f, s) = L_\infty(\mathrm{sym}^j f, s) L(\mathrm{sym}^j f, s)$$

is an entire function on the whole complex plane $\mathbb{C}$, and satisfies the functional equation

$$\Lambda(\mathrm{sym}^j f, s) = \epsilon_{\mathrm{sym}^j f} \Lambda(\mathrm{sym}^j f, 1-s),$$

where $\epsilon_{\mathrm{sym}^j f} = \pm 1$.

Proof This lemma follows from Gelbart and Jacquet [5] for $k = 2$ and from the recent works of Kim and Shahidi [11, 12] and Kim [10] when $k = 3, 4$. The current explicit version of this lemma can be found in [17]. ∎

From the famous works of Gelbart and Jacquet [5], Kim and Shahidi [11, 12], and Kim [10], we learn that for $1 \le j \le 4$ the j-th symmetric power L-function $L(\mathrm{sym}^j f, s)$ agrees with the L-function associated with an automorphic cuspidal self-dual representation $\mathrm{sym}^j \pi_f$ of $\mathrm{GL}_{j+1}(\mathbb{A}_{\mathbb{Q}})$. Then from the works of Jacquet and Shalika [8, 9], Shahidi [20–24], and the reformulation of Rudnick and Sarnak [18], we know the analytic properties for the Rankin–Selberg L-functions $L(\mathrm{sym}^i f \times \mathrm{sym}^j g, s)$ with $i, j = 1, 2, 3, 4$. Therefore, corresponding to Lemmas 2.1–2.5, we have the following results.

Lemma 2.7 *Let $f \in S_k(\Gamma)$ be a Hecke eigencuspform of even integral weight k. Then*

$$L_1(s) = \sum_{n=1}^{\infty} \frac{\lambda_f^5(n)}{n^s}$$

can be extended to be an entire function in the half plane $\mathrm{Re}\, s > 1/2$.

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Lemma 2.8 *Let $f \in S_k(\Gamma)$ be a Hecke eigencuspform of even integral weight k. Then*

$$L_2(s) = \sum_{n=1}^{\infty} \frac{\lambda_f^7(n)}{n^s}$$

can be extended to be an entire function in the half plane $\operatorname{Re} s > 1/2$.

Lemma 2.9 *Let $f, g \in S_k(\Gamma)$ be Hecke eigencuspforms of even integral weight k such that* $\operatorname{sym}^3\pi_f \not\cong \operatorname{sym}^3\pi_g$. *Then*

$$L_3(s) = \sum_{n=1}^{\infty} \frac{\lambda_f^3(n)\lambda_g^3(n)}{n^s}$$

can be extended to be an entire function in the half plane $\operatorname{Re} s > 1/2$.

Lemma 2.10 *Let $f, g \in S_k(\Gamma)$ be Hecke eigencuspforms of even integral weight k such that* $\operatorname{sym}^2\pi_f \not\cong \operatorname{sym}^2\pi_g$. *Then*

$$L_4(s) = \sum_{n=1}^{\infty} \frac{\lambda_f^4(n)\lambda_g^2(n)}{n^s},$$

can be extended to be a meromorphic function in the half plane $\operatorname{Re} s > 1/2$ *with only a pole $s = 1$ of order* 2.

Lemma 2.11 *Let $f, g \in S_k(\Gamma)$ be Hecke eigencuspforms of even integral weight k such that* $\operatorname{sym}^2\pi_f \not\cong \operatorname{sym}^2\pi_g$ *and* $\operatorname{sym}^4\pi_f \not\cong \operatorname{sym}^4\pi_g$. *Then*

$$L_5(s) = \sum_{n=1}^{\infty} \frac{\lambda_f^4(n)\lambda_g^4(n)}{n^s},$$

can be extended to be a meromorphic function in the half plane $\operatorname{Re} s > 1/2$ *with only a pole $s = 1$ of order* 4.

To prove our results, we also need the following two folklore results about the convexity bound and mean square value for nice L-functions.

Lemma 2.12 *Let $j = 1, 2, 3, 4$. Then for any $\varepsilon > 0$ and $0 \le \sigma \le 1$, we have*

$$L(\operatorname{sym}^j f, \sigma + it) \ll_{f,\varepsilon} (1 + |t|)^{\frac{j+1}{2}(1-\sigma)+\varepsilon},$$

and

$$L(\operatorname{sym}^i f \times \operatorname{sym}^j g, \sigma + it) \ll_{f,g,\varepsilon} (1 + |t|)^{\frac{(i+1)(j+1)}{2}(1-\sigma)+\varepsilon}.$$

Lemma 2.13 *Let $L(f, s)$ be a Dirichlet series with Euler product of degree $m \ge 2$, which means*

$$L(f, s) = \sum_{n=1}^{\infty} \lambda_f(n) n^{-s} = \prod_{p<\infty} \prod_{j=1}^{m} \left(1 - \frac{\alpha_f(p, j)}{p^s}\right)^{-1},$$

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where $\alpha_f(p,j)$, $j = 1,\dots,m$ are the local parameters of $L(f,s)$ at prime p and $\lambda_f(n) \ll n^{\varepsilon}$. Assume that this series and its Euler product are absolutely convergent for $\operatorname{Re} s > 1$. *Assume also that it is entire except possibly for simple poles at $s = 0, 1$ and satisfies a functional equation of Riemann type. Then we have that for $T \geq 1$*

$$\int_T^{2T} |L(f, 1/2+\varepsilon+it)|^2 \, dt \ll T^{\frac{m}{2}+\varepsilon}.$$

3 Proofs of Theorems

In this section we give the proof of Theorem 1.1. The proofs of Theorems 1.2–1.5 are similar to that of Theorem 1.1. In order to avoid repetition, we omit the proofs of Theorems 1.2–1.5.

