Bull. Aust. Math. Soc. **81** (2010), 274–280
doi:10.1017/S0004972709000914

SIMPLE PROOFS FOR UNIVERSAL BINARY HERMITIAN LATTICES

POO-SUNG PARK

(Received 1 June 2009)

Abstract

If a positive definite Hermitian lattice represents all positive integers, we call it universal. Several mathematicians, including the author, have between them found 25 universal binary Hermitian lattices. But their *ad hoc* proofs are complicated. We give simple and unified proofs of universality.

2000 *Mathematics subject classification*: primary 11E39; secondary 11E20, 11E41.

Keywords and phrases: universal Hermitian lattice, binary Hermitian lattice.

1. Introduction

It has been a central problem in the theory of quadratic forms to find integers represented by quadratic forms. The celebrated *four square theorem* by Lagrange [10] was an outstanding result in this study. Ramanujan generalized this theorem and found 54 positive definite quaternary diagonal quadratic forms which represent all positive integers [13]. We call a positive definite quadratic form *universal*, if it represents all positive integers. The classification of nondiagonal universal classical quadratic forms was completed by Conway and Schneeberger using their *fifteen theorem* in 2000 [5]. The theorem was newly and beautifully proved by Bhargava [1]. The theorem states that if a positive definite classical quadratic form (with four or more variables) represents all positive integers up to 15, it is universal.

In 1997 Earnest and Khosravani defined universal positive definite Hermitian forms as ones representing all positive integers and they found 13 universal *binary* Hermitian forms over imaginary quadratic fields of class number one [6]. For instance, $x\overline{x} + y\overline{y}$ over $\mathbb{Q}(\sqrt{-1})$ represents all positive integers. Iwabuchi extended the result to imaginary quadratic fields of class number bigger than one and he found nine binary Hermitian lattices (as a generalization of Hermitian forms) [7]. Kim and the current author completed the list by appending 3 universal binary Hermitian forms [9]. Moreover, Kim, Kim and the current author found an analogous result to the fifteen theorem: if a positive definite Hermitian lattice represents up to 15, then it represents

This work was supported by the Kyungnam University Foundation Grant, 2009.

all positive integers [8]. The proof was more complicated than that of the Conway–Schneeberger theorem for it contains nonclassical quadratic forms. The criterion for universal nonclassical quadratic forms (the 290 theorem) was recently proved by Bhargava and Hanke [2].

In the present paper we give simple and unified proofs for universal binary Hermitian lattices. Although the three papers [6, 7, 9] contained proofs, they were complicated and used local properties of Hermitian forms. The 290 theorem can also be used to verify the universalities, since a binary Hermitian lattice can be converted to a quaternary (nonclassical) quadratic form. But it is unnatural to use such a big theorem for proofs.

2. Notation

Let $E = \mathbb{Q}(\sqrt{-m})$ for a positive square-free integer m. Denote the $\mathbb{Q}$-involution by $\bar{\cdot}$ and the ring of integers by $\mathcal{O} = \mathcal{O}_E$. The generators of $\mathcal{O}$ are 1 and ω over $\mathbb{Z}$ where $\omega = \sqrt{-m}$ if $m \not\equiv 3 \bmod 4$ and $\omega = (1 + \sqrt{-m})/2$ otherwise.

A finitely generated $\mathcal{O}$-module L is called a *Hermitian lattice* over $\mathcal{O}$ if there exists a nondegenerate Hermitian space $(E \otimes_{\mathcal{O}} L, H)$ over E. We consider only positive definite integral lattices. That is, we assume that $H(\mathbf{x}, \mathbf{y}) \in \mathcal{O}$ for all $\mathbf{x}, \mathbf{y} \in L$ and $H(\mathbf{x}) := H(\mathbf{x}, \mathbf{x}) > 0$ if $0 \neq \mathbf{x} \in L$.

If the class number of E is one, all Hermitian lattices L are free and we can write

$$L = \mathcal{O}\mathbf{v}_1 + \mathcal{O}\mathbf{v}_2 + \cdots + \mathcal{O}\mathbf{v}_n$$

where $n = \operatorname{rank} L = \dim_E E \otimes L$. Then the Gram matrix for L is defined as $M_L = [H(\mathbf{v}_i, \mathbf{v}_j)]_{1 \le i,j \le n}$.

