PROJECTIVE GEOMETRY IN THE ONE-DIMENSIONAL AFFINE GROUP

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Introduction. The idea of considering the set of the elements of a group as a space, provided with a topology, measure, or metric, connected somehow with the group operation, has been used often in the work of E. Cartan and others. In the present paper we shall study a very special group whose space can be embedded naturally into a projective plane and where the straight lines have a very simple group-theoretical interpretation. It remains to be seen whether this geometrical embedding in a projective space can be extended to other classes of groups and whether the method could become an instrument of geometrical investigation, like co-ordinates or reflections. In the final section it is shown how a geometrical theorem may lead to relations within the group.

1. Let $\mathfrak{G}$ denote the affine group on a real straight line (x-axis), i.e. the group of all transformations

$$x \to ax + \alpha, \qquad a, \alpha \text{ real}, \ a \neq 0.$$

Each of these mappings may be represented by an ordered pair of real numbers $A = (a, \alpha)$. If $B = (b, \beta)$ represents also an element of $\mathfrak{G}$, then the product of A and B corresponding to the composition of the two mappings (first B, then A) is given by

$$AB = (ab, a\beta + \alpha).$$

The unit element of $\mathfrak{G}$, corresponding to the identity map, is represented by the symbol $I = (1, 0)$ and $A^{-1} = (a^{-1}, -a^{-1}\alpha)$ is the inverse of A. It will be shown how the structure of $\mathfrak{G}$ can be described geometrically.

As pairs of real numbers the elements $A = (a, \alpha)$ of $\mathfrak{G}$ may be interpreted as points in a cartesian co-ordinate plane with a horizontal a-axis ($\alpha = 0$) and a vertical α-axis ($a = 0$). The points of the second axis do not represent group elements; with regard to $\mathfrak{G}$ it is therefore an exceptional line in the plane. It will be of advantage to turn the plane into a projective plane by completing it with a straight line at infinity which, with regard to the group $\mathfrak{G}$, must also be considered as exceptional. The plane minus these two exceptional lines will be called a $\mathfrak{G}$-plane.

Two straight lines $\mathfrak{L}_1, \mathfrak{L}_2$ are said to be parallel, or rather ∞-parallel, $\mathfrak{L}_1 \parallel_\infty \mathfrak{L}_2$, if they are parallel in the usual sense, i.e. if their common point lies

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on the line at infinity. They are said to be 0-parallel, $\mathfrak{L}_1 \,||_0\, \mathfrak{L}_2$, if their common point lies on the α-axis. With regard to these parallelisms the two exceptional lines may be considered as the "absolute" or the "infinite" of the geometry of the plane of the group $\mathfrak{G}$. (By removing from the plane all straight lines through the common point of the two exceptional lines one obtains the "*frammento autoduale*" introduced by K. Menger **(1)**.)

By adding the two exceptional lines to the plane of $\mathfrak{G}$ the latter becomes a projective plane. By applying a suitable projective transformation it will be possible to reduce the two exceptional lines into any two preassigned lines of the plane; we may call them the ∞-line $\mathfrak{L}_\infty$ and the 0-line $\mathfrak{L}_0$ of the $\mathfrak{G}$-plane. Two straight lines will be said to be ∞-parallel ($||_\infty$) if they meet on $\mathfrak{L}_\infty$, and 0-parallel ($||_0$) if they meet on $\mathfrak{L}_0$.

Points on the exceptional lines will be denoted by bold face letters. The point $\mathbf{U}$ is the intersection of these two lines.

The following propositions are easily established:

1. *Any two points in the $\mathfrak{G}$-plane can be joined by a unique straight line in the $\mathfrak{G}$-plane.*

2. *Any two straight lines in the $\mathfrak{G}$-plane have exactly one common point in the $\mathfrak{G}$-plane except when they are ∞-parallel or 0-parallel.*

3. *To any line passing through* $\mathbf{U}$, *there is one and only one parallel line through each point of the $\mathfrak{G}$-plane (both parallelisms coincide).*

4. *To any line $\mathfrak{L}$ not passing through* $\mathbf{U}$, *there is exactly one 0-parallel and exactly one ∞-parallel line through each point not on $\mathfrak{L}$; both are always distinct.*

5. *Each of the two parallelisms is reflexive, symmetric, and transitive.*

6. *To any two non-parallel lines $\mathfrak{L}_1$, $\mathfrak{L}_2$ none of which passes through* $\mathbf{U}$ *there is exactly one line $\mathfrak{L}$ such that*

$$\mathfrak{L} \,||_\infty\, \mathfrak{L}_1 \quad \textit{and} \quad \mathfrak{L} \,||_0\, \mathfrak{L}_2.$$

7. *With regard to either of the two parallelisms the theorem of Desargues is valid: If ABC and $A'B'C'$ are two triangles and D another point such that*

$$\overline{AA'D}, \qquad \overline{BB'D}, \qquad \overline{CC'D}$$

are collinear sets of three points, if further

$$\overline{AB} \,||\, \overline{A'B'}, \qquad \overline{BC} \,||\, \overline{B'C'},$$

then

$$\overline{CA} \,||\, \overline{C'A'}.$$

Apart from 7 it may be pointed out that, of course, the theorem of Desargues and all the other theorems of projective geometry can be applied freely even if their interpretation in terms of the $\mathfrak{G}$-geometry proper is not immediate.

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2. We shall now define straight lines and parallelisms in $\mathfrak{G}$ algebraically. Let $A = (a, \alpha)$, $a \neq 0$, $a \neq 1$. We consider the normalizer $\mathfrak{N}_A$ of A in $\mathfrak{G}$, that is the subgroup of all T such that $TAT^{-1} = A$. Obviously, $I \in \mathfrak{N}_A$ and $A \in \mathfrak{N}_A$. Let $T = (t, \tau)$. The condition $TA = AT$ yields

$$t\alpha + \tau = a\tau + \alpha, \qquad \text{i.e. } \alpha(t-1) = (a-1)\tau;$$

hence

$$T = (t, \nu(t-1)), \qquad \nu = \alpha/(a-1),\ t \neq 0.$$

Thus $\mathfrak{N}_A$ consists of all the points of the straight line through I and A.

Every normalizer $\mathfrak{N}_A$ is the normalizer of each of its elements T, $T \neq I$ (i.e. $t \neq 1$):

$$\mathfrak{N}_A = \mathfrak{N}_T \qquad \text{if } T \in \mathfrak{N}_A.$$

Indeed, if $S = (s, \sigma) \in \mathfrak{N}_T$, then $ST = TS$ yields $\sigma = \nu(s-1)$; thus $S \in \mathfrak{N}_A$.

