INTERPOLATION AND INEQUALITIES FOR FUNCTIONS OF EXPONENTIAL TYPE: THE ARENS IRREGULARITY OF AN EXTREMAL ALGEBRA

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For any compact convex set $K \subset \mathbb{C}$ there is a unital Banach algebra $Ea(K)$ generated by an element h in which every polynomial in h attains its maximum norm over all Banach algebras subject to the numerical range $V(h)$ being contained in K, [**1**]. In the case of K a line segment in $\mathbb{R}$, we show here that $Ea(K)$ does not have Arens regular multiplication. We also show that ideas about $Ea(K)$ give simple proofs of, and extend, two inequalities of C. Frappier [**4**].

For $K=[-\tau,\tau]$, $\tau>0$, the generator h is Hermitian; equivalently, $\|e^{ith}\|=1$ $(t \in \mathbb{R})$, with $V(h)=\mathrm{Sp}(h)=[-\tau,\tau]$. For the inequalities we use the fact that the operator D of differentiation on B_τ, the space of entire functions f such that $|f(z)|e^{-\tau|\mathrm{Im}\, z|}$ is bounded for $z \in \mathbb{C}$, is i times a Hermitian, where we give $f \in B_\tau$ the norm $\|f\|=\sup\{|f(x)|: x \in \mathbb{R}\}$. This is all we require here; in fact with $h=-iD$ this gives a realization of $Ea[-\tau,\tau]$, [**2**].

We may assume, replacing h by $\alpha h+\beta$ for suitable $\alpha, \beta \in \mathbb{C}$, that $K=[-2\pi, 2\pi]$, so that h is Hermitian: that is, $\|e^{ith}\|=1$ $(t \in \mathbb{R})$. As in [**2**], any entire function f such that $|f(z)|\exp(-2\pi\,|\mathrm{Im}\, z|)$ is bounded for $z \in \mathbb{C}$ gives a functional $\phi \in Ea(K)'$, by $\phi(e^{izh})=f(z)$ $(z \in \mathbb{C})$. By [**5**] a Banach algebra A is Arens irregular if, for some bounded sequences a_m, b_n in A and ϕ in A', the two repeated limits of $\phi(a_m b_n)$ exist and differ.

Proposition 1. *$Ea[-2\pi, 2\pi]$ is Arens irregular.*

Proof. Let $0<\beta<1$ and define, for $z \in \mathbb{C}$,

$$f(z)=\pi^2/[\Gamma(z)^2\Gamma(1+\beta-z)\Gamma(1-\beta-z)].$$

Since $1/\Gamma$ is entire, so is f. By [**3**, p. 47(5)], for $\mathrm{Re}\, z>0$ and $\alpha \in \mathbb{C}$, $z^\alpha\Gamma(z)/\Gamma(z+\alpha)\to 1$ uniformly as $|z|\to\infty$. Hence $\Gamma(z-\beta)\Gamma(z+\beta)/\Gamma(z)^2\to 1$, and so we obtain here $f(z)/[\sin\pi(z-\beta)\sin\pi(z+\beta)]\to 1$ as $|z|\to\infty$, $\mathrm{Re}\, z>0$. For $\mathrm{Re}\, z<0$ we find similarly that $f(z)/\sin^2\pi z\to 1$ as $|z|\to\infty$. These imply that f satisfies the above mentioned conditions so that $\phi(e^{izh})=f(z)$ defines $\phi \in Ea[-2\pi, 2\pi]'$.

Take $a_m=e^{imh}$ and $b_n=e^{-inh}(m, n \in \mathbb{N})$, so that $\|a_m\|=\|b_n\|=1$. Then $\phi(a_m b_n)=f(m-n)$. The limits involving $f(z)$ give $\lim_{k\to\infty, k\in\mathbb{Z}} f(k)=-\sin^2\pi\beta\neq 0=\lim_{k\to-\infty, k\in\mathbb{Z}} f(k)$. Hence by Pym's criterion $Ea[-2\pi, 2\pi]$ is Arens irregular.

Turning to the inequalities, we rephrase theorem 3 of [**4**] equivalently as follows.

Proposition 2. *If all the zeros z_k of $p(z)=\alpha z^2+\beta z+\gamma$ have $\mathrm{Im}\, z_k\leq 0$, then for $f \in B_\tau$,*

$$|\alpha f''(x)+\beta f'(x)+\gamma f(x)|\leq|p(i\tau)|\sup\{|f(t)|: t \in \mathbb{R}\} \qquad (x \in \mathbb{R}).$$

Proof. Since the operator iD is Hermitian, with $\mathrm{Sp}(D)=[-\tau i, \tau i]$, [**6**] gives that for any $\zeta \in \mathbb{C}$, $D-\zeta I$ has norm equal to spectral radius, and so $\|D-\zeta I\|=\max\{|\zeta+\tau i|, |\zeta-\tau i|\}$. Hence $\|D-z_k I\|=|\tau i-z_k|$ for each k, and so $\|p(D)\|\leq|p(\tau i)|$. Since $p(\tau i)\in\mathrm{Sp}(p(D))$ we have $\|p(D)\|-|p(\tau i)|$, which is the required result.

This argument shows that the proposition holds for polynomials of any degree. The inequality of theorem 4 of [**4**] follows similarly: we restrict D to the subspace of B_τ of functions f with $|f(z)|$ bounded for $\operatorname{Im} z \geqslant 0$, which gives now $\operatorname{Sp}(D) = [0, \tau i]$.

The Arens regularity problem when K has interior remains open. We are grateful to J. Duncan for useful discussions on this topic.

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