A nonfree Hermitian lattice L can be written as

$$L = \mathcal{O}\mathbf{v}_1 + \mathcal{O}\mathbf{v}_2 + \cdots + \mathcal{O}\mathbf{v}_{n-1} + \mathcal{A}\mathbf{v}_n$$

with a nonprincipal ideal $\mathcal{A}$ in $\mathcal{O}$ [12]. Since $\mathcal{A}$ is generated by two elements $\alpha, \beta \in \mathcal{O}$, we can rewrite

$$L = \mathcal{O}\mathbf{v}_1 + \mathcal{O}\mathbf{v}_2 + \cdots + \mathcal{O}\mathbf{v}_{n-1} + \mathcal{O}(\alpha\mathbf{v}_n) + \mathcal{O}(\beta\mathbf{v}_n).$$

So we can deal with L as if L were a *free* Hermitian lattice of rank $n + 1$. The (formal) Gram matrix is defined as the $(n + 1) \times (n + 1)$ matrix

$$M_L = \begin{bmatrix} H(\mathbf{v}_1, \mathbf{v}_1) & \cdots & H(\mathbf{v}_1, \mathbf{v}_{n-1}) & H(\mathbf{v}_1, \alpha\mathbf{v}_n) & H(\mathbf{v}_1, \beta\mathbf{v}_n) \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ H(\alpha\mathbf{v}_n, \mathbf{v}_1) & \cdots & H(\alpha\mathbf{v}_n, \mathbf{v}_{n-1}) & H(\alpha\mathbf{v}_n, \alpha\mathbf{v}_n) & H(\alpha\mathbf{v}_n, \beta\mathbf{v}_n) \\ H(\beta\mathbf{v}_n, \mathbf{v}_1) & \cdots & H(\beta\mathbf{v}_n, \mathbf{v}_{n-1}) & H(\beta\mathbf{v}_n, \alpha\mathbf{v}_n) & H(\beta\mathbf{v}_n, \beta\mathbf{v}_n) \end{bmatrix}$$

whose rank, however, is still n.

If $L = L_1 \oplus L_2$ and $H(L_1, L_2) = \{0\}$, then we write $L = L_1 \perp L_2$. If L is a Hermitian lattice generated by only one vector $\mathbf{v}$, then we write $L = \langle H(\mathbf{v}) \rangle$. Also $\langle H(\mathbf{v}_1) \rangle \perp \cdots \perp \langle H(\mathbf{v}_n) \rangle$ is written as $\langle H(\mathbf{v}_1), \ldots, H(\mathbf{v}_n) \rangle$. From now on we identify a Hermitian lattice L and its (formal) Gram matrix M_L.

https://doi.org/10.1017/S0004972709000914 Published online by Cambridge University Press

3. Main result

Earnest and Khosravani [6], Iwabuchi [7], and Kim and the current author [9] found all universal binary Hermitian lattices over imaginary quadratic fields.

THEOREM 3.1. *There are* 25 *universal binary Hermitian lattices over imaginary quadratic fields up to isometry.*

