A NEW REPRESENTATION AND INVERSION THEORY FOR THE LAPLACE TRANSFORMATION

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1. Introduction. In the literature, considerable attention has been devoted to the study of inversion operators for the Laplace transformation. In particular, much interest attaches to "real" inversion operators, i.e., operators which make use of the values of the generating function arising only from real values of the independent variable. Several of these operators are known (see for example Widder [**3**, chap. 7, §6; chap. 8, §25], Hirschman [**2**]).

In this paper we shall develop the inversion and representation theory for a new "real" inversion operator. If[1]

$$\text{I} \qquad f(s) = \int_0^\infty e^{-st}\phi(t)dt,$$

and

$$\text{II} \qquad L_{k,t}[f(s)] = (ke^{2k}(\pi t)^{-1})\int_0^\infty x^{-\frac{1}{2}} \cos (2kx^{\frac{1}{2}})f(k(x+1)/t)dx,$$

then we shall show that under certain conditions

$$\lim_{k\to\infty} L_{k,t}[f(s)] = \phi(t).$$

This operator was given by A. Erdélyi [**1**]. However, the resulting inversion and representation theory were not developed there.

There is another operator related to II which is given by

$$L_{k,t}[f(s)] = (ke^{2k}(\pi t)^{-1})\int_0^\infty \sin (2kx^{\frac{1}{2}})f(k(x+1)/t)dx.$$

This operator and the operator II are special cases of another operator,

$$L_{k,t}[f(s)] = [2tK_\nu(2k)]^{-1}k\int_0^\infty x^{\frac{1}{2}\nu}J_\nu(2kx^{\frac{1}{2}})f(k(x+1)/t)dx,$$

which is also given in Professor Erdélyi's paper. The inversion and representation theory for this last operator has also been investigated by the author, and it was found that the resulting theory is similar in every respect to that for the operator II.

The operator II has some points of resemblance to Phragmén's operator [**3**, chap. 7, §2] in that both are "real," involve only the values of $f(s)$ for large values of s, and involve only elementary functions. Unlike Phragmén's operator though, II is an integral operator.

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[1]The notations introduced at this point will be used consistently throughout the rest of this paper.

To avoid unessential difficulties we shall restrict our discussion to the operator II. We shall show that the operator II inverts the transformation provided only that $e^{-st}t^{-\frac{1}{2}}\phi(t)$ is absolutely integrable from zero to infinity for some value of s. Preliminary to the representation theory we shall show that

$$\lim_{k\to\infty}\int_0^\infty e^{-\sigma t}L_{k,t}[f(s)]dt = f(\sigma)$$

under the conditions that $s^{-1}f(s) \in L(\delta, \infty)$ for all positive δ, that

$$\int_x^\infty y^{-1}f(y^{-1})dy$$

be suitably restricted in its behaviour at zero and infinity, and that

$$e^{-\sigma t}L_{k,t}[f(s)] \in L(0, \infty)$$

for some σ. Representation theorems are then established for $L_{k,t}$ $[f(s)]$ in various classes of functions. Lastly the Laplace-Stieltjes transformation is treated in a similar manner.

This paper embodies the results of a portion of a study being carried out on the inversion of the Laplace transformation. Other portions of the study deal with the Laplace transformation of Banach-valued functions of a real variable.

2. Inversion theorem.

THEOREM 1. *If* $e^{-st}t^{-\frac{1}{2}}\phi(t) \in L(0, \infty)$ *for all* $s > \gamma$, *then* $f(s)$ *exists for* $s > \gamma$, *and for* $t > 0$,

(i) $$\lim_{k\to\infty} L_{k,t}[f(s)] = \tfrac{1}{2}\{\phi(t+) + \phi(t-)\}$$

at every point at which $\phi(t+)$ *and* $\phi(t-)$ *both exist.*

(ii) $$\lim_{k\to\infty} L_{k,t}[f(s)] = \phi(t)$$

at every point of the Lebesgue set of $\phi(t)$.