Every line through I, except the one through $\mathbf{U}$, is a conjugate of $\mathfrak{N}_A$. Indeed, for any $B = (b, \beta) \in \mathfrak{G}$ the conjugate subgroup $B\mathfrak{N}_A B^{-1}$ consists of the elements

$$BTB^{-1} = (t, (b\nu - \beta)(t-1)) \qquad (T \in \mathfrak{N}_A).$$

Conversely, by a suitable choice of B, the factor $b\nu - \beta$ can be given any prescribed value. In order to obtain all those lines through I as conjugates to $\mathfrak{N}_A$, it is sufficient to choose $b = 1$, that is, to take B on the line $\mathfrak{H}$ through I and $\mathbf{U}$. Every conjugate of $\mathfrak{N}_A$ is, of course, a normalizer; indeed

$$B\mathfrak{N}_A B^{-1} = \mathfrak{N}_{BAB^{-1}}.$$

The elements $H = (1, \eta)$ of $\mathfrak{H}$ form a normal subgroup of $\mathfrak{G}$. This $\mathfrak{H}$ is the only line through I which is not a normalizer $\mathfrak{N}_A$. The composition law in $\mathfrak{H}$ is the addition in the second component:

$$(1, \eta_1)(1, \eta_2) = (1, \eta_1 + \eta_2).$$

All the other straight lines in the $\mathfrak{G}$-plane are defined as the left or right cosets of subgroups $\mathfrak{N}_A$ ($A \in \mathfrak{H}$) and of $\mathfrak{H}$. Indeed, if B is any fixed element of $\mathfrak{G}$, then the elements of $B\mathfrak{N}_A$ are

$$BT = (bt, \nu b(t-1) + \beta) = (t', \nu(t' - b) + \beta) \qquad (t' = bt),$$

evidently the points of the line passing through B (put $t' = b$) and ∞-parallel to $\mathfrak{N}_A$. Indeed the lines carrying $\mathfrak{N}_A$ and $B\mathfrak{N}_A$ have a common point in $\mathfrak{G}$ if and only if

$$\frac{\beta}{b-1} = \frac{\alpha}{a-1} = \nu,$$

that is, if $B \in \mathfrak{N}_A$, or $B\mathfrak{N}_A = \mathfrak{N}_A$. They have also no common point on the 0-line. Hence they meet on the line $\mathfrak{L}_\infty$.

We notice that there is a unique element $H = (1, \eta) \in \mathfrak{H}$ such that

$$B\mathfrak{N}_A = H\mathfrak{N}_A$$

because there is a unique $H \in \mathfrak{H}$ where $B\mathfrak{N}_A$ meets $\mathfrak{H}$. Thus, each line, not parallel to $\mathfrak{H}$, meets $\mathfrak{H}$ in a unique point of the $\mathfrak{G}$-plane.

The right coset $\mathfrak{N}_A B$ consists of the elements

$$TB = (tb, t\beta + \nu(t-1)) = \left(t', \frac{\beta+\nu}{b}t' - \nu\right),$$

which are the points of the straight line that meets the line carrying $\mathfrak{N}_A$ on the 0-line (put $t = 0$). Hence $\mathfrak{N}_A \parallel_0 \mathfrak{N}_A B$. Again there is a unique element $H \in \mathfrak{H}$ such that

$$\mathfrak{N}_A B = \mathfrak{N}_A H.$$

Finally, we have the straight lines carrying $\mathfrak{H}$ and the cosets of $\mathfrak{H}$. They all pass through the point $\mathbf{U}$. The elements of the coset $\mathfrak{H}B$ are

$$HB = (1, \eta)(b, \beta) = (b, \beta + \eta).$$

We point out that $\mathfrak{H}B = B\mathfrak{H}$ is also the class of all elements conjugate to B, if $B \notin \mathfrak{H}$. Indeed

$$XBX^{-1} = (x, \xi)(b, \beta)(x^{-1}, -x^{-1}\xi) = (b, (1-b)\xi + x\beta) \qquad (x \neq 0),$$

and for each conjugate XBX^{-1} of B ($B \notin \mathfrak{H}$, i.e. $b \neq 1$) there is a unique $H \in \mathfrak{H}$ such that

$$XBX^{-1} = HBH^{-1}.$$

In fact, there is a unique η such that $(1-b)\xi + x\beta = (1-b)\eta + \beta$. Among the cosets of $\mathfrak{H}$ we mention, in particular, the class $\mathfrak{H}J$ of all involutory elements of $\mathfrak{G}$, that is the set of all $J \neq I$ such that $J^2 = I$. Evidently $J = (-1, \omega)$; we put $(-1, 0) = I'$.

We also notice that $\mathfrak{H}$ is the commutator group of $\mathfrak{G}$. Indeed every commutator

$$ABA^{-1}B^{-1} = HBH^{-1}B^{-1}$$

with a suitable $H \in \mathfrak{H}$, whatever $B \in \mathfrak{G}$. Conversely, if B is some fixed element of $\mathfrak{G}$, $B \notin \mathfrak{H}$, then every $K \in \mathfrak{H}$ can be respresented as a commutator $HBH^{-1}B^{-1}$. Indeed, $KB \in \mathfrak{H}B$ is a conjugate of B and there is a unique $H \in \mathfrak{H}$ such that

$$KB = HBH^{-1}.$$

From these properties follows that $\mathfrak{G}$ is a so-called Frobenius group, more precisely, a T_1-group; cf. **(2)**.

Since the factor group $\mathfrak{G}/\mathfrak{H}$ is abelian, we conclude that for all $A, B \in \mathfrak{G}$:

$$\mathfrak{H}AB = \mathfrak{H}BA;$$

thus AB and BA are elements of the same coset of $\mathfrak{H}$. This also follows from the relation $B(AB)B^{-1} = BA$.

Concentrating entirely on the algebraic definitions of points as group elements and straight lines as cosets of normalizers and of the greatest normal subgroup $\mathfrak{H}$, we may replace the group $\mathfrak{G}$, that is the one-dimensional affine group over the real field, by the one-dimensional affine group $\mathfrak{G}_F$ over an arbitrary field F. The $\mathfrak{G}_F$-plane is the set of the elements (a, α), a, $\alpha \in F$, $a \neq 0$, of $\mathfrak{G}_F$. By adjoining to it the two exceptional lines $\mathfrak{L}_\infty$ (the "line at infinity") and $\mathfrak{L}_0$ (the set of all symbols $(0, \eta)$, this plane becomes the projective plane over the field F. Operating with the same composition law as in $\mathfrak{G}$ the whole discussion can be carried over to the more general case of the group $\mathfrak{G}_F$.

This group can be represented as the semi-direct product of the multiplication group $F^\times$ and the addition group F^+ of the field F. Indeed if $A_0 = (a, 0)$, $a \neq 0$, $a \neq 1$, then

$$F^\times \simeq \mathfrak{N}_{A_0} \simeq \mathfrak{N}_A \qquad \text{for any } A \notin \mathfrak{H}_F, \qquad F^+ \simeq \mathfrak{H}_F,$$

and since $(a, 0)(b, 0) = (ab, 0)$, every element of $\mathfrak{G}_F$ can be written in the form

$$A = (a, \alpha) = (1, \alpha)(a, 0) = HA_0.$$

It follows that

$$\mathfrak{G}_F = \mathfrak{H}_F \mathfrak{N}_{A_0} = \mathfrak{H}_F \mathfrak{N}_A = \mathfrak{N}_A \mathfrak{H}_F.$$

3. In the following discussion we shall emphasize the equivalence between geometrical constructions with the ruler and group operations in the $\mathfrak{G}$-plane. For this reason we restrict ourselves to the case of the group $\mathfrak{G}$ over the real field. Its elements are the points of an ordinary projective plane from which two lines $\mathfrak{L}_\infty$ and $\mathfrak{L}_0$ have been removed. A point I off these lines is distinguished by its association with the unit element. On L_0, not on $\mathfrak{L}_\infty$, we mark a certain point 0; the straight line through 0 and I carries the normalizer $\mathfrak{N}^{(0)} = \mathfrak{N}_{A_0}$ ($A_0 = (a, 0)$, $a \neq 0$, $a \neq 1$). Put $\mathfrak{L}_\infty \cap \mathfrak{N}^{(0)} = \infty$, $\mathfrak{L}_0 \cap \mathfrak{L}_\infty = \mathbf{U}$.

