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Equisingular Deformations of Sandwiched Singularities

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Abstract. We relate the equisingular deformation theory of plane curve singularities and sandwiched surface singularities. We show the existence of a smooth map between the two corresponding deformation functors and study the kernel of this map. In particular we show that the map is an isomorphism when a certain invariant is large enough.

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Key words. deformations, equisingularity, plane curve singularities, rational surface singularities, sandwiched singularities.

Introduction

In this paper we connect the theory of equisingular deformations of plane curve singularities and sandwiched surface singularities, see below for the definition of sandwiched singularities. From a simultaneous embedded deformation of an equisingular family of plane curve singularities we identify a simultaneous resolution of a family of sandwiched surface singularities.

We consider deformations in the sense of [12], and equisingular deformations in the sense of [15, 16]. In this language we prove that there is a smooth map of deformation functors $\mathrm{ES}_C \to \mathrm{ES}_X$ where ES_C is the functor of equisingular deformations of a plane curve C and ES_X is the deformation functor of a particular surface singularity X. We identify the kernel (on the tangent level) of this map as a subspace of the tangent space of ES_C generated by the image of a complete ideal and a cohomology group. The surface singularity X is constructed from the plane curve singularity C and an integer vector a. We prove that the map above is an isomorphism when a is large enough.

Theorems 4.1 and 4.2 of this paper was proved in my PhD thesis [7] under the assumption that C is irreducible, see also [6]. Theo de Jong later gave a proof of the second part of Theorem 4.1 and of Theorem 4.2 in the context of RC-deformations (see [4]), without assuming that C is irreducible. In the present paper, we give our original proof but without assuming C irreducible.

1. Notation

DEFINITION 1.1. A sandwiched singularity is a normal surface singularity which lies on a projective surface V which admits a birational morphism to a smooth projective surface S.

Cyclic quotient singularities and, more generally, rational surface singularities with reduced fundamental cycle, are sandwiched singularities, see [5, 13]. As remarked by Spivakovsky [13], there is also a general method of constructing sandwiched singularities from plane curve singularities as follows.

Let Spec R be an open subset of $\mathbb{A}^2 = \operatorname{Spec} \mathbb{C}[x, y]$ such that $C = \operatorname{Spec} R/(f)$ has an isolated singularity at the origin (corresponding to the maximal ideal $\mathfrak{m} = \mathfrak{m}_R = (x, y)$) and such that the complement of the origin is smooth. We will refer to C as a plane curve singularity. Let

$$\begin{array}{ccccccc} Z_n & \longrightarrow & Z_{n-1} & \longrightarrow & \cdots & \longrightarrow & Z_0 = \operatorname{Spec} R \\ \uparrow & & \uparrow & & & & \uparrow \\ C_n & \longrightarrow & C_{n-1} & \longrightarrow & \cdots & \longrightarrow & C_0 = C \end{array} \qquad (1.1)$$

be an embedded (possibly nonminimal) good resolution of C (i.e. the components of the inverse image of C in Z_n meet transversally in pairs and have only normal crossings). Let $\{p_{i1}, \ldots, p_{ik}\}$ be the intersection of the strict transform C_i of C in Z_i with the exceptional set of $Z_i \to Z_0$. Then $Z_{i+1} \to Z_i$ is the blow up of Z_i in some of these points. Let π be the composition of the upper row of maps in (1.1), and let $E = \pi^{-1}(0) = \bigcup_{i=1}^m E_i$ be the decomposition of the exceptional set into its irreducible components. Assume $E_r, \ldots, E_m$ are those with $E_i^2 = -1$. Then intersection matrix of $E_1, \ldots, E_{r-1}$ will be negative definite, so we may blow these curves down to obtain a surface Y. This surface is in fact algebraic, see [1, 11]. Let C^* denote the total transform of C in Z. Then $C^* = D + \tilde{C}$, where $D = \sum m_i E_i$, and we put $\mathfrak{q} = H^0(Z, \mathcal{O}_Z(-D)) = H^0(\pi_* \mathcal{O}_Z(-D)) \subset R$. The ideal $\mathfrak{q}$ is a complete ideal, in the sense of [17]. We denote by $\tilde{C} = C_n \subset Z = Z_n$ the strict transform of C. Put $\tilde{X} = Z \backslash \tilde{C}$. Then $\tilde{X}$ is the minimal good resolution of some affine subset $X \subset Y$.

DEFINITION 1.2. Let C be a plane curve singularity. Let k be the number of analytic branches of C, and let $a = (a_1, \ldots, a_k) \in \mathbb{N}^k$, $a_i > 0$. Let $Z = Z_n$ be as above such that Z is obtained from the minimal good resolution Z_m of C as follows. Setting $i = m$, there are k points p_{ij}, in the notation above. Repeatedly blow up p_{ij} if $a_j > 0$, and decrease a_j by one when p_{ij} is blown up.