Proof. We shall use Widder [**3**, p. 25, theorem 15c; pp. 278-280, theorem 2b and corollaries]. Operating formally we have

$$\begin{aligned} L_{k,t}[f(s)] &= (ke^{2k}(\pi t)^{-1})\int_0^\infty x^{-\frac{1}{2}}\cos(2kx^{\frac{1}{2}})f(k(x+1)/t)dx \\ &= (ke^{2k}(\pi t)^{-1})\int_0^\infty x^{-\frac{1}{2}}\cos(2kx^{\frac{1}{2}})dx\int_0^\infty e^{-k(x+1)u/t}\phi(u)du \\ &= (ke^{2k}(\pi t)^{-1})\int_0^\infty e^{-ku/t}\phi(u)du\int_0^\infty e^{-kxu/t}x^{-\frac{1}{2}}\cos(2kx^{\frac{1}{2}})dx \\ &= (2k^{\frac{1}{2}}e^{2k}(\pi t^{\frac{1}{2}})^{-1})\int_0^\infty e^{-ku/t}u^{-\frac{1}{2}}\phi(u)du\int_0^\infty e^{-v^2}\cos\{2(kt/u)^{\frac{1}{2}}v\}dv \end{aligned}$$

(where $v^2 = kxu/t$)

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$$= (k(\pi t)^{-1})^{\frac{1}{2}} e^{2k} \int_0^\infty e^{-k(ut^{-1}+tu^{-1})} u^{-\frac{1}{2}} \phi(u) du$$
$$\sim (k(\pi t)^{-1})^{\frac{1}{2}} e^{2k} (t^2/k)^{\frac{1}{2}} \phi(t) t^{-\frac{1}{2}} e^{-2k} \Gamma(\tfrac{1}{2})$$
$$= \phi(t) \qquad \text{as } k \to \infty.$$

If the conditions of the above-mentioned theorems are fulfilled, i.e., the conditions for the interchange of integrations and for the asymptotic evaluation, the theorem will be proved.

All the conditions for the interchange of integrations are fulfilled except possibly

$$\int_0^\infty e^{-ku/t} |u^{-\frac{1}{2}} \phi(u)| du \int_0^\infty |e^{-v^2} \cos (2(kt/u)^{\frac{1}{2}} v)| dv < \infty.$$

However, this last condition is also fulfilled since

$$\int_0^\infty |e^{-v^2} \cos (2(kt/u)^{\frac{1}{2}} v)| dv \leqslant \int_0^\infty e^{-v^2} dv = \tfrac{1}{2}\sqrt{\pi},$$

and thus

$$\int_0^\infty e^{-ku/t} |u^{-\frac{1}{2}} \phi(u)| du \int_0^\infty |e^{-v^2} \cos (2(kt/u)^{\frac{1}{2}} v)| dv \leqslant \tfrac{1}{2}\sqrt{\pi} \int_0^\infty e^{-ku/t} |u^{-\frac{1}{2}} \phi(u)| du < \infty$$

if $k/t > \gamma$. To justify the asymptotic evaluation, it must be shown that

$$I_1 = \left| k^{\frac{1}{2}} e^{2k} \int_0^{t-\delta} e^{-k(ut^{-1}+tu^{-1})} u^{-\frac{1}{2}} \phi(u) du \right|,$$

and

$$I_2 = \left| k^{\frac{1}{2}} e^{2k} \int_{t+\delta}^\infty e^{-k(ut^{-1}+tu^{-1})} u^{-\frac{1}{2}} \phi(u) du \right|$$

tend to zero as $k \to \infty$.

For the latter, choose $k_0 > \gamma t$. Then for $k > k_0$,

$$I_2 \leqslant k^{\frac{1}{2}} e^{2k} \int_{t+\delta}^\infty e^{-k(ut^{-1}+tu^{-1})} |u^{-\frac{1}{2}} \phi(u)| du$$
$$= k^{\frac{1}{2}} e^{2k} \int_{t+\delta}^\infty e^{-k_0(ut^{-1}+tu^{-1})} e^{-(k-k_0)(ut^{-1}+tu^{-1})} |u^{-\frac{1}{2}} \phi(u)| du$$
$$\leqslant k^{\frac{1}{2}} e^{2k} e^{-(k-k_0)(t(t+\delta)^{-1}+(t+\delta)t^{-1})} \int_{t+\delta}^\infty e^{-k_0(ut^{-1}+tu^{-1})} |u^{-\frac{1}{2}} \phi(u)| du$$
$$= A(t) k^{\frac{1}{2}} e^{2k_0} e^{-(k-k_0)[t^2+(t+\delta)^2-2t(t+\delta)]/[t(t+\delta)]}$$
$$= B(t) k^{\frac{1}{2}} e^{-k\delta^2/(t(t+\delta))} \to 0 \qquad \text{as } k \to \infty \text{ for } t > 0.$$

Similarly $I_1 \to 0$ as $k \to \infty$.