We shall now deal with a number of problems concerning group operations and the corresponding geometrical constructions in the modified projective plane of the group $\mathfrak{G}$.

(i) *Group multiplication.* For two elements $A, B \in \mathfrak{G}$, find the product AB and the inverse A^{-1}.

(a) Neither A nor B is an element of $\mathfrak{H}$ nor are they both elements of the same normalizer $\mathfrak{N}_A$. Then AB is obtained as the intersection

$$AB = (A\mathfrak{N}_B) \cap (\mathfrak{N}_A B).$$

Indeed $A\mathfrak{N}_B$ and $\mathfrak{N}_A B$ both contain the element AB and they have one and only one point in common. For the actual construction (cf. Fig. 1) draw the lines through I and A, and through I and B; then the line through A, ∞-parallel to $\mathfrak{N}_B$, and the line through B, 0-parallel to $\mathfrak{N}_A$; these two lines meet at AB.

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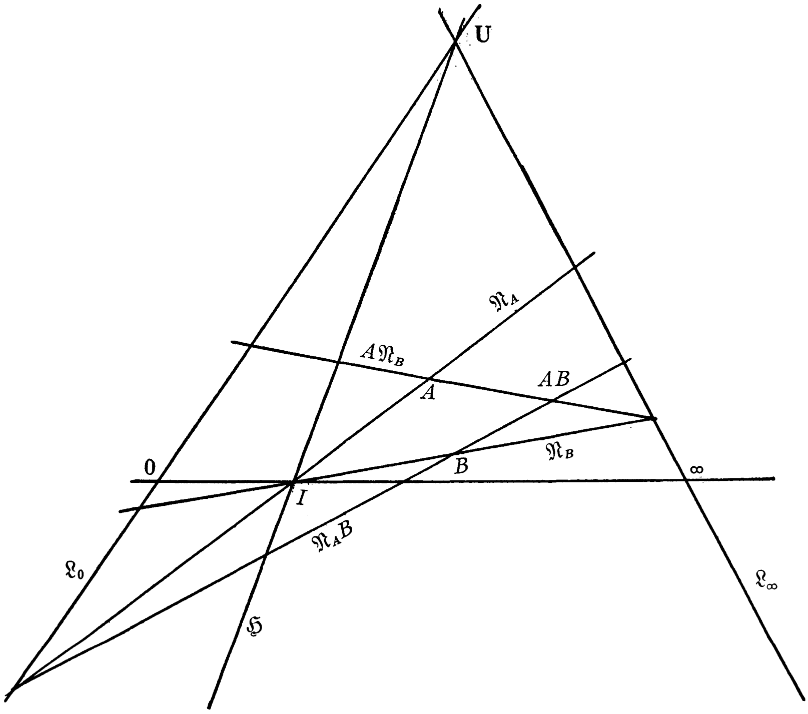

FIGURE 1

(*b*) $A \in \mathfrak{H}$, $B \notin \mathfrak{H}$, $B \notin \mathfrak{N}_A$. Then, similarly,

$$AB = (A\ \mathfrak{N}_B) \cap (\mathfrak{H}B).$$

(*c*) $A \in \mathfrak{H}$, $B \in \mathfrak{H}$. The multiplication corresponds to the addition of the second co-ordinates α, β. This can be carried out by means of the projective segment addition process (**3**, pp. 141–144) effected by the parallel displacement of a triangle over the segment $\overline{IA}$ into the position over the segment $\overline{BC}$; then $C = AB = BA$. The process can be carried out on any of the lines parallel to $\mathfrak{H}$, e.g. on $\mathfrak{H}$ itself (cf. Fig. 2). Select a point $A' \notin \mathfrak{H}$, find $A'\mathfrak{H}$. Determine the triangle $IA'A$ with its sides on $\mathfrak{N}_{A'}$, $A\mathfrak{N}^{(0)}$, and $\mathfrak{H}$. The parallel displacement is carried out by drawing the three ∞-parallel sides $B\mathfrak{N}_{A'}$, $B'\mathfrak{N}^{(0)}$, and $\mathfrak{H}$ respectively, where $B' = (A'\ \mathfrak{H}) \cap (B\ \mathfrak{N}_{A'})$. Then

$$(B'\ \mathfrak{N}^{(0)}) \cap \mathfrak{H} = C = AB.$$

Similarly, we find by segment subtraction the inverse A^{-1} of a given $A \in \mathfrak{H}$: Let $D' = \mathfrak{N}^{(0)} \cap (A'\ \mathfrak{H})$; then

$$(D'\ \mathfrak{N}_{A'}) \cap \mathfrak{H} = A.$$

Here the triangle $IA'A$ is shifted into the position $A^{-1}D'I$.

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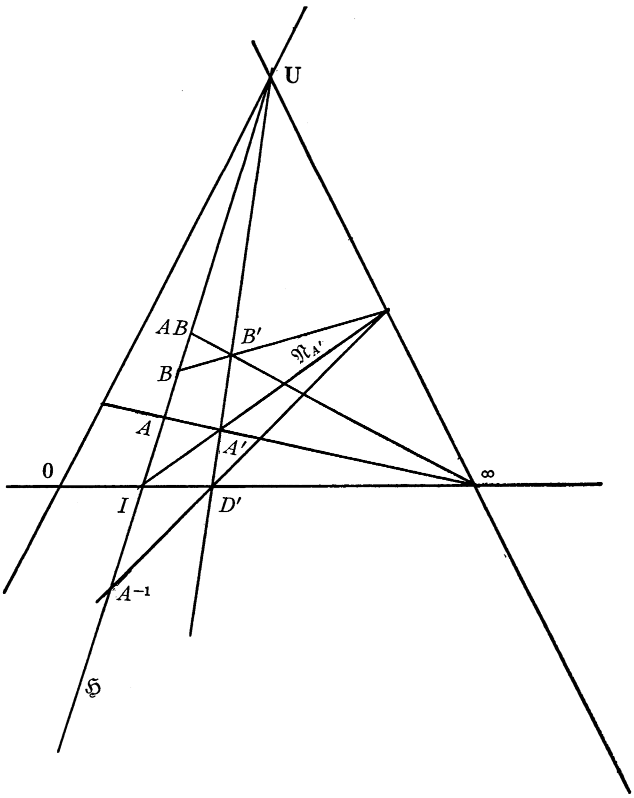

FIGURE 2

(d) Now let $A \notin \mathfrak{H}$ and $B \in \mathfrak{N}_A$. Then $AB = BA \in \mathfrak{N}_A$ and the product is defined by its first component ab. In this case one has to apply the projective segment multiplication on the line $\overline{0\infty}$, i.e. on $\mathfrak{N}^{(0)}$ (**3**, pp. 144–149). For the construction draw an arbitrary line through 0, e.g. through A, that is the coset $\mathfrak{N}^{(0)}A$. Let

$$(\mathfrak{N}^{(0)} A) \cap \mathfrak{H} = E, \qquad (A\mathfrak{H}) \cap \mathfrak{N}^{(0)} = A_0$$