$$\mathbb{Q}(\sqrt{-1}) : \langle 1,1\rangle,\ \langle 1,2\rangle,\ \langle 1,3\rangle$$
$$\mathbb{Q}(\sqrt{-2}) : \langle 1,1\rangle,\ \langle 1,2\rangle,\ \langle 1,3\rangle,\ \langle 1,4\rangle,\ \langle 1,5\rangle$$
$$\mathbb{Q}(\sqrt{-3}) : \langle 1,1\rangle,\ \langle 1,2\rangle$$
$$\mathbb{Q}(\sqrt{-5}) : \langle 1,2\rangle,\ \langle 1\rangle \perp \begin{bmatrix} 2 & -1+\omega_5 \\ -1+\overline{\omega}_5 & 3 \end{bmatrix}$$
$$\mathbb{Q}(\sqrt{-6}) : \langle 1\rangle \perp \begin{bmatrix} 2 & \omega_6 \\ \overline{\omega}_6 & 3 \end{bmatrix}$$
$$\mathbb{Q}(\sqrt{-7}) : \langle 1,1\rangle,\ \langle 1,2\rangle,\ \langle 1,3\rangle$$
$$\mathbb{Q}(\sqrt{-10}) : \langle 1\rangle \perp \begin{bmatrix} 2 & \omega_{10} \\ \overline{\omega}_{10} & 5 \end{bmatrix}$$
$$\mathbb{Q}(\sqrt{-11}) : \langle 1,1\rangle,\ \langle 1,2\rangle$$
$$\mathbb{Q}(\sqrt{-15}) : \langle 1\rangle \perp \begin{bmatrix} 2 & \omega_{15} \\ \overline{\omega}_{15} & 2 \end{bmatrix}$$
$$\mathbb{Q}(\sqrt{-19}) : \langle 1,2\rangle$$
$$\mathbb{Q}(\sqrt{-23}) : \langle 1\rangle \perp \begin{bmatrix} 2 & \omega_{23} \\ \overline{\omega}_{23} & 3 \end{bmatrix},\quad \langle 1\rangle \perp \begin{bmatrix} 2 & -1+\omega_{23} \\ -1+\overline{\omega}_{23} & 3 \end{bmatrix}$$
$$\mathbb{Q}(\sqrt{-31}) : \langle 1\rangle \perp \begin{bmatrix} 2 & \omega_{31} \\ \overline{\omega}_{31} & 4 \end{bmatrix},\quad \langle 1\rangle \perp \begin{bmatrix} 2 & -1+\omega_{31} \\ -1+\overline{\omega}_{31} & 4 \end{bmatrix}.$$

We can associate a $2n$-dimensional quadratic space $(\widetilde{V}, B_H)$ over $\mathbb{Q}$ with an n-dimensional Hermitian space (V, H) over E by considering V as a vector space over $\mathbb{Q}$ and defining a bilinear map $B_H(\mathbf{x}, \mathbf{y}) = \frac{1}{2}\,\mathrm{Tr}_{E/\mathbb{Q}}\, H(\mathbf{x}, \mathbf{y})$. Thus, to prove the universality of a given Hermitian lattice, we may show that the associated quadratic form represents all positive integers.

For $m \not\equiv 3 \bmod 4$ the quadratic forms associated with *free* Hermitian lattices are diagonal. So their universalities can be checked by Ramanujan's list. The quadratic forms associated with

$$\langle 1\rangle \perp \begin{bmatrix} 2 & \omega_6 \\ \overline{\omega}_6 & 3 \end{bmatrix} \quad \text{and} \quad \langle 1\rangle \perp \begin{bmatrix} 2 & \omega_{10} \\ \overline{\omega}_{10} & 5 \end{bmatrix}$$

are also diagonal, $x^2 + 2y^2 + 3z^2 + 6w^2$ and $x^2 + 2y^2 + 5z^2 + 10w^2$, and they are universal.

The two Hermitian lattices $\langle 1, 1\rangle$ and $\langle 1, 2\rangle$ over $\mathbb{Q}(\sqrt{-3})$ are associated with quadratic forms

$$x^2 + xy + y^2 + z^2 + zw + w^2 \quad \text{and} \quad x^2 + xy + y^2 + 2z^2 + 2zw + 2w^2.$$

They represent universal quadratic forms

$$x^2 + z^2 + 3y^2 + 3w^2 \quad \text{and} \quad x^2 + 2z^2 + 3y^2 + 6w^2,$$

respectively. Also,

$$\langle 1\rangle \perp \begin{bmatrix} 2 & -1+\omega_5 \\ -1+\overline{\omega}_5 & 3 \end{bmatrix}$$

over $\mathbb{Q}(\sqrt{-5})$ contains a universal lattice $\langle 1, 2\rangle$.