3. Some lemmas. In this section we shall prove some lemmas that will be needed in the representation theory.

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LEMMA 1. *If*

(1) $$s^{-1}f(s) \in L(\delta, \infty) \qquad \textit{for all } \delta > 0,$$

(2) $$F(x) = \int_x^\infty y^{-1}|f(y)|dy = O(x^{-m}) \quad \textit{with } m > 0, \textit{ as } x \to \infty,$$

$$F(x) = O(e^{\gamma/x}) \qquad \textit{with } \gamma \geqslant 0, \textit{ as } x \to 0+,$$

(3) $$m + n > 0,$$

then

(i) $$s^{n-1}f(s^{-1}) \in L(0, R) \qquad \textit{for all } R > 0,$$

(ii) $$G(t) = \int_0^t u^{n-1}|f(u^{-1})|du = O(t^{m+n}) \qquad \textit{as } t \to 0+,$$

(iii) $$G(t) = O(t^n e^{\gamma t}) \qquad \textit{as } t \to \infty \textit{ if either } \gamma > 0 \textit{ or } n \geqslant 0$$

$$= O(1) \qquad \textit{as } t \to \infty \textit{ if } \gamma = 0 \textit{ and } n < 0,$$

(iv) $$\int_0^\infty e^{-st}t^{n-1}f(t^{-1})dt$$

is absolutely convergent for $s > \gamma$, and is $O(s^{-m-n})$ as $s \to \infty$.

Proof. (i) Clearly $u^{n-1}|f(u^{-1})| \in L(\delta, R)$ for all $R > \delta > 0$. Thus

$$\int_\epsilon^t u^{n-1}|f(u^{-1})|du = \int_\epsilon^t u^n dF(u^{-1})$$

$$= t^n F(t^{-1}) - \epsilon^n F(\epsilon^{-1}) - n\int_\epsilon^t u^{n-1}F(u^{-1})du,$$

and by (2), the right-hand side tends to a finite limit as $\epsilon \to 0$ since $m + n > 0$. Thus

$$G(t) = t^n F(t^{-1}) - n\int_0^t u^{n-1}F(u^{-1})du.$$

(ii) $$G(t) = t^n F(t^{-1}) - n\int_0^t u^{n-1}F(u^{-1})du = O(t^{m+n})$$

by the last equation and (2).

(iii) Let $m + n > 0$ and either $\gamma > 0$ or $n \geqslant 0$ so that

$$\int_0^t u^{n-1}F(u^{-1})du \to \infty \qquad \text{as } t \to \infty.$$

$$G(t) = t^n O(e^{\gamma t}) - n\int_0^t u^{n-1}O(e^{\gamma u})du = O(t^n e^{\gamma t}).$$

If $\gamma = 0$, $n < 0$, $G(t)$ is clearly bounded.

(iv) Clearly $e^{-st}t^{n-1}|f(t^{-1})| \in L(\delta, R)$ for all $R > \delta > 0$.

$$\int_\delta^R e^{-st}t^{n-1}|f(t^{-1})|dt = e^{-sR}G(R) - e^{-s\delta}G(\delta) + s\int_\delta^R e^{-st}G(t)dt.$$

Convergence as $\delta \to 0$ follows from (ii) since $m + n > 0$; convergence as $R \to \infty$ follows from (iii) if $s > \gamma$. Moreover, from Widder [**1**, p. 181, theorem 1], the integral is $O(s^{-m-n})$ as $s \to \infty$ (and is $O((s-\gamma)^{-n})$ as $s \to \gamma+$ if $n \geqslant 0$).

LEMMA 2. *If $f(s)$ satisfies the reqiurements of Lemma 1 with $m > \frac{1}{2}$ then, for each $k > 0$, $L_{k,t}[f(s)]$ exists for almost all $t > 0$. In particular, $L_{k,t}[f(s)]$ exists when $k, t > 0$ and k/t is in the Lebesgue set of $f(s)$.*

Proof. We have to show convergence of the integral II,

$$L_{k,t}[f(s)] = (ke^{2k}(\pi t)^{-1})\int_0^\infty x^{-\frac{1}{2}} \cos(2kx^{\frac{1}{2}}) f(k(x+1)/t)dx,$$

at the origin and at infinity.