We put $Z_{(C,a)} = Z$ and let $Y_{(C,a)} = Y$ be the corresponding contraction. Let X be the open subscheme of Y obtained as above. We define $X_{(C,a)} = X$, and $\mathfrak{q}_{(C,a)} = \mathfrak{q}$ where $\mathfrak{q}$ is obtained as above.

In this setting we have the following properties, see, for instance, [13].

PROPOSITION 1.3. *Let* $Z = Z_{(C,a)}$, $Y = Y_{(C,a)}$ *and* $X = X_{(C,a)}$ *be as above. Then*

(1) *Y has a unique isolated singular point which is rational,*
(2) *X is an affine open subscheme of Y containing the singularity,*
(3) *up to analytic equivalence the singularity of Y only depends on the isomorphism class of C and on the integer vector a,*
(4) *up to analytic equivalence all sandwiched singularities are obtained as the singularity of $Y_{(C,a)}$ for some C and a,*
(5) *Y is the blow up of* Spec R *in* $\mathfrak{q}_{(C,a)}$.

Remark 1.4. Note that the a_i used here are the same as $l(i) - M(i)$ in the notation of [5].

Remark 1.5. Note that $X_{(C,a)}$ is an affine scheme of finite type over $\mathbb{C}$, and thus an algebraic *representative* of a normal surface singularity. We will however speak of $X_{(C,a)}$ as a surface singularity.

2. Relating ES_C, ES_X and ES_Y

Let C, $a \in \mathbb{N}^k$, $a_i > 0$, $Z = Z_{(C,a)}$, $X = X_{(C,a)}$ and $Y = Y_{(C,a)}$ be as defined above.

2.1. DEFINITION OF ES_C

We introduce the deformation functor ES_C. Consider a good embedded resolution of C, as in 1.1. Let s_{ij}: Spec $\mathbb{C} \to Z_i$ define the points which are blown up. Let also t_j: Spec $\mathbb{C} \to Z$ define the ordinary double points of the reduced total transform of C. (The reduced total transform of C is $E + C_n$ where E is the reduced exceptional divisor.) We define a deformation of C over $A \in \mathcal{C}$ with simultaneous embedded resolution, to be a deformation $\bar{C}$ of C, a commutative diagram

$$\begin{array}{ccccccc} \bar{Z} = \bar{Z}_n & \longrightarrow & \bar{Z}_{n-1} & \longrightarrow & \cdots & \longrightarrow & \bar{Z}_0 \\ \uparrow & & \uparrow & & & & \uparrow \\ Z = Z_n & \longrightarrow & Z_{n-1} & \longrightarrow & \cdots & \longrightarrow & Z_0 \end{array} \tag{2.1}$$

and A-sections $\bar{s}_{ij}$: Spec $A \to \bar{Z}_i$ inducing s_{ij}, A-sections $\bar{t}_j$: Spec $A \to \bar{Z}$ inducing t_j such that

(1) in (2.1), the sections $\bar{s}_{ij}$ and $\bar{t}_j$, $\bar{C}_i$ (the strict transform of $\bar{C}$ in $\bar{Z}_i$) and $\bar{E}$ (the unique Cartier divisor in $\bar{Z}_n$ inducing E) give a deformation of the situation given by 1.1, the sections s_{ij} and t_j, C_i and E as above, in particular, such that $\bar{Z}_{i+1}$ is the blow up of $\bar{Z}_i$ in the sections $\bar{s}_{ij}$
(2) all the obvious diagrams commute and all sections are compatible
(3) all A-sections are normally flat (i.e. they are defined by an ideal $\bar{\mathfrak{m}}$ such that $\bar{\mathfrak{m}}^n$ is A-flat for all $n > 0$.)

Two deformations

$$\{\bar{C}, \bar{Z}_i, \bar{s}_{ij}, \bar{t}_j\} \quad \text{and} \quad \{\bar{\bar{C}}, \bar{\bar{Z}}_i, \bar{\bar{s}}_{ij}, \bar{\bar{t}}_j\}$$

are said to be isomorphic if there is an isomorphism (of deformations) of $\bar{Z}_0$ and $\bar{\bar{Z}}_0$ such that it

(1) sends $\bar{C}$ to $\bar{\bar{C}}$,
(2) induces an isomorphism of (2.1) and the corresponding diagram for $\bar{\bar{Z}}_0$,
(3) the sections are compatible with the isomorphisms.