The quadratic forms associated with $\langle 1, 1\rangle$ over $\mathbb{Q}(\sqrt{-7})$ and $\langle 1, 2\rangle$ over $\mathbb{Q}(\sqrt{-11})$ lie in one-class genera as listed in [11].

$$\mathbb{Q}(\sqrt{-7}) : \langle 1, 1\rangle \text{ corresponds to } x^2 + y^2 + 2z^2 + 2w^2 + xz + yw$$
$$\mathbb{Q}(\sqrt{-11}) : \langle 1, 2\rangle \text{ corresponds to } x^2 + 2y^2 + 3z^2 + 6w^2 + xz + 2yw.$$

Thus nine lattices remain:

$$\mathbb{Q}(\sqrt{-7}) : \langle 1, 2\rangle \text{ corresponds to } f_{7,2} = x^2 + 2y^2 + 2z^2 + 4w^2 + xy + 2zw,$$
$$\langle 1, 3\rangle \text{ corresponds to } f_{7,3} = x^2 + 2y^2 + 3z^2 + 6w^2 + xy + 3zw,$$
$$\mathbb{Q}(\sqrt{-11}) : \langle 1, 1\rangle \text{ corresponds to } f_{11} = x^2 + y^2 + 3z^2 + 3w^2 + xz + yw,$$
$$\mathbb{Q}(\sqrt{-15}) : \langle 1\rangle \perp \begin{bmatrix} 2 & \omega_{15} \\ \overline{\omega}_{15} & 2 \end{bmatrix}$$
$$\text{corresponds to } f_{15} = x^2 + 2y^2 + 2z^2 + 4w^2 + xw + yz,$$
$$\mathbb{Q}(\sqrt{-19}) : \langle 1, 2\rangle \text{ corresponds to } f_{19} = x^2 + 2y^2 + 5z^2 + 10w^2 + xz + 2yw,$$
$$\mathbb{Q}(\sqrt{-23}) : \langle 1\rangle \perp \begin{bmatrix} 2 & \omega_{23} \\ \overline{\omega}_{23} & 3 \end{bmatrix}, \quad \langle 1\rangle \perp \begin{bmatrix} 2 & -1+\omega_{23} \\ -1+\overline{\omega}_{23} & 3 \end{bmatrix}$$
$$\text{correspond to } f_{23} = x^2 + 2y^2 + 3z^2 + 6w^2 + xw + yz,$$
$$\mathbb{Q}(\sqrt{-31}) : \langle 1\rangle \perp \begin{bmatrix} 2 & \omega_{31} \\ \overline{\omega}_{31} & 4 \end{bmatrix}, \quad \langle 1\rangle \perp \begin{bmatrix} 2 & -1+\omega_{31} \\ -1+\overline{\omega}_{31} & 4 \end{bmatrix}$$
$$\text{correspond to } f_{31} = x^2 + 2y^2 + 4z^2 + 8w^2 + xw + yz.$$

Note that the quadratic forms associated with each of the two lattices over $\mathbb{Q}(\sqrt{-23})$ and $\mathbb{Q}(\sqrt{-31})$ coincide. So we will show the universalities of the seven quaternary quadratic forms.

The key idea of the proofs is to find a genus whose classes are all represented by the associated quaternary quadratic form. Then we can use the local conditions for representation of numbers.

Gerena for plenty of ternary quadratic forms are listed in the Brandt–Intrau–Schiemann tables [3]. Refer to [4, Ch. 8] for the local representations.

3.1. Universality of $f_{7,2}$. Let $g = x^2 + 2y^2 + 2z^2 + xy$. Then $f_{7,2}$ represents $g(x, y, z) + 2 \cdot 7w^2$. Note that g represents two forms

$$g_1 = g(x + y + z, -2y - 2z, y - 2z) = x^2 + 9y^2 + 15z^2 + 6yz,$$
$$g_2 = g(x + 2y + z, -2z, x - y) = 3x^2 + 6y^2 + 7z^2$$

which compose a whole genus. So g represents all positive integers which are represented by the genus of g_1 and g_2. Thus, g represents n unless $n \equiv 2 \bmod 3$ or $7 \mid n$.

Since $7 = (-1 + 2\omega)(-1 + 2\overline{\omega})$, $7^s \cdot 3t$ with $7 \nmid t$ and $7^s(3t + 1)$ with $7 \nmid (3t + 1)$ are represented by $f_{7,2}$.

If $n = 7^s(3t + 2)$ with $7 \nmid (3t + 2)$, then $(3t + 2) - 2 \cdot 7 \not\equiv 2 \bmod 3$ and $7 \nmid (3t + 2) - 2 \cdot 7$. Thus if $3t + 2 \geq 2 \cdot 7$, then $3t + 2$ is represented by $f_{7,2}$. So is $7^s(3t + 2)$.