If k/t is in the Lebesgue set of $f(s)$, we have

$$J(h) = \int_0^h |f(k(x+1)/t) - f(k/t)|dx = o(h), \qquad h \to 0.$$

Thus

$$\begin{aligned}\int_\epsilon^\delta |x^{-\frac{1}{2}} \cos(2kx^{\frac{1}{2}}) f(k(x+1)/t)|dx \\ &\leqslant \int_\epsilon^\delta x^{-\frac{1}{2}}dx \cdot f(k/t) + \int_\epsilon^\delta x^{-\frac{1}{2}}|f(k(x+1)/t) - f(k/t)|dx \\ &= o(1) + \int_\epsilon^\delta x^{-\frac{1}{2}}dJ(x) = o(1) + \tfrac{1}{2}\int_\epsilon^\delta x^{-3/2}J(x)dx \\ &= o(1) + \tfrac{1}{2}\int_\epsilon^\delta x^{-3/2}o(x)dx = o(1)\end{aligned}$$

as $\epsilon, \delta \to 0+$, and II converges absolutely at the origin.

From Lemma 1 we have

$$\int_0^\epsilon u^{-3/2}|f(u^{-1})|du < \infty.$$

Here we put $u^{-1} = k(x+1)/t$ and choose $\epsilon < t/k$. We then have

$$(k/t)^{\frac{1}{2}}\int_{(t/(k\epsilon)-1)}^\infty (1+x)^{-\frac{1}{2}}|f(k(x+1)/t)|dx < \infty$$

and thus II converges absolutely at infinity.

4. Fundamental theorem.

THEOREM 2. *If*

(1) $\qquad s^{-1}f(s) \in L(\delta, \infty) \qquad$ *for all* $\delta > 0$,

(2) $\qquad F(x) = \int_x^\infty y^{-1}|f(y)|dy = O(x^{-m}) \qquad$ *with* $m > \frac{1}{2}$, *as* $x \to \infty$,

$\qquad F(x) = O(e^{\gamma/x}), \qquad$ *with* $\gamma \geqslant 0$, *as* $x \to 0+$,

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(3) $$e^{-\sigma t}L_{k,t}[f(s)] \in L(0, \infty), \qquad \sigma > \gamma_1 \text{ for all } k > k_0,$$

then

$$\lim_{k\to\infty} \int_0^\infty e^{-\sigma t}L_{k,t}[f(s)]dt = f(\sigma)$$

at every point of the Lebesgue set of $f(\sigma)$, $\sigma > \gamma_1$.

Proof. $L_{k,t}[f(s)]$ exists and has a Laplace transform when $\sigma > \gamma_1$. To prove the assertion we shall use the same theorems of Widder [**3**], that were used in the proof of Theorem 1. Operating formally we have,

$$\int_0^\infty e^{-\sigma t}L_{k,t}[f(s)]dt = (ke^{2k}/\pi)\int_0^\infty e^{-\sigma t}t^{-1}dt\int_0^\infty x^{-\frac{1}{2}} \cos (2kx^{\frac{1}{2}})f(k(x+1)/t)dx$$

$$= (2ke^{2k}/\pi)\int_0^\infty e^{-\sigma t}t^{-1}dt\int_0^\infty \cos (2ky)f(k(y^2+1)/t)dy \qquad (\text{where } y^2 = x)$$

$$= (2ke^{2k}/\pi)\int_0^\infty \cos (2ky)dy\int_0^\infty e^{-\sigma t}t^{-1}f(k(y^2+1)/t)dt$$

$$= (2ke^{2k}/\pi)\int_0^\infty \cos (2ky)dy\int_0^\infty e^{-k\sigma u(y^2+1)}u^{-1}f(u^{-1})du \quad (\text{where } u^{-1}=k(y^2+1)/t)$$

$$= (2ke^{2k}/\pi)\int_0^\infty e^{-k\sigma u}u^{-1}f(u^{-1})du\int_0^\infty e^{-k\sigma uy^2} \cos (2ky)dy$$