We denote by $\mathcal{C}$ the category of local artinian $\mathbb{C}$-algebras (with residue field $\mathbb{C}$.) The functor of equisingular deformations of C is defined as follows. Let for $A \in \mathcal{C}$,

$$\mathrm{ES}_C(A) = \begin{cases} \text{Set of isomorphism classes of deformations } \bar{C} \text{ over Spec } A \\ \text{of } C \text{ with simultaneous embedded resolution.} \end{cases}$$

Working with algebraic representatives, this is analogous to the definition given in [16], 2.7, 2.12, 3.2, 7.3. In particular is ES_C independent of the choice of embedded resolution, 1.1.

The functor of equisingular deformation of a plane curve possesses the following property:

THEOREM 2.1 ([16], 7.4). $\mathrm{ES}_C \subset \mathrm{Def}_C$ *is a smooth subfunctor and has a hull.*

2.2. DEFINITION OF ES_X

For normal surface singularities the concept of equisingular deformations is more difficult than for plane curves. Wahl, [16] tries to define equisingular deformation of normal surface singularities through special deformations of the minimal good resolution. In the case of rational surface singularities the definition reduces to the following simple one.

Let $X = \mathrm{Spec}\, B$ be a rational surface singularity and let $\tilde{X}$ be its minimal good resolution. Let $E_X = \bigcup E_{X,i}$ be the exceptional set in $\tilde{X}$. The functor of equisingular deformations of X, is for $A \in \mathcal{C}$ defined by

$$\mathrm{ES}_X(A) = \begin{cases} \text{Set of isomorphism clases of deformations } \bar{\tilde{X}} \text{ over Spec } A \\ \text{of } \tilde{X} \text{ to which the } E_{X,i} \text{ lift (locally trivially).} \end{cases}$$

Remark 2.2. If the $E_{X,i}$ lift to $\bar{\tilde{X}}$, they lift uniquely.

This is also (for rational surface singularities) the functor of simultaneous resolutions along normally flat sections, see [16, Theorem 5.16]. It has the following properties.

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THEOREM 2.3 ([16], 4.6). *Let X be a rational surface singularity and let $\tilde{X}$ be the minimal good resolution with exceptional set E. Then $\mathrm{ES}_X \subset \mathrm{Def}_X$ is a smooth subfunctor and has a hull. Moreover $\mathrm{ES}_X(\mathbb{C}[\epsilon]) = H^1(\tilde{X}, \theta(\log E)) \subset H^1(\tilde{X}, \theta)$ where $\theta(\log E)$ is the dual of the sheaf of one forms with logarithmic poles along the exceptional set.*

2.3. DEFINITION OF ES_Y

Inspired by the definition of ES_X, we define ES_Y for $A \in \mathcal{C}$, by

$$\mathrm{ES}_Y(A) = \left\{ \begin{array}{l} \text{Set of isomorphism classes of deformations } \bar{Z} \text{ over Spec } A \\ \text{of } Z \text{ to which the } E_i \text{ lift (locally trivially).} \end{array} \right.$$

To justify this definition it is shown in ([7], Prop. 4.5.28) that ES_Y may be viewed as the subfunctor of Def_Y corresponding to deformations which restricts to equisingular deformations of X.

2.4. CONNECTING THE FUNCTORS ON THE TANGENT LEVEL AND SMOOTHNESS OF ES_Y

The tangent space for the functor ES_Y is described by a cohomology group. Let $\theta = \theta_Z$ be the tangent sheaf on Z, and let $\theta(\log E) \subset \theta$ be the dual of the sheaf of one forms with logarithmic poles along the exceptional set E of $Z \to \mathrm{Spec}\, R$. Then from general theory we know that there is an identification $\mathrm{ES}_Y(\mathbb{C}[\epsilon]) = H^1(Z, \theta(\log E))$.

Remark 2.4. Since the exceptional sets of $Z \to \mathrm{Spec}\, R$ and $Y \to \mathrm{Spec}\, R$ are one-dimensional, then if $\mathcal{F}$ is any quasi-coherent sheaf on Y or on Z, we may calculate cohomology using a Čech complex with respect to a covering $\{U_i\}$ of an open neighbourhood of the exceptional set in Y or Z, respectively, where the U_i are open affine (or Stein analytic) subsets such that three of them never intersect, see [8], III.4.5, III.8.1, III.8.5. In particular

$$H^1(\mathcal{F}) = \bigoplus_{i,j} H^0(U_i \cap U_j, \mathcal{F}) / \bigoplus_i H^0(U_i, \mathcal{F})$$

and $H^2(\mathcal{F}) = 0$. Note also that the $H^i(\mathcal{F})$ are R-modules.