(cf. Fig. 3). The line through E and A_0 is given by $E\mathfrak{N}_{E^{-1}A_0}$; cf. (i) ($c$) and (ii). Now shift the triangle IEA_0 0-parallel to itself so that the point originally at I falls into

$$B_0 = (B\mathfrak{H}) \cap \mathfrak{N}^{(0)}.$$

Further, find

$$F = (\mathfrak{N}^{(0)} A) \cap (B\mathfrak{H}), \qquad (F\mathfrak{N}_{E^{-1}A_0}) \cap \mathfrak{N}^{(0)} = C_0.$$

Then

$$\mathfrak{N}_A \cap (C_0\,\mathfrak{H}) = C = AB.$$

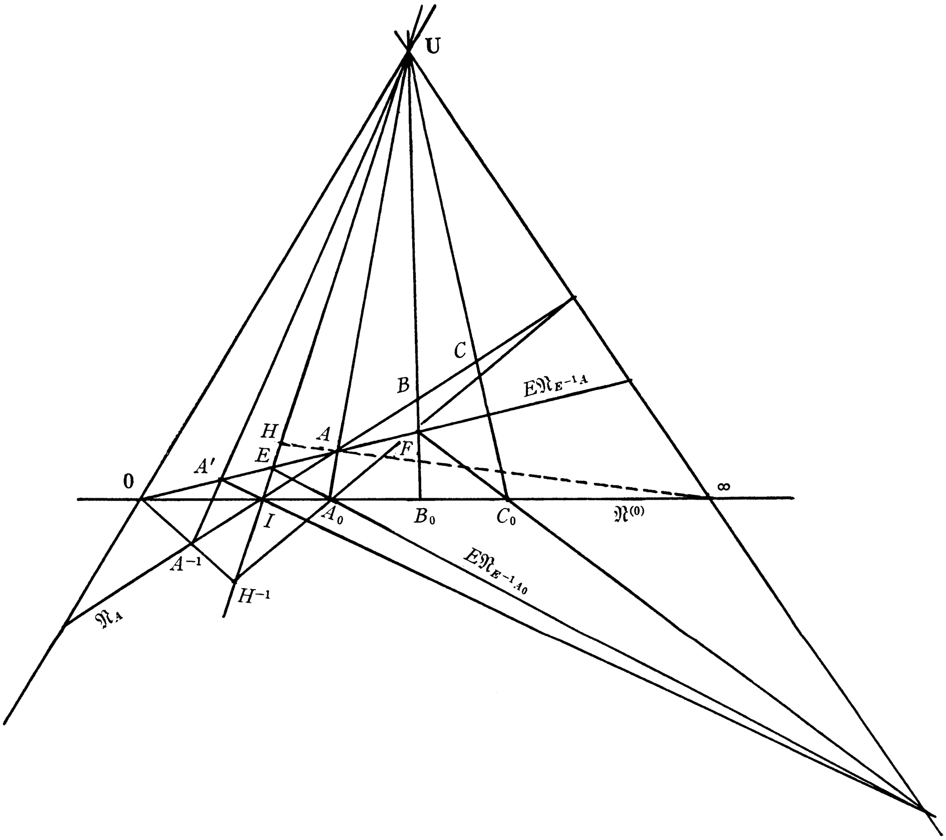

FIGURE 3

For the inverse of A $(\notin \mathfrak{H})$, one has the following construction: Find

$$A' = E\mathfrak{N}_{E^{-1}A_0} \cap (\mathfrak{N}^{(0)}\, A);$$

then

$$A^{-1} = (A'\, \mathfrak{H}) \cap \mathfrak{N}_A.$$

The following construction of A^{-1} is often preferable: Write $A = HA_0$ with (unique) $H \in \mathfrak{H}$, $A_0 \in \mathfrak{N}^{(0)}$. Then $A^{-1} = A_0^{-1} H^{-1}$, and since $\mathfrak{N}_{A^{-1}} = \mathfrak{N}_A$, it follows that $H^{-1} \in A_0\, \mathfrak{N}_A$. Hence, construct

$$A_0 = \mathfrak{N}^{(0)} \cap (\mathfrak{H}A), \qquad H^{-1} = \mathfrak{H} \cap (A_0\, \mathfrak{N}_A).$$

Then

$$A^{-1} = (\mathfrak{N}^{(0)}\, H^{-1}) \cap \mathfrak{N}_A.$$

Thus, the construction of product and inverse is settled in all cases so that all group operations can be carried out graphically in the $\mathfrak{G}$-plane, completed by the two exceptional lines, with a ruler only.

The solution of the following problem was used in (i) (*d*). We state it for reference:

(ii) Given two elements $A, B \in \mathfrak{G}$. Find the coset through A and B.

Let $A^{-1} B \notin \mathfrak{H}$. Then the line through A and B is given by

$$A\mathfrak{N}_{A^{-1}B} = B\mathfrak{N}_{A^{-1}B} = \mathfrak{N}_{AB^{-1}}B = \mathfrak{N}_{AB^{-1}}A.$$

(Note that $\mathfrak{N}_{A^{-1}B} = \mathfrak{N}_{B^{-1}A}$. If $A^{-1}B \in \mathfrak{H}$, then $A\mathfrak{H} = B\mathfrak{H}$ is the line through A and B.)

As an application, we construct $\mathfrak{N}_{A^{-1}B} \,||_{\infty}\, A\mathfrak{N}_{A^{-1}B}$ and

$$A^{-1}B = (\mathfrak{N}_A B) \cap \mathfrak{N}_{A^{-1}B},$$

observing that $A^{-1} \in \mathfrak{N}_A$.

It has been noted that all the normalizers of elements $A \notin \mathfrak{H}$ are conjugate subgroups in $\mathfrak{G}$. Thus the following question arises:

(iii) Given two normalizers $\mathfrak{N}$ and $\mathfrak{N}'$. Find all elements T such that $T\mathfrak{N}T^{-1} = \mathfrak{N}'$. (Cf. Fig. 4.)

Using a notation, not quite accurate, but as we hope, readily understood, let

$$\mathfrak{N} \cap \mathfrak{L}_\infty = \mathbf{B}, \qquad \mathfrak{N}' \cap \mathfrak{L}_0 = \mathbf{A}.$$

Then every T on the line $\overline{\mathbf{AB}}$ has the property

$$T\mathfrak{N} = \mathfrak{N}'T = \overline{\mathbf{AB}} \cap \mathfrak{G}.$$

Indeed, $T\mathfrak{N}$ is the unique line which is ∞-parallel to $\mathfrak{N}$ and at the same time 0-parallel to $\mathfrak{N}'$. On the line $\overline{\mathbf{AB}}$, there is a unique element $H \in \mathfrak{H}$ such that

$$H\mathfrak{N} = \mathfrak{N}'H = \overline{\mathbf{AB}}.$$

For the sake of symmetry we consider also the line $\mathbf{A}'\mathbf{B}'$, where

$$\mathfrak{N} \cap \mathfrak{L}_0 = \mathbf{B}', \qquad \mathfrak{N}' \cap \mathfrak{L}_\infty = \mathbf{A}'.$$

Then

$$\mathfrak{N}T^{-1} = T^{-1}\mathfrak{N}' = \mathfrak{N}H^{-1} = H^{-1}\mathfrak{N}' = \overline{\mathbf{A}'\mathbf{B}'} \cap \mathfrak{G}.$$

There is an intersection

$$\overline{\mathbf{AB}} \cap \overline{\mathbf{A}'\mathbf{B}'} = W.$$

We shall show that

$$W^2 = I.$$

For this purpose consider an arbitrary point $T = HN$ on the line $\overline{\mathbf{AB}} = H\mathfrak{N}$. It defines its normalizer $\mathfrak{N}_T$, the line through I and T which meets $\overline{\mathbf{A}'\mathbf{B}'} = \mathfrak{N}H^{-1}$ at $T' = N'H^{-1}$; $\mathfrak{N}_T$ contains T^{-1} and so does $\mathfrak{N}H^{-1}$. Hence, $T' = T^{-1}$. Since W lies on both lines $\overline{\mathbf{AB}}$ and $\overline{\mathbf{A}'\mathbf{B}'}$, it follows that $W = W^{-1}$, and the statement is proved.