Recall that we have defined

$$L_1(s) = \sum_{n=1}^{\infty} \frac{\lambda_f^5(n)}{n^s} \tag{3.1}$$

for $\operatorname{Re} s > 1$. From Lemma 2.7, we learn that

$$L_1(s) = L^4(f,s) L^3(\operatorname{sym}^3 f, s) L(\operatorname{sym}^2 f \times \operatorname{sym}^3 f, s) U_1(s)$$

can be analytically continued to be an entire function in the half plane $\operatorname{Re} s > 1/2$.

By (3.1) and Perron's formula (see [7, Proposition 5.54]), we have

$$\sum_{n \leq x} \lambda_f^5(n) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} L_1(s) \frac{x^s}{s} ds + O\Big(\frac{x^{1+\varepsilon}}{T}\Big),$$

where $b = 1+\varepsilon$ and $1 \leq T \leq x$ is a parameter to be chosen later. Here we have used (1.1).

Then we move the integration to the parallel segment with $\operatorname{Re} s = \frac{1}{2}+\varepsilon$. By Cauchy's theorem, we have

$$\begin{aligned} \sum_{n \leq x} \lambda_f^5(n) &= \frac{1}{2\pi i} \left\{ \int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT} + \int_{\frac{1}{2}+\varepsilon+iT}^{b+iT} + \int_{b-iT}^{\frac{1}{2}+\varepsilon-iT} \right\} L_1(s) \frac{x^s}{s} ds \\ &\quad + O\left(\frac{x^{1+\varepsilon}}{T}\right) \\ &:= J_1 + J_2 + J_3 + O\left(\frac{x^{1+\varepsilon}}{T}\right). \end{aligned} \tag{3.2}$$

To go further, we recall that $L^4(f,s)L^3(\operatorname{sym}^3 f,s)L(\operatorname{sym}^2 f \times \operatorname{sym}^3 f, s)$ is a Riemann-type nice L-function with Euler product of degree $m = 32$.

For J_1, from Lemma 2.1 we have

$$J_1 \ll x^{\frac{1}{2}+\varepsilon} \int_1^T |\{L^4(f,s)L^3(\operatorname{sym}^3 f,s)L(\operatorname{sym}^2 f \times \operatorname{sym}^3 f,s)\}|_{s=1/2+\varepsilon+it}| t^{-1} dt + x^{\frac{1}{2}+\varepsilon}.$$

Then by Cauchy's inequality, we have

(3.3)

$$
\begin{aligned}
J_1 \ll x^{1/2+\varepsilon} \log T \max_{T_1 \le T} \Bigg\{ \frac{1}{T_1} \Bigg(\int_{T_1/2}^{T_1} \Big| \big\{ L^4(f,s) L^3(\mathrm{sym}^3 f, s) \big\} \big|_{s=1/2+\varepsilon+it} \Big|^2 dt \Bigg)^{\frac{1}{2}} \\
\times \Bigg(\int_{T_1/2}^{T_1} \big| L(\mathrm{sym}^2 f \times \mathrm{sym}^3 f, 1/2+\varepsilon+it) \big|^2 dt \Bigg)^{\frac{1}{2}} \Bigg\} + x^{\frac{1}{2}+\varepsilon} \\
\ll x^{\frac{1}{2}+\varepsilon} T^{7+\varepsilon},
\end{aligned}
$$

where we have used Lemma 2.13 in the following forms

$$
\int_{T_1/2}^{T_1} \Big| \big\{ L^4(f,s) L^3(\mathrm{sym}^3 f, s) \big\} \big|_{s=1/2+\varepsilon+it} \Big|^2 dt \ll T^{10+\varepsilon},
$$

and

$$
\int_{T_1/2}^{T_1} \big| L(\mathrm{sym}^2 f \times \mathrm{sym}^3 f, 1/2+\varepsilon+it) \big|^2 dt \ll T^{6+\varepsilon}.
$$

For the integral over the horizontal segments, we use Lemma 2.12 to bound

$$
\begin{aligned}
(3.4)\quad J_2 + J_3 &\ll \int_{\frac{1}{2}+\varepsilon}^{b} x^{\sigma} \left| L^4(f,s) L^3(\mathrm{sym}^3 f, s) L(\mathrm{sym}^2 f \times \mathrm{sym}^3 f, s) \} |_{s=\sigma+iT} \right| T^{-1} d\sigma \\
&\ll \max_{\frac{1}{2}+\varepsilon \le \sigma \le b} x^{\sigma} T^{16(1-\sigma)+\varepsilon} T^{-1} = \max_{\frac{1}{2}+\varepsilon \le \sigma \le b} \left(\frac{x}{T^{16}} \right)^{\sigma} T^{15+\varepsilon} \\
&\ll \frac{x^{1+\varepsilon}}{T} + x^{\frac{1}{2}+\varepsilon} T^{7+\varepsilon}.
\end{aligned}
$$

From (3.2), (3.3), and (3.4), we have

$$
\sum_{n \le x} \lambda_f^5(n) \ll \frac{x^{1+\varepsilon}}{T} + x^{\frac{1}{2}+\varepsilon} T^{7+\varepsilon}.
$$

On taking $T = x^{\frac{1}{16}}$ in (3.6), we have

$$
\sum_{n \le x} \lambda_f^5(n) \ll x^{\frac{15}{16}+\varepsilon}.
$$

This completes the proof of Theorem 1.1.

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