Let $\mathbb{T}^1_Y$ be the tangent space of Def_Y, and let T^1_X be the tangent space of Def_X. From Remark 2.4, all H^2 are zero, and in our situation we have from the local global spectral sequence, see, for instance, [9], the short exact sequence

$$0 \to H^1(Y, \theta_Y) \to \mathbb{T}^1_Y \to T^1_X \to 0, \tag{2.2}$$

where θ_Y is the tangent sheaf on Y. Here $H^1(Y, \theta_Y)$ is the subspace of $\mathbb{T}_Y$ corresponding to the locally trivial deformations of Y. We claim that there is a similar sequence involving the tangent spaces for ES_Y and ES_X.

PROPOSITION 2.5. *There is an exact sequence*

$$0 \to H^1(Y, \theta_Y) \to \mathrm{ES}_Y(\mathbb{C}[\epsilon]) \to \mathrm{ES}_X(\mathbb{C}[\epsilon]) \to 0.$$

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Proof. Let $E' = \bigcup_{i=1}^{r-1} E_i$. From Theorem 2.3 we have

$$\mathrm{ES}_X(\mathbb{C}[\epsilon]) = H^1(\tilde{X}, \theta(\log E')).$$

Recall that there are exact sequences

$$0 \to \theta(\log E) \to \theta_Z \to \bigoplus_{i=1}^{m} N_{E_i} \to 0$$

and

$$0 \to \theta(\log E') \to \theta_Z \to \bigoplus_{i=1}^{r-1} N_{E_i} \to 0$$

of sheaves on Z, see [16], Prop. 2.2. We have $H^0(N_{E_i}) = 0$ for all i, so $H^1(Z, \theta(\log E')) = \ker(H^1(Z, \theta_Z) \to \bigoplus_{i=1}^{r-1} H^1(Z, N_{E_i}))$ and $H^1(Z, \theta(\log E)) = \ker(H^1(Z, \theta_Z) \to \bigoplus_{i=1}^{m} H^1(Z, N_{E_i}))$. By Riemann–Roch we have $H^1(Z, N_{E_i}) = 0$ for $r \leqslant i \leqslant m$, which shows that $H^1(Z, \theta(\log E)) = H^1(Z, \theta(\log E'))$. In our situation

$$\begin{array}{ccccc}
0 & & 0 & & \\
\downarrow & & \downarrow & & \\
H^1(Z, \theta(\log E')) & \xrightarrow{\alpha} & H^1(\tilde{X}, \theta(\log E')) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
H^1(Z, \theta) & \xrightarrow{\beta} & H^1(\tilde{X}, \theta) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
\bigoplus H^1(Z, N_{E_i}) & \xrightarrow{\gamma} & \bigoplus H^1(\tilde{X}, N_{E_i}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
0 & & 0 & &
\end{array}$$

Consider the map γ. Since N_{E_i} has support on E_i and the sum is taken over $E_i \subset \tilde{X}$, it follows that γ is an isomorphism, and thus that $\ker \alpha \simeq \ker \beta$. We claim that $\ker \beta \simeq H^1(Y, \theta_Y)$. Let π_1 be the restriction of π to $\tilde{X}$. From the Leray spectral sequence we get

$$0 \to H^1(Y, (\pi_1)_* \theta) \to H^1(Z, \theta) \to H^0(Y, R^1(\pi_1)_* \theta) \to 0.$$

Burns and Wahl [3], Prop. 1.2, state that $(\pi_1)_* \theta_Z = \theta_Y$, and

$$H^0(Y, R^1(\pi_1)_* \theta_Z) = H^1(\tilde{X}, \theta),$$

see [8], Prop. III.8.2.

From the proposition and from Theorem 2.3, it follows that

$$\mathrm{ES}_Y(\mathbb{C}[\epsilon]) = H^1(Z, \theta(\log E)) \subset \mathbb{T}^1_Y.$$

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PROPOSITION 2.6. *The functor* ES_Y *is smooth.*

Proof. It is clear that the obstructions for smoothness sit in the cohomology group $H^2(Z, \theta(\log E))$, see [16], Prop. 2.5(iii). But from Remark 2.4, $H^2(Z, \theta(\log E)) = 0$.

3. The Maps ρ and σ and their Kernels

By Theorem 2.1, to every $\bar{C}$ representing an element in $\mathrm{ES}_C(A)$, $(A \in \mathcal{C})$, there corresponds a deformation $\bar{Z}$. Since $\bar{Z}$ is constructed by blowing up sections, this represents in fact an element in $\mathrm{ES}_Y(A)$. In this way we get a well defined map of deformation functors $\rho\colon \mathrm{ES}_C \to \mathrm{ES}_Y$. It is also clear from the definitions that by restriction we have a map σ of deformation functors $\sigma\colon \mathrm{ES}_Y \to \mathrm{ES}_X$. Recall that a morphism $F \to G$ of deformation functors is said to be smooth, see [12], if for any surjection $A_2 \to A_1$ in $\mathcal{C}$, the morphism

$$F(A_2) \to F(A_1) \times_{G(A_1)} G(A_2)$$

is surjective. Note in particular that a smooth morphism is surjective on the level of sets.