Let us repeat the construction for a second normalizer $\mathfrak{N}^*$ in place of $\mathfrak{N}$ (with the same $\mathfrak{N}'$). We find the points

$$\mathbf{C} = \mathfrak{N}^* \cap \mathfrak{L}_\infty, \qquad \mathbf{C}' = \mathfrak{N}^* \cap \mathfrak{L}_0.$$

Let

$$V = \overline{\mathbf{AC}} \cap \overline{\mathbf{A'C'}}.$$

Then $V^2 = I$. Thus the three points $\mathbf{U}$, V, W are collinear on the involutory line $\mathfrak{H}I'$.

This can also be obtained as a consequence of the (general) theorem of Desargues. Indeed, a look at Fig. 4 shows that the two triangles **ABC** and **A′B′C′** are perspective with the centre I:

$$\overline{\mathbf{AA'}} \cap \overline{\mathbf{BB'}} \cap \overline{\mathbf{CC'}} = I$$

according to the construction. Thus the three points

$$\overline{\mathbf{BC}} \cap \overline{\mathbf{B'C'}} = \mathbf{U}, \qquad \overline{\mathbf{CA}} \cap \overline{\mathbf{C'A'}} = V, \qquad \overline{\mathbf{AB}} \cap \overline{\mathbf{A'B'}} = W$$

are collinear.

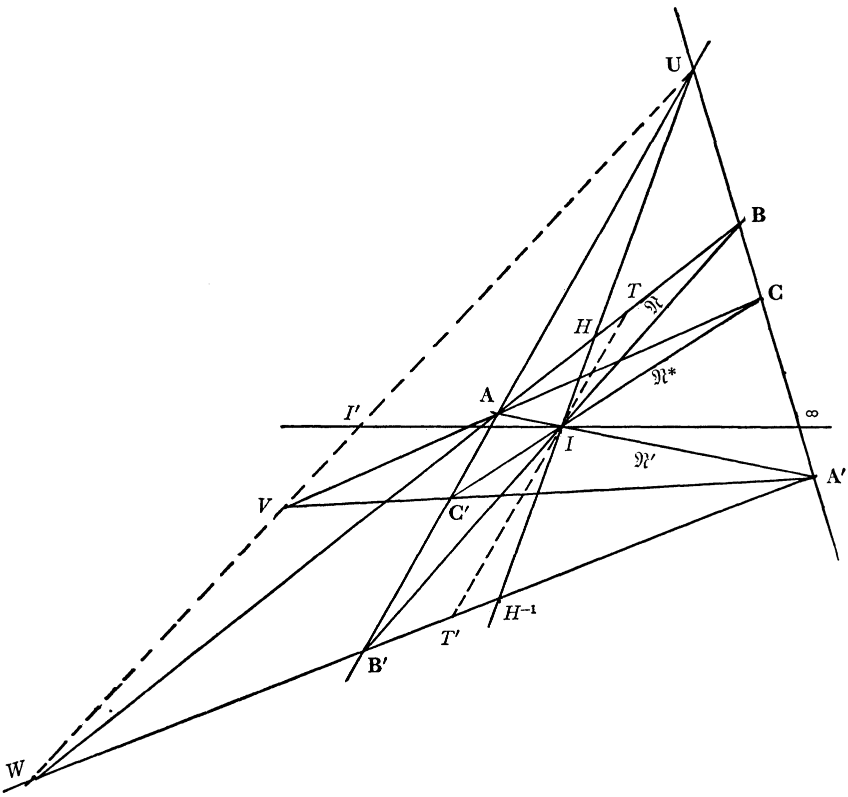

FIGURE 4

(iv) For given normalizer $\mathfrak{N}$ and element T, to find $T\mathfrak{N}T^{-1}$, find $\overline{\mathbf{B}T} \cap \mathfrak{L}_0 = \mathbf{A}$. Then $\overline{\mathbf{A}I} = \mathfrak{N}' = T\mathfrak{N}T^{-1}$.

(v) For given A and T find the point TAT^{-1}.

(a) Let $A \notin \mathfrak{H}$. Draw $\mathfrak{N}_A$ and $T\mathfrak{N}_A T^{-1}$. Then

$$TAT^{-1} = (A\mathfrak{H}) \cap (T\mathfrak{N}_A T^{-1}).$$

(b) If $A \in \mathfrak{H}$, the construction is as follows (Fig. 5): Find $T\mathfrak{H}$ and

$$TA = (T\mathfrak{H}) \cap (\mathfrak{N}_T A), \qquad TAT^{-1} = (\mathfrak{N}_{TA}\, T^{-1}) \cap \mathfrak{H}.$$