$$= (2k^{\frac{1}{2}}e^{2k}(\pi\sigma^{\frac{1}{2}})^{-1})\int_0^\infty e^{-k\sigma u}u^{-3/2}f(u^{-1})du\int_0^\infty e^{-v^2} \cos \{2(k(\sigma u)^{-1})^{\frac{1}{2}}v\}dv$$

$$(\text{where } v^2 = k\sigma uy^2)$$

$$= e^{2k}(k(\sigma\pi)^{-1})^{\frac{1}{2}}\int_0^\infty e^{-k(\sigma u+(\sigma u)^{-1})}u^{-3/2}f(u^{-1})du \to f(\sigma) \qquad \text{as } k \to \infty.$$

These formal calculations will be justified if two interchanges of integrations are justified and the conditions for the asymptotic evaluation are met.

For the first interchange of integrations we must show that

$$\int_0^\infty |\cos (2kx)|dx\int_0^\infty e^{-k\sigma u(x^2+1)}|u^{-1}f(u^{-1})|du < \infty.$$

But, by assumption (2) and Lemma 1, if $k\sigma > \gamma$ then the inner integral is $O(x^{-2m})$ as $x \to \infty$ and $m > \frac{1}{2}$. Thus the interchange is justified.

For the second interchange we must show that

$$\int_0^\infty e^{-k\sigma u}|u^{-3/2}f(u^{-1})|du\int_0^\infty e^{-v^2}|\cos \{2(k(\sigma u)^{-1})^{\frac{1}{2}}v\}|dv$$

exists. But this is obvious since the inner integral is less than $\frac{1}{2}\sqrt{\pi}$ and since

$$\int_0^\infty e^{-k\sigma u}u^{-3/2}f(u^{-1})du$$

converges absolutely by assumption (2) and Lemma 1.

The verification of the conditions for the asymptotic evaluation is exactly the same as in Theorem 1.

5. Representation theorems.

THEOREM 3. *If $f(s)$ satisfies conditions* (1) *and* (2) *of Theorem* 2, *and*

$$(3) \qquad |L_{k,t}[f(s)]| < M, \qquad k > k_0,\ 0 \leqslant t < \infty,$$

then there exists a $\phi(t)$, bounded in $0 \leqslant t < \infty$, and such that

$$f(s) = \int_0^\infty e^{-st}\phi(t)dt.$$

Proof. We shall use Widder [**3**, p. 33, theorem 17b]. By this theorem, there exists an increasing and unbounded sequence of numbers $\{k_i\}$, and a bounded function $\phi(t)$ such that

$$\lim_{i\to\infty}\int_0^\infty e^{-\sigma t}L_{k_i,t}[f(s)]dt = \int_0^\infty e^{-\sigma t}\phi(t)dt.$$

But, because of (3), $f(s)$ satisfies all the postulates of Theorem 2. Thus

$$\lim_{i\to\infty}\int_0^\infty e^{-\sigma t}L_{k_i,t}[f(s)]dt = f(\sigma),$$

so that

$$f(\sigma) = \int_0^\infty e^{-\sigma t}\phi(t)dt.$$

THEOREM 4. *If $f(s)$ satisfies conditions* (1) *and* (2) *of Theorem* 2, *and*

$$(3) \qquad \int_0^\infty |L_{k,t}[f(s)]|^p dt < M^p, \qquad k > k_0,\ p > 1,$$

then, there exists a $\phi(t) \in L_p(0, \infty)$ such that

$$f(s) = \int_0^\infty e^{-st}\phi(t)dt.$$

Proof. This theorem is proved in exactly the same manner as Theorem 3, but using Widder [**3**, p. 33, theorem 17a].

6. Inversion and representation theory for the Laplace-Stieltjes transform. In this section we shall regard $f(s)$ as defined by

$$\text{III} \qquad f(s) = \int_0^\infty e^{-st}d\alpha(t)$$

THEOREM 5. *If*

(1) $\alpha(t)$ *is of bounded variation in* $(0,T)$ *for all* $T > 0$, $\alpha(0) = \alpha(0+) = 0$,

(2) $e^{-st}t^{-3/2}\alpha(t) \in L(0, \infty)$ *for* $s > \gamma$,

then

$$\lim_{k\to\infty}\int_0^t L_{k,t}[f(s)]dt = \tfrac{1}{2}\{\alpha(t+) + \alpha(t-)\}$$

almost everywhere for $t > 0$.