THEOREM 3.1. *The maps* ρ *and* σ *are smooth.*

Proof. See also [16], Prop. 4.9, Th. 4.2. Let $A_2 \to A_1$ be a small surjection in $\mathcal{C}$. Assume $\rho(A_1)(\bar{C}_1) = \bar{Z}_1$. Assume $\bar{Z}_2 \in \mathrm{ES}_Y(A_2)$ and maps to $\bar{Z}_1$. We must prove that there is $\bar{C}_2 \in \mathrm{ES}_C(A_2)$, lifting $\bar{C}_1$, such that $\rho(A_2)(\bar{C}_2) = \bar{Z}_2$. But the total transform $\bar{C}_1^* \subset \bar{Z}_1$ may be lifted locally trivially to a divisor $\bar{C}_2^* \subset \bar{Z}_2$ in the following way: Let D be such that $\mathfrak{q}\mathcal{O}_Z = \mathcal{O}_Z(-D)$, see [11], Section 18. As the E_i lift to $\bar{Z}_1$ and $\bar{Z}_2$, then so does D and its liftings are $\bar{D}_j = \Sigma r_i \bar{E}_i^{(j)}$, where $D = \Sigma r_i E_i$ and $\bar{E}_i^{(j)}$ are the liftings of the E_i to $\bar{Z}_j$. Moreover, $\bar{C}_1^* = \bar{D}_1 + \bar{C}_1'$, where $\bar{C}_1'$ lifts the strict transform of C. Now $\bar{C}_1'$ may be lifted trivially to $\bar{C}_2' \subset \bar{Z}_2$, and we may put $\bar{C}_2^* = \bar{D}_2 + \bar{C}_2'$.

There is a map $\bar{\pi}\colon \bar{Z}_2 \to \operatorname{Spec} R \otimes_{\mathbb{C}} A_2$, and the ideal

$$\bar{\mathfrak{a}} := \bar{\pi}_* \mathcal{O}_{\bar{Z}_2}(-\bar{C}_2^*) \subset \bar{\mathfrak{q}} := \bar{\pi}_* \mathcal{O}_{\bar{Z}_2}(-\bar{D}_2)$$

may be shown to be a principal ideal. This ideal gives the lifting $\bar{C}_2$ of $\bar{C}_1$. The contraction of the maximal ideals which are blown up in the resolution of C determine complete ideals $\mathfrak{q}_k$. In the same way as for $\mathfrak{q}$, the liftings $\bar{E}_i^{(j)}$ of the E_i to $\bar{Z}_j$, determine uniquely liftings $\bar{\mathfrak{q}}_k^j$ of $\mathfrak{q}_k$ such that $\bar{\mathfrak{q}}_k^{(1)}$ correspond to the sections of $\bar{C}_1$ and $\bar{\mathfrak{q}}_k^{(2)}$ determine liftings of these sections to sections for $\bar{C}_2$. This shows that $\bar{C}_2$ is in $\mathrm{ES}_C(A_2)$, and that $\bar{C}_2$ maps to $\bar{Z}_2$.

For the smoothness of σ, it is enough to remark, see [2], that from Proposition 2.5 the map $\mathrm{ES}_Y(\mathbb{C}[\epsilon]) \to \mathrm{ES}_X(\mathbb{C}[\epsilon])$ is surjective.

Let K denote the kernel of $\rho(\mathbb{C}[\epsilon])$, and recall that

$$\mathrm{ES}_C(\mathbb{C}[\epsilon]) \subset T_C^1 = \frac{\mathbb{C}[[x, y]]}{\left(f, \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}\right)},$$

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see Theorem 2.1.

THEOREM 3.2. *The kernel* $K \subset T^1_C$ *of* $\rho(\mathbb{C}[\epsilon])$ *is given by the image of* $\mathfrak{q}_{(C,a)}$ *in* T^1_C, *and* $H^1(Y, \theta_Y)$ *is the kernel of* $\sigma(\mathbb{C}[\epsilon])$.