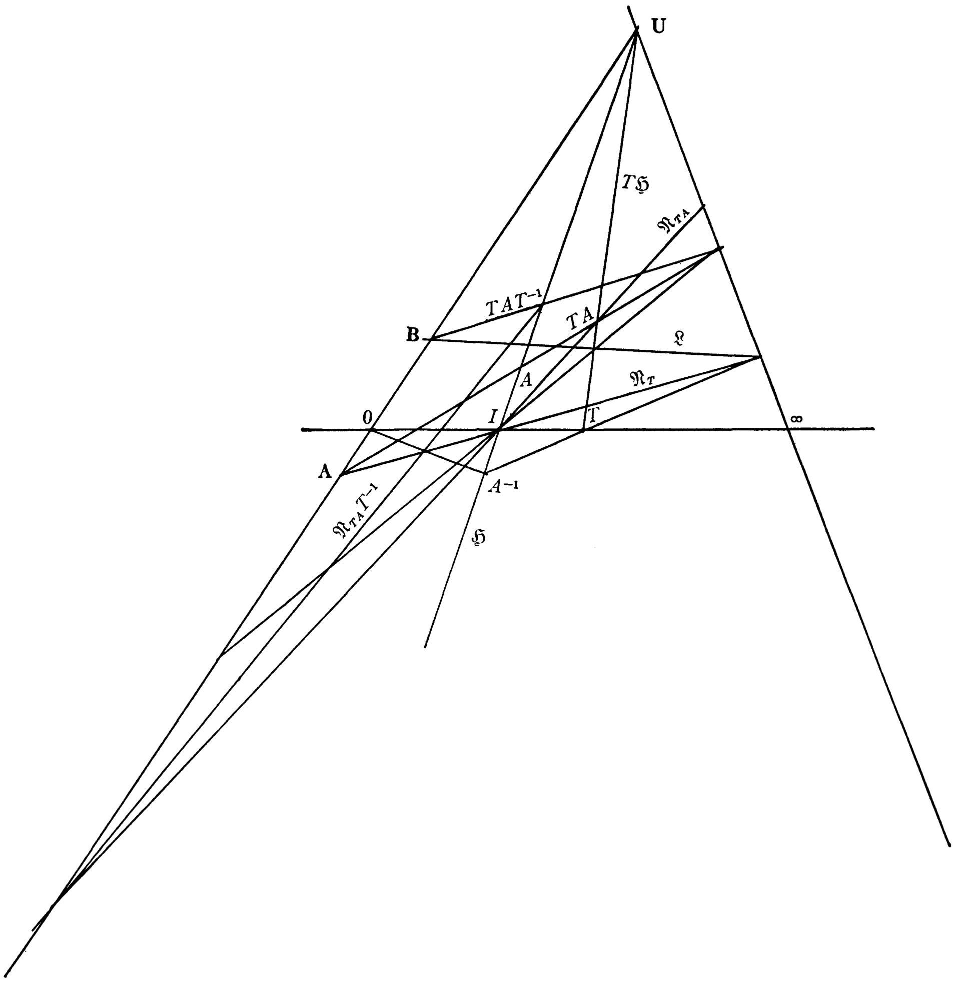

FIGURE 5

(vi) For given A and T, find the commutator

$$C = [T, A] = TAT^{-1}A^{-1}.$$

(a) $A \notin \mathfrak{H}$. With reference to Fig. 6, find

$$(T\mathfrak{N}_A T^{-1}) \cap (\mathfrak{H}A) = TAT^{-1};$$

then

$$(TAT^{-1}\mathfrak{N}_A) \cap \mathfrak{H} = TAT^{-1}A^{-1} = C.$$

Any element of $T\mathfrak{N}_A$ used instead of T gives the same result.

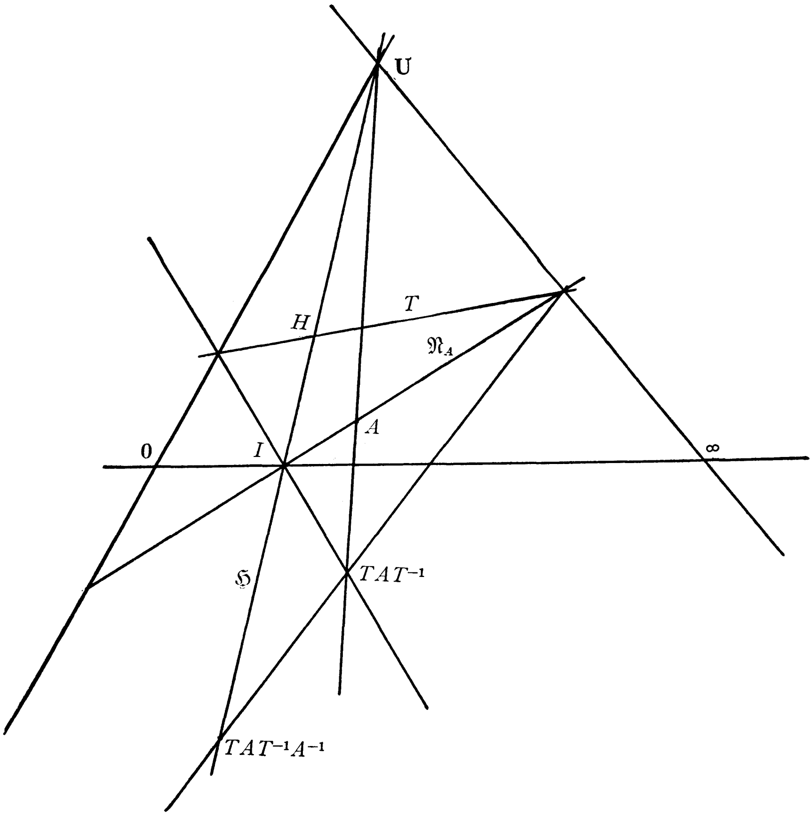

FIGURE 6

Here we may mention a relation between C and A^{-1}:

$$(T\mathfrak{N}_A T^{-1} C) \cap \mathfrak{N}_A = A^{-1}$$

or

$$(T\mathfrak{N}_A T^{-1}A^{-1}) \cap \mathfrak{H} = C.$$

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(*b*) $A \in \mathfrak{H}$. With reference to Fig. 5, multiply TAT^{-1} graphically by A^{-1} from the right (or left). This, according to (i) (*c*), is done by parallel displacement of the triangle $I\mathbf{A}A$ (where $\mathbf{A} = \mathfrak{N}_T \cap \mathfrak{L}_0$) into the position $C\mathbf{B}(TAT^{-1})$, where

$$\mathbf{B} = (TAT^{-1} A^{-1} \mathfrak{N}_T A) \cap \mathfrak{L}_0.$$

If, then, we denote by $\mathfrak{L}$ the line which is ∞-parallel with $\mathfrak{N}_T$, and 0-parallel with $(TAT^{-1})(A^{-1} \mathfrak{N}_T A)$, then

$$C = \mathfrak{H} \cap \mathfrak{L}.$$

The element T may be replaced by any element of $\mathfrak{H}T$.

From the discussion in (iii) we obtain the following general fact. Let $\mathfrak{L}_1$, $\mathfrak{L}_2$ be two lines, not parallel in any sense and neither of them passing through $\mathbf{U}$; let

$$\mathfrak{L} \parallel_\infty \mathfrak{L}_1 \quad \text{and} \quad \mathfrak{L} \parallel_0 \mathfrak{L}_2.$$

Then

$$\mathfrak{L} \cap \mathfrak{H} = H, \qquad \text{where } H\mathfrak{N}_1 = \mathfrak{N}_2 H$$

if $\mathfrak{N}_1$, $\mathfrak{N}_2$ are two normalizers such that

$$\mathfrak{L}_1 = H_1 \mathfrak{N}_1, \qquad \mathfrak{L}_2 = \mathfrak{N}_2 H_2, \qquad H_1, H_2 \in \mathfrak{H}.$$

In the present case, we have

$$\mathfrak{L}_1 = \mathfrak{N}_1 = \mathfrak{N}_T, \qquad \mathfrak{L}_2 = (TAT^{-1})(A^{-1} \mathfrak{N}_T A).$$

Hence,

$$\mathfrak{N}_2 = (TAT^{-1})(A^{-1} \mathfrak{N}_T A)(TAT^{-1})^{-1} = C \mathfrak{N}_T C^{-1}.$$

Thus, $H = C$.

4. It was seen that by group multiplication in $\mathfrak{G}$ every straight line is turned into a straight line. Hence for every fixed $A \in \mathfrak{G}$, the mappings $X \to AX$ and $X \to XA$ are collineations. They are even affine transformations in the sense that they turn ∞-parallel and 0-parallel lines respectively into ∞- and 0-parallel lines. In fact, $B(A\mathfrak{N}) = (BA)\mathfrak{N} \parallel_\infty \mathfrak{N}$; hence $B(A\mathfrak{N}) \parallel_\infty A\mathfrak{N}$. Also $(\mathfrak{N}A)B = \mathfrak{N}(AB) \parallel_0 \mathfrak{N}$; hence $(\mathfrak{N}A)B \parallel_0 \mathfrak{N}A$.