It is easily checked that $f_{7,2}$ represents 2, 5, 8 and 11. Hence, $f_{7,2}$ represents all positive $7^s(3t + 2)$ and $f_{7,2}$ is universal.

3.2. Universality of $f_{7,3}$. The quadratic form $f_{7,3}$ represents two ternary forms

$$g_1 = f_{7,3}(x + z, -2z, y, 0) = x^2 + 3y^2 + 7z^2,$$
$$g_2 = f_{7,3}(2z, x, y, 0) = 2x^2 + 3y^2 + 4z^2 + 2xz$$

which compose a whole genus. So $f_{7,3}$ represents all positive integers not divisible by 3.

Since $\langle 1, 3\rangle$ represents $3\langle 1, 3\rangle = \langle 3, 3^2\rangle$, $f_{7,3}$ represents all positive integers.

3.3. Universality of f_{11}. The quadratic form f_{11} represents two ternary forms

$$g_1 = f_{11}(y, x + z, 0, -2z) = x^2 + y^2 + 11z^2,$$
$$g_2 = f_{11}(2z, x, y, 0) = x^2 + 3y^2 + 4z^2 + 2yz$$

which compose a whole genus. Thus f_{11} represents all positive integers not divisible by 11.

Since $11 = (-1 + 2\omega)(-1 + 2\overline{\omega})$, f_{11} represents all positive integers.

3.4. Universality of f_{15}. The quadratic form f_{15} represents

$$g_1 = f_{15}(x, 2z, y, 0) = x^2 + 2y^2 + 8z^2 + 2yz,$$
$$g_2 = f_{15}(x, y + z, -y + z, 0) = x^2 + 3y^2 + 5z^2$$

which compose a whole genus. So f_{15} represents all positive integers not divisible by 5.

Since

$$f_{15}(2y + 3z, x + 3w, x - 2w, y - z) = 5 f_{15}(x, y, z, w),$$

f_{15} represents all positive integers.

3.5. Universality of f_{19}. Let $g = x^2 + 2y^2 + 5z^2 + xz$. Then f_{19} represents $g(x, y, z) + 2 \cdot 19w^2$. Note that g represents two forms

$$\begin{aligned} g_1 &= g(5z, x, y) = 2x^2 + 5y^2 + 25z^2 + 5yz, \\ g_2 &= g(x + 3z, x + y - z, -y - z) \\ &= 3x^2 + 7y^2 + 13z^2 + 3yz + xz + 3xy \end{aligned}$$

which compose a whole genus. So g represents all positive integers n unless $n \equiv 1, 4 \bmod 5$ or $2 \mid n$.

Since $\langle 1, 2\rangle$ represents $2\langle 1, 2\rangle$, f_{19} represents all positive integers $n = 2^s(5t + k)$ with $k = 2, 3$ and $2 \nmid (5t + k)$.

Suppose that $n = 5t + 1$ with $t \geq 8$ is odd. Then $n - 2 \cdot 19 = 5(t - 8) + 3$ is represented by g.

If $n = 5t + 4$ with $t \geq 30$ is odd, then $n - 2 \cdot 19 \cdot 2^2 = 5(t - 30) + 2$ is represented by g.

Since $\langle 1, 2\rangle$ represents all positive integers up to $5 \cdot 30 + 4$, f_{19} is universal.

3.6. Universality of f_{23}. Let $g = x^2 + 2y^2 + 3z^2 + yz$. Then f_{23} represents $g(x, y, z) + 23w^2$. Note that g represents two forms

$$\begin{aligned} g_1 &= g(x, 2y, 2z) = x^2 + 8y^2 + 12z^2 + 4yz, \\ g_2 &= g(x - y, -2z, x + y + z) = 4x^2 + 4y^2 + 9z^2 + 4yz + 4xz + 4xy \end{aligned}$$

which compose a whole genus. Thus g represents all positive integers n unless $n \equiv 2, 3 \bmod 4$ or $23 \mid n$.

Since

$$f_{23}(y, 2z, 2x, w) = 2f_{23}(x, y, z, w) \quad \text{and} \quad 23 = (-1 + 2\omega)(-1 + 2\overline{\omega}),$$

f_{23} represents $n = 2^r \cdot 23^s(4t + 1)$ with $23 \nmid (4t + 1)$.