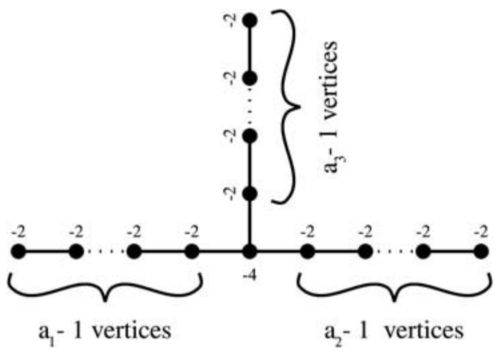
Proof. Let $\bar{C} \in \mathrm{ES}_C(\mathbb{C}[\epsilon])$. Assume that $C = V(f) \subset \operatorname{Spec} R$ and that $\bar{C}$ corresponds to a lifting $\bar{f} \in R \otimes_{\mathbb{C}} \mathbb{C}[\epsilon]$ of f. Let $\bar{Z}$ represent $\rho(\mathbb{C}[\epsilon])(\bar{C})$. As in the proof of Theorem 3.1, one constructs uniquely a corresponding lifting $\bar{\mathfrak{q}}$ of $\mathfrak{q}$, with $\bar{f} \in \bar{\mathfrak{q}} \subset R \otimes_{\mathbb{C}} \mathbb{C}[\epsilon]$. Now, if $\bar{Z}$ represents a trivial deformation of Z, then we may assume (after an infinitesimal automorphism) that $\bar{\mathfrak{q}}$ is the trivial lifting of $\mathfrak{q}$. So if $\bar{f} = f + \epsilon g$ we must have $g \in \mathfrak{q}$. This shows that K is contained in the image of $\mathfrak{q}$ in T^1_C.

On the other hand, assume $g \in \mathfrak{q}$ and consider the lifting $\bar{f} = f + \epsilon g$ of f. Denote by $\bar{Z}$ and $\bar{\mathfrak{q}}$ the trivial liftings of Z and $\mathfrak{q}$ respectively. Then $\bar{f} \in \bar{\mathfrak{q}}$ and in fact the total transform of $\bar{C}$ in $\bar{Z}$ is of the form $\bar{C}' + \bar{D}$ where $\bar{D}$ is as in the proof of Theorem 3.1, and where $\bar{C}'$ is the strict transform of $\bar{C}$ in $\bar{Z}$. Again as in the proof of Theorem 3.1, we may uniquely construct sections for $\bar{C}$ to show that ρ maps $\bar{C} \in \mathrm{ES}_C(\mathbb{C}[\epsilon])$ to $\bar{Z}$.

Finally, the kernel of $\sigma(\mathbb{C}[\epsilon])$ is identified by Proposition 2.5.

The following example illustrates how the theorem connects the theory of equisingular deformations of plane curve singularities and surface singularities.

EXAMPLE 3.3. Let $C = \operatorname{Spec} \mathbb{C}[x, y]/(x^3 + y^3)$ be three lines intersecting in a point. This has $\mathrm{ES}_C(\mathbb{C}[\epsilon]) = 0$. Thus for any $a = (a_1, a_2, a_3) \in \mathbb{N}^3$, it follows from Theorem 3.1 that $X = X_{(C,a)}$ is equisingular rigid. Indeed $X_{(C,a)}$ has the dual graph



and this dual graph is known to correspond to a unique equisingular rigid surface singularity, see [10].

We now consider the case when $a \in \mathbb{N}^k$ is large.

LEMMA 3.4. *For every* $n \gg 0$, *there exists an* $a^* \in \mathbb{N}^k$ *depending only on the topological type of* C, *such that* $\mathfrak{q}_{(C,a)} \subset (f) + \mathfrak{m}^n$ *for all* $a \geqslant a^*$.

Proof. Since every complete ideal may be written as a product of simple complete ideals, it is enough to consider the case when $\mathfrak{q}$ is simple and C irreducible. Then the

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lemma follows from [14]. In fact, there are numerical invariants g, $\bar{\beta}_0, \ldots, \bar{\beta}_{g+1}$ and $n_0, \ldots, n_{g+1}$, computable from the graph, such that

— $\bar{\beta}_0, \ldots, \bar{\beta}_g$ generates the semigroup of C and do not depend on a
— $n_0, \ldots, n_g$ do not depend on a
— $\bar{\beta}_{g+1}$ is linear in a.