Also the conjugacy $X \to TXT^{-1}$ represents a collineation for every fixed T, but not an affine one. Indeed,

$$T(A\mathfrak{N})T^{-1} = (TAT^{-1})(T\mathfrak{N}T^{-1}). \tag{4.1}$$

Thus, the conjugate of a left coset of $\mathfrak{N}$ is a left coset of the corresponding conjugate of $\mathfrak{N}$. There is obviously a corresponding statement for right cosets.

Every normalizer $\mathfrak{N}$ is invariant by transformation with any one of its own elements and no others. For which elements T is

$$T(A\mathfrak{N})T^{-1} = A\mathfrak{N}$$

if $A\mathfrak{N} \neq \mathfrak{N}$? By (4.1) this condition implies that $A\mathfrak{N} \parallel_\infty T\mathfrak{N}T^{-1}$; hence $T\mathfrak{N}T^{-1} \parallel_\infty \mathfrak{N}$, and therefore $T\mathfrak{N}T^{-1} = \mathfrak{N}$; thus $T \in \mathfrak{N}$. But then

$$T(A\mathfrak{N})T^{-1} = T(A\mathfrak{N}) = A\mathfrak{N},$$

which means that $TA \in A\mathfrak{N}$, i.e.

$$A^{-1}TA \in \mathfrak{N},$$

whence either $A \in \mathfrak{N}$ or $T = I$. Thus the (proper) cosets of a normalizer are not invariant under any conjugacy except the identity.

Geometrically this is evident since by conjugacy the elements $X \in A\mathfrak{N}$ are moved within their cosets $\mathfrak{H}X$. It also follows that if $T(A\mathfrak{N})T^{-1} = S(A\mathfrak{N})S^{-1}$ $(A \notin \mathfrak{N})$, then $T = S$.

This might suggest the following theorem.

THEOREM. *Let $\mathfrak{L}$, $\mathfrak{M}$ be two cosets of normalizers in $\mathfrak{G}$, both not normalizers themselves. Then there is always a unique $T \in \mathfrak{G}$ such that*

$$T\mathfrak{L}T^{-1} = \mathfrak{M}.$$

This will be proved algebraically. It is sufficient to show that for $A \in \mathfrak{N}^{(0)}$ and arbitrary $\mathfrak{M}$ there is a unique T such that

$$T(A\mathfrak{N}^{(0)})T^{-1} = \mathfrak{M}. \tag{4.2}$$

For a suitable $H = (1, \eta) \in \mathfrak{H}$, we have $A\mathfrak{N}^{(0)} = H\mathfrak{N}^{(0)}$. Thus, an arbitrary element of this coset, ∞-parallel to $\mathfrak{N}^{(0)}$, is given in the form (x, η), where x is variable, η fixed. If $T = (t, \tau)$, then

$$T(x, \eta)T^{-1} = (x, -\tau(x-1) + t\eta).$$

On the other hand, with a suitable normalizer $\mathfrak{N}$ and a certain element $K = (1, \kappa) \in \mathfrak{H}$ the line $\mathfrak{M}$ can be represe..ted by the coset $K\mathfrak{N}$. Thus, its elements are

$$(1, \kappa)(x, \nu(x-1)) = (x, \nu(x-1) + \kappa).$$

In order to have (4.2) satisfied with $\mathfrak{M} = K\mathfrak{N}$, we have to determine t and τ such that

$$t\eta = \kappa, \qquad -\tau = \nu,$$

which is always possible in exactly one way because $\eta \neq 0$, that is $H \neq I$.

Under the same assumptions concerning $\mathfrak{L}$ and $\mathfrak{M}$ we now consider the intersection

$$C = \mathfrak{L} \cap \mathfrak{M}.$$

Since $TCT^{-1} = \mathfrak{M} \cap (\mathfrak{H}C)$, we conclude that

$$TCT^{-1} = C, \tag{4.3}$$

if T is the element mentioned in the theorem. Hence, $T \in \mathfrak{N}_C$ or $C \in \mathfrak{N}_T$.

Now let $\mathfrak{N}$, $\mathfrak{N}'$ be normalizers such that

$$\mathfrak{L} = C\mathfrak{N}', \qquad \mathfrak{M} = \mathfrak{N}C.$$

According to §3 (iii), we find all S such that $S\mathfrak{N}' = \mathfrak{N}S$ on the line passing through the points $\mathfrak{L}_0 \cap \mathfrak{N} = \mathfrak{L}_0 \cap \mathfrak{M}$ and $\mathfrak{L}_\infty \cap \mathfrak{N}' = \mathfrak{L}_\infty \cap \mathfrak{L}$. In particular, let T_0 be the point where this line meets the normalizer $\mathfrak{N}_C$. Then also

$$\mathfrak{N} = T_0 \, \mathfrak{N}' \, T_0^{-1},$$

and since T_0 is the only S commuting with C, we have

$$\begin{aligned} \mathfrak{M} &= \mathfrak{N}C = T_0 \, \mathfrak{N}' \, T_0^{-1} \, C = T_0 \, \mathfrak{N}' \, CT_0^{-1} \\ &= T_0 \, C^{-1} \, C \, \mathfrak{N}' \, CT_0^{-1} = T_0 \, C^{-1} \, \mathfrak{L} \, CT_0^{-1} \\ &= T_0 \, C^{-1} \, \mathfrak{L}(T_0 \, C^{-1})^{-1}. \end{aligned}$$

Thus, we have

$$T = T_0 \, C^{-1}$$

as an expression for the unique element T transforming $\mathfrak{L}$ into $\mathfrak{M}$. Therefore

$$C = T^{-1} \, T_0. \tag{4.4}$$

Another expression for T can be found by means of the following construction. Let A, B be two points on $\mathfrak{L}$; find

$$A^* = (\mathfrak{H}A) \cap \mathfrak{M}, \qquad B^* = (\mathfrak{H}B) \cap \mathfrak{M},$$

so that

$$A^* = TAT^{-1}, \qquad B^* = TBT^{-1}.$$

Let

$$\begin{aligned} \mathfrak{N}_A \cap \mathfrak{L}_\infty &= \mathbf{A}, & \mathfrak{N}_{A*} \cap \mathfrak{L}_0 &= \mathbf{A}', \\ \mathfrak{N}_B \cap \mathfrak{L}_\infty &= \mathbf{B}, & N_{B*} \cap \mathfrak{L}_0 &= \mathbf{B}'. \end{aligned}$$

Then

$$T = \overline{\mathbf{AA}'} \cap \overline{\mathbf{BB}'}.$$

We note the principal result of these considerations: Whereas all normalizers of elements $A \in \mathfrak{G}$, $A \notin \mathfrak{H}$ form a complete set of conjugate subgroups and any two of them are conjugate in many ways (cf. §3 (iii)), all non-normalizer cosets of normalizers also form a family of conjugate sets and any two of them are conjugate in one and only one way.

5. In order to have full reciprocity between the group and the corresponding geometry it is necessary to express the intersection C of two lines $\mathfrak{L}, \mathfrak{M}$ by means of group operations. A first approach in this direction is contained in formula (4.4).