Suppose that $n = 4t + 3$ is not divisible by 23. Then $n - 23 = 4(t - 5)$ is represented by g since $23 \nmid (t - 5)$.

Since f_{23} represents all positive integers up to $4 \cdot 5 + 3$, f_{23} is universal.

3.7. Universality of f_{31}. Let $g = x^2 + 2y^2 + 4z^2 + yz$. Then f_{31} represents $g(x, y, z) + 31w^2$. Note that g represents three forms

$$\begin{aligned} g_1 &= g(x, 4z, y) = x^2 + 4y^2 + 32z^2 + 4yz, \\ g_2 &= g(x, 2y, 2z) = x^2 + 8y^2 + 16z^2 + 4yz, \\ g_3 &= g(2x + z, y - z, -y - z) = 4x^2 + 5y^2 + 8z^2 + 4yz + 4xz \end{aligned}$$

which compose a whole genus. This genus represents all positive integers n unless $n \equiv 2, 3 \bmod 4$ or $31 \mid n$.

Since

$$f_{31}(2y, w, x, 2z) = 2f_{31}(x, y, z, w) \quad \text{and} \quad 31 = (-1 + 2\omega)(-1 + 2\overline{\omega}),$$

f_{31} represents $n = 2^r \cdot 31^s(4t + 1)$ with $31 \nmid (4t + 1)$.

https://doi.org/10.1017/S0004972709000914 Published online by Cambridge University Press

Suppose that $n = 4t + 3$ is not divisible by 31. Then $n - 31 = 4(t - 7)$ is represented by g since $31 \nmid (t - 7)$.

Since f_{31} represents all positive integers up to $4 \cdot 7 + 3$, f_{31} is universal.

References

[1] M. Bhargava, 'On the Conway–Schneeberger fifteen theorem', *Contemp. Math.* **272** (2000), 27–37.

[2] M. Bhargava and J. Hanke, 'Universal quadratic forms and the 290-theorem', http://www.math.duke.edu/~jonhanke/290/Universal-290.html.

[3] H. Brandt, O. Intrau and A. Schiemann, 'The Brandt–Intrau–Schiemann tables', http://www.research.att.com/~njas/lattices/Brandt_1.html, http://www.research.att.com/~njas/lattices/Brandt_2.html.

[4] J. W. S. Cassels, *Rational Quadratic Forms*, London Mathematical Society Monographs, 13 (Academic Press, New York, 1978).

[5] J. H. Conway, 'Universal quadratic forms and the fifteen theorem', *Contemp. Math.* **272** (2000), 23–26.

[6] A. G. Earnest and A. Khosravani, 'Universal binary Hermitian forms', *Math. Comp.* **66**(219) (1997), 1161–1168.

[7] H. Iwabuchi, 'Universal binary positive definite Hermitian lattices', *Rocky Mountain J. Math.* **30**(3) (2000), 951–959.

[8] B. M. Kim, J. Y. Kim and P.-S. Park, 'The fifteen theorem for universal Hermitian lattices over imaginary quadratic fields', *Math. Comp.*, to appear.

[9] J.-H. Kim and P.-S. Park, 'A few uncaught universal Hermitian forms', *Proc. Amer. Math. Soc.* **135** (2007), 47–49.

[10] J. L. Lagrange, 'Démonstration d'un théorème d'arithmétique', *Œuvres* **3** (1770), 189–201.

[11] G. L. Nipp, 'Gordon Nipp's tables of quaternary quadratic forms', http://www.research.att.com/~njas/lattices/nipp.html.

[12] O. T. O'Meara, *Introduction to Quadratic Forms* (Spinger, New York, 1973).

[13] S. Ramanujan, 'On the expression of a number in the form $ax^2 + by^2 + cz^2 + dw^2$', *Proc. Cambridge Philos. Soc.* **19** (1917), 11–21.

POO-SUNG PARK, Department of Mathematics Education, Kyungnam University, Masan, 631-701, Korea
e-mail: pspark@kyungnam.ac.kr

https://doi.org/10.1017/S0004972709000914 Published online by Cambridge University Press