In [14], it is shown that in R (localizing if necessary) $\mathfrak{q}$ is generated by

$$\left\{ \prod_{j=0}^{g+1} Q_j^{\gamma_j} \middle| \sum_{j=0}^{g} \gamma_j \bar{\beta}_j \geqslant \bar{\beta}_{g+1} \right\}$$

where in particular Q_0 and Q_1 are parameters for the maximal ideal of R at the origin and Q_{g+1} defines C. Now let $\mathfrak{a}$ be the ideal generated by the set

$$\left\{ \prod_{j=0}^{g} Q_j^{\gamma_j} \middle| \sum_{j=0}^{g} \gamma_j \bar{\beta}_j \geqslant \bar{\beta}_{g+1} \right\}$$

and set $f = Q_{g+1}$ to be the defining element of C. Then we have $\mathfrak{q}_{(C,a)} = (f) + \mathfrak{a}$. From [14], we also have that

$$\mathrm{mult}_{m_R} \mathfrak{a} = \min \left\{ \sum_{j=0}^{g} \gamma_j \prod_{i=0}^{j-1} n_i \middle| \sum_{j=0}^{g} \gamma_j \bar{\beta}_j \geqslant \bar{\beta}_{g+1} \right\} \tag{3.1}$$

and since $\bar{\beta}_{g+1} = (a-1)e_g + \bar{\beta}_g n_g$ and $e_g, \bar{\beta}_0, \ldots, \bar{\beta}_g$ and $n_0, \ldots, n_g$ do not depend on a, we see that the multiplicity of $\mathfrak{a}$ only depends on the semigroup of C and on a. Furthermore, the multiplicity of a increases with a, that is, we may increase the multiplicity of $\mathfrak{a}$ beyond any limit, by increasing a.

THEOREM 3.5. *There exists an $a^* \in \mathbb{N}^k$ (k being the number of analytic branches of C) which depends only on the topological type of C, such that if $a \geqslant a^*$ and $Y = Y_{(C,a)}$, then* $\mathrm{ES}_C(\mathbb{C}[\epsilon]) = \mathrm{ES}_Y(\mathbb{C}[\epsilon])$.

Proof. From Theorem 3.2, we know that $\mathrm{ES}_Y(\mathbb{C}[\epsilon])$ is the image of $\mathrm{ES}_C(\mathbb{C}[\epsilon]) \subset T^1_C$ in $\mathbb{C}[x,y]/(f, \partial f/\partial x, \partial f/\partial y) + \mathfrak{q}$. Here f defines C and $\mathfrak{q} = \mathfrak{q}_{(C,a)}$. We will prove that there exist an a^* such that $a \geqslant a^*$ implies

$$\mathfrak{q} \subset \left(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right).$$

Counting dimensions, we must have $(x,y)^{\mu} \subset (\partial f/\partial x, \partial f/\partial y)$, where

$$\mu = \dim_{\mathbb{C}} \frac{\mathbb{C}[[x,y]]}{\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)}.$$

From Lemma 3.4, we may choose a^* such that $a \geqslant a^*$ gives $\mathfrak{q} \subset (f) + (x,y)^{\mu}$. It is known that μ is a topological invariant of C.

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THEOREM 3.6. *There exists an* $a^* \in \mathbb{N}^k$ (*k being the number of analytic branches of* C) *which depends only on the topological type of* C, *such that if* $a \geqslant a^*$ *and* $Y = Y_{(C,a)}$, *then* $H^1(Y, \theta_Y) = 0$.

Proof. We calculate $H^1(Y, \theta_Y)$ as

$$H^1(Y, \theta_Y) = H^0(U \cap \tilde{X}, \theta_Z)/H^0(\tilde{X}, \theta_Z) + H^0(U, \theta_Z),$$

where $Z = U \cup \tilde{X}$ and U is affine. This identity is deduced from the fact that $\pi_* \theta_Z = \theta_Y$, where $\pi: Z \to Y$ is the minimal resolution, see [3], Prop. 1.2. To each analytic branch of C there corresponds a chain of -2 exceptional curves in Z of length a_i, $1 \leqslant i \leqslant k$. Each of these chains end with a -1-curve. Consider one of these chains. The -1-curve may be covered by two copies of $\mathbb{C}^2$ having coordinates (x, y) and (x', y'), respectively, such that, locally, U has coordinates (x, y) and $U \cap \tilde{X}$ is given by $x \neq 0$. On the other end of the chain the last -2-curve is covered by copies of $\mathbb{C}^2$. Denote one of them by U_l with coordinates (u_l, v_l) where l is the length of the chain. We may assume that

$$x = \frac{1}{u_l^l v_l^{l+1}}, \qquad u_l = xy^{l+1}, \qquad y = u_l v_l, \qquad v_l = \frac{1}{xy^l}$$

$(x' = v_0, y' = u_0.)$ From this one calculates that

$$\frac{\partial}{\partial x} = (u_l v_l)^{l+1} \frac{\partial}{\partial u_l} - u_l^l v_l^{l+2} \frac{\partial}{\partial v_l}, \qquad \frac{\partial}{\partial y} = \frac{l+1}{v_l} \frac{\partial}{\partial u_l} - \frac{l}{u_l} \frac{\partial}{\partial v_l}.$$