Assuming, again, $\mathfrak{L}$ and $\mathfrak{M}$ to be two non-parallel cosets of normalizers $\mathfrak{N}'$ and $\mathfrak{N}$ respectively, we shall determine two points $A \in \mathfrak{N}$, $B \in \mathfrak{N}'$ such that

$$C = AB. \tag{5.1}$$

Indeed this relation will be satisfied if we take

$$A = \mathfrak{L} \cap \mathfrak{N}, \qquad B = \mathfrak{M} \cap \mathfrak{N}',$$

where again $\mathfrak{L} = C\mathfrak{N}'$, $\mathfrak{M} = \mathfrak{N}C$. This then involves that $\mathfrak{N} = \mathfrak{N}_A$ and $\mathfrak{N}' = \mathfrak{N}_B$; hence

$$\mathfrak{L} = A\mathfrak{N}_B, \qquad \mathfrak{M} = \mathfrak{N}_A B,$$

whence (5.1) (cf. §3 (ii)).

To complete the solution of the problem, it remains to determine A and B. This is always possible by the co-ordinate method. Let $\mathfrak{N}$, $\mathfrak{N}'$ be given by the slopes ν, ν' so that their elements are

$$\mathfrak{N}: X = (x, \nu(x-1)), \qquad \mathfrak{N}': Y = (y, \nu'(y-1)).$$

Let $\mathfrak{L} = H\mathfrak{N}'$, where $H = (1, \eta)$. Then every element of $\mathfrak{L}$ appears in the form

$$Z = (z, \nu'(z-1) + \eta).$$

Hence, $A = \mathfrak{L} \cap \mathfrak{N}$ implies that $z = x$ and

$$\nu'(x-1) + \eta = \nu(x-1)$$

or

$$x = 1 + \frac{\eta}{\nu + \nu'},$$

and thus

$$A = \left(\frac{\nu - \nu' + \eta}{\nu - \nu'}, \frac{\nu\eta}{\nu - \nu'}\right).$$

Similarly, we find if $\mathfrak{M} = \mathfrak{N}H'$, $H' = (1, \eta')$:

$$B = \left(\frac{\nu - \nu'}{\nu - \nu' + \eta'}, -\frac{\nu\eta'}{\nu - \nu' + \eta'}\right).$$

On the basis of these formulae it is possible to carry out all the geometrical constructions with ruler by group operations within the group $\mathfrak{G}$.

6. As an example for the transfer into $\mathfrak{G}$ of a more involved statement of projective geometry we mention the theorem of Desargues as stated under §1, Proposition 7 (cf. Fig. 7). The assumptions concerning the two triangles ABC and $A'B'C'$ are expressed as follows:

$$\mathfrak{N}_{D^{-1}A'} = \mathfrak{N}_{D^{-1}A}, \qquad \mathfrak{N}_{D^{-1}B'} = \mathfrak{N}_{D^{-1}B}, \qquad \mathfrak{N}_{D^{-1}C'} = \mathfrak{N}_{D^{-1}C}. \tag{6.1}$$

Assuming, further, that no two of the points A, B, C are together in one and the same coset of $\mathfrak{H}$, we conclude from

$$\mathfrak{N}_{B'^{-1}C'} = \mathfrak{N}_{B^{-1}C} \quad \text{and} \quad \mathfrak{N}_{C'^{-1}A'} = \mathfrak{N}_{C^{-1}A} \tag{6.2}$$

that

$$\mathfrak{N}_{A'^{-1}B'} = \mathfrak{N}_{A^{-1}B}.$$

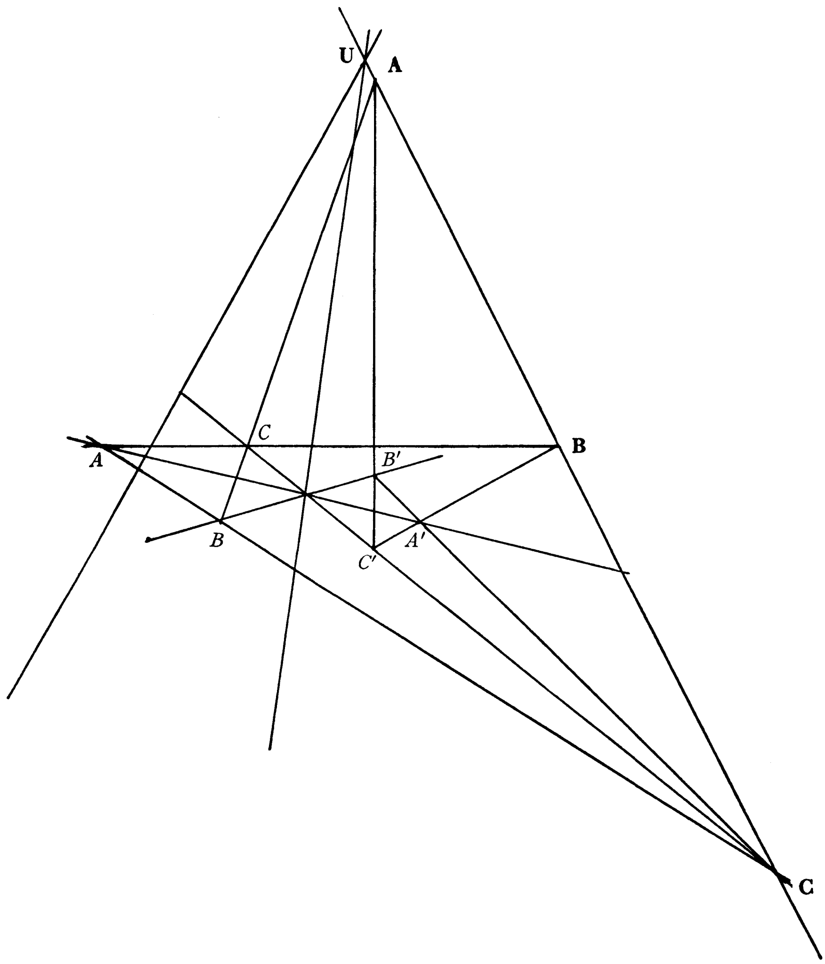

FIGURE 7

It will be no restriction to put D into I. Then (6.1) becomes

$$\mathfrak{N}_{A'} = \mathfrak{N}_A, \qquad \mathfrak{N}_{B'} = \mathfrak{N}_B, \qquad \mathfrak{N}_{C'} = \mathfrak{N}_C. \tag{6.3}$$

An application arises if we choose

$$A' = A^{-1}, \qquad B' = B^{-1}, \qquad C' = C^{-1}.$$

Then the relations (6.3) are obviously satisfied and instead of (6.2) we have the conditions

$$\mathfrak{N}_{BC^{-1}} = \mathfrak{N}_{B^{-1}C}, \qquad \mathfrak{N}_{CA^{-1}} = \mathfrak{N}_{C^{-1}A}. \tag{6.4}$$

The first relation (6.4) is equivalent to $BC^{-1} \in \mathfrak{N}_{B^{-1}C}$, i.e.

$$(BC^{-1})(B^{-1}C) = (B^{-1}C)(BC^{-1}).$$

Thus, we obtain the following commutator relation in $\mathfrak{G}$:

If $[B, C^{-1}] = [B^{-1}, C]$ and $[C, A^{-1}] = [C^{-1}, A]$, then also $[A, B^{-1}] = [A^{-1}, B]$.

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