Proof. Clearly $f(s)$ exists for $s > \gamma$. To prove the assertion we shall use the same theorems of Widder used in Theorem 1.

$$f(s) = \int_0^\infty e^{-st}d\alpha(t) = s\int_0^\infty e^{-st}\alpha(t)dt.$$

Operating formally we have,

$$L_{k,t}[f(s)] = (ke^{2k}(\pi t)^{-1})\int_0^\infty x^{-\frac{1}{2}}\cos(2kx^{\frac{1}{2}})(k(x+1)/t)dx\int_0^\infty e^{-k(x+1)u/t}\alpha(u)du$$

$$= (k^2e^{2k}(\pi t^2)^{-1})\int_0^\infty e^{-ku/t}\alpha(u)du\int_0^\infty e^{-kxu/t}x^{-\frac{1}{2}}(x+1)\cos(2kx^{\frac{1}{2}})dx$$

$$= (2k^{3/2}e^{2k}(\pi t^{3/2})^{-1})\int_0^\infty e^{-ku/t}\alpha(u)u^{-\frac{1}{2}}du\int_0^\infty e^{-v^2}(tv^2(ku)^{-1}+1)\cos\{2(ktu^{-1})^{\frac{1}{2}}v\}dv$$

(where $v^2 = kxu/t$)

$$= (2k^{3/2}e^{2k}(\pi^{\frac{1}{2}}t^{3/2})^{-1})\int_0^\infty e^{-k(ut^{-1}+tu^{-1})}\alpha(u)u^{-\frac{1}{2}}(1+t(2ku)^{-1}-t^2u^{-2})du$$

$$= e^{2k}(k/\pi)^{\frac{1}{2}}\int_0^\infty \frac{d}{dt}(e^{-k(ut^{-1}+tu^{-1})}t^{\frac{1}{2}}u^{-3/2}\alpha(u))du.$$

Thus

$$\int_0^t L_{k,t}[f(s)]dt = e^{2k}(kt/\pi)^{\frac{1}{2}}\int_0^\infty e^{-k(ut^{-1}+tu^{-1})}u^{-3/2}\alpha(u)du \to \tfrac{1}{2}(\alpha(t+)+\alpha(t-))$$

as $k\to\infty$.

These formal calculations will be justified if the two interchanges of integrations are justified and the asymptotic evaluation is justified.

The first interchange of integrations is justified in almost the same manner as in Theorem 1, as is also the asymptotic evaluation.

For the second interchange of integrations, consider

$$\int_0^\infty e^{-k(ut^{-1}+tu^{-1})}u^{-5/2}\alpha(u)du.$$

By assumption (2), this integral converges uniformly and absolutely in any bounded closed interval for which $k/t > \gamma$.
Thus

$$\frac{d}{dt}\int_0^\infty e^{-k(ut^{-1}+tu^{-1})}\alpha(u)t^{\frac{1}{2}}u^{-3/2}du = \int_0^\infty \frac{d}{dt}\{e^{-k(ut^{-1}+tu^{-1})}\alpha(u)t^{\frac{1}{2}}u^{-3/2}\}du,$$

and the second interchange is justified.

THEOREM 6. *If $f(s)$ satisfies conditions* (1) *and* (2) *of Theorem* 2, *and*

$$\int_0^t |L_{k,t}[f(s)]|dt < M, \qquad k > k_0, 0 < t < \infty, \tag{3}$$

then there exists $\alpha(t)$, of bounded variation in $0 \leqslant t < \infty$ such that

$$f(s) = \int_0^\infty e^{-st}d\alpha(t).$$

Proof. This theorem is proved in exactly the same manner as Theorem 3, but using Widder [**3**, p. 31, theorem 16.4].

REFERENCES

1. A. Erdélyi, *The inversion of the Laplace transformation*, Math. Mag., vol. 29 (1950-51).
2. I. I. Hirschman Jr., *A new representation and inversion theory for the Laplace integral*, Duke Math. J., vol. 15 (1948).
3. D. V. Widder, *The Laplace transformation* (Princeton, 1941).

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