We also have $y^i/x^j = u_l^{i+jl} v_l^{i+j(l+1)}$. For a given natural number n, we may choose a such that l is large enough so that $(y^i/x^j)\,\partial/\partial x$ and $(y^i/x^j)\,\partial/\partial y$ are zero in $\theta_Z(U_l)/\mathfrak{m}^n \theta_Z(U_l)$. From this we deduce that $(y^i/x^j)\,\partial/\partial x$ and $(y^i/x^j)\,\partial/\partial y$ map to zero in $H^1(Z, \theta(\log E)) \otimes_R R/\mathfrak{m}^n$ via the composition of the map $H^1(Y, \theta_Y) \to H^1(Z, \theta(\log E))$ in (the proof of) Proposition 2.5 and the natural map. We may do similar considerations for the other chains of -2-curves to prove that all the potential generators of $H^1(Y, \theta_Y)$ map to zero.

Let b be the a^* of Theorem 3.5. Then it follows that for $a > b$, $H^1(Z_{(C,a)}, \theta(\log E))$ and $H^1(Z_{(C,b)}, \theta(\log E))$ are isomorphic as (finitely generated) $\mathbb{C}$-vectorspaces, but then they must also be isomorphic as R-modules, since, by using the exact sequence from the proof of [3], 1.13, one can show that there is a surjective R-module homomorphism, $H^1(Z_{(C,a)}, \theta(\log E)) \to H^1(Z_{(C,b)}, \theta(\log E))$, see [7], p. 161, for details. Since $H^1(Z_{(C,b)}, \theta(\log E))$ is finitely generated as a $\mathbb{C}$-vectorspace, we may find n such that $\mathfrak{m}^n$ annihilates it, and so $\mathfrak{m}^n H^1(Z_{(C,a)}, \theta(\log E)) = 0$ for all $a > b$. Thus for this fixed n, the natural map $H^1(Z, \theta(\log E)) \to H^1(Z, \theta(\log E)) \otimes_R R/\mathfrak{m}^n$ is an isomorphism when $a > b$. The argument above then shows that, after increasing a further, the injective map $H^1(Y, \theta_Y) \to H^1(Z, \theta(\log E))$ is the zero-map, and this forces $H^1(Y, \theta_Y) = 0$.

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4. The Main Results

We may now summarize in the following theorem which is of interest form the point of view of rational surface singularities.

THEOREM 4.1. *Let C be a plane curve singularity and let $a \in \mathbb{N}^k$ where k is the number of analytic branches of C. Let $X = X_{(C,a)}$ and $Y = Y_{(C,a)}$. Then there is a smooth map $\mathrm{ES}_C \to \mathrm{ES}_X$ of deformation functors, with the following properties:*

(1) *The map is the composition of smooth maps $\mathrm{ES}_C \xrightarrow{\rho} \mathrm{ES}_Y$ and $\mathrm{ES}_Y \xrightarrow{\sigma} \mathrm{ES}_X$, where the kernel of $\rho(\mathbb{C}[\epsilon])$ is given by the image of $\mathfrak{q}_{(C,a)}$ in $\mathrm{ES}_C(\mathbb{C}[\epsilon])$ and the kernel of $\sigma(\mathbb{C}[\epsilon])$ is $H^1(Y, \theta)$.*
(2) *There exist an $a^* \in \mathbb{N}^k$ which depends only on the topological type of C, such that if $a \geqslant a^*$ then the map is an isomorphism.*

Proof. For the proof of the first part, see Theorems 3.1, 3.2, 3.5 and 3.6. We now consider the second part. Since $\mathrm{ES}_C \to \mathrm{ES}_X$ is smooth, we need only prove injectivity. From Theorems 3.6 and 3.5 this is clear on tangent level. Consider two equisingular deformations $\bar{C}_1$ and $\bar{C}_2$ over $A \in C$. By induction we may assume that they induce the same deformation class over $A' \in C$ where $A \to A'$ is a small morphism. The obstruction to lift the isomorphism over A' to an isomorphism over A sits in T^1_C and, since the deformations are equisingular, in fact in $\mathrm{ES}_C(\mathbb{C}[\epsilon])$. In view of Theorems 3.6 and 3.5 the obstruction will vanish, for details see, [7], proof of theorem 4.5.29.

Lastly, we mention the following theorem.

THEOREM 4.2. *Assume C is a plane curve singularity with topological type Φ and let Γ be the dual graph of $X = X_{(C,a)}$. Then there exists an a^*, depending only on Φ such that if $a \geqslant a^*$ the isomorphism classes of plane curve singularities with topological type Φ are in one to one correspondence with the isomorphism classes of* (*the complete local ring of*) *normal surface singularities with dual graph Γ, by the construction in Section* 1.

This theorem is proved in [6], in the case where C is irreducible, and in general case in [4].

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