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FINITE HILBERT TRANSFORMS AND COMPACTNESS

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It is shown that for the finite Hilbert transform T_p on the Banach space $\mathcal{L}^p(]-1,1[)$, $1 < p < \infty$, the linear operator $T_p^n + I$ is not strictly singular whenever n is a positive integer.

1. Introduction

Let $1 < p < \infty$. The Hilbert transform H_p on the space $\mathcal{L}^p(\mathbb{R})$ is defined by the Cauchy principal value

$$(H_p f)(t) = \lim_{\varepsilon \downarrow 0} \left[\int_{-\infty}^{t-\varepsilon} + \int_{t+\varepsilon}^{\infty} \right] \frac{f(\tau)}{\pi(\tau - t)} d\tau, \quad t \in \mathbb{R},$$

for every $f \in \mathcal{L}^p(\mathbb{R})$. Then H_p is a continuous linear operator satisfying the M. Riesz identity: $H_p^2 + I = 0$ on $\mathcal{L}^p(\mathbb{R})$, [6, p.239].

Let Ω denote the open interval $]-1,1[$. It is clear that the identity $T_p^2 + I = 0$ does not hold for the finite Hilbert transform T_p on $\mathcal{L}^p(\Omega)$ (for the definition of T_p, see Section 2). For example,

$$\left(T_p^2 + I\right)\left(\left(1 - \mathrm{x}^2\right)^{1/2}\right) = -\pi^{-1}\left(2 + \mathrm{x}\ln\left(1 - \mathrm{x}\right)\left(1 + \mathrm{x}\right)^{-1}\right) + \left(1 - \mathrm{x}^2\right)^{1/2} \neq 0,$$

x denoting the identity function on Ω. If one believes that the finite Hilbert transform behaves like the Hilbert transform, then $T_p^2 + I$ ought to be a "small" operator. Therefore it would be natural to see whether or not $T_p^2 + I$ is compact. This question has been raised by M. Cowling.

The aim of this note is to show that, given a positive integer n, the operator $T_p^n + I$ on $\mathcal{L}^p(\Omega)$ is not strictly singular, and hence it is not compact; see Theorem 3.

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2. The main result

Let X be a Banach space. A continuous linear operator $S\colon X \to X$ with closed range is called a *Noether* or *Fredholm* operator if the dimension of its null space $\mathcal{N}(S)$ and the codimension of its range $\mathcal{R}(S)$ are both finite. The *index* $\kappa(S)$ of a Noether operator S is defined as

$$\kappa(S) = \dim \mathcal{N}(S) - \operatorname{codim} \mathcal{R}(S).$$

A continuous linear operator $A\colon X \to X$ is called *strictly singular* if the restriction of A to any infinite-dimensional subspace of X is not an isomorphism onto its range. In particular, compact operators are strictly singular.

The following result can be found in [**3**, Propositions 2.c.7 and 2.c.10], for example.

Lemma 1. *Let S be a Noether operator from a Banach space X into X. Then the following statements hold.*

(i) *For every positive integer n, the n-th power S^n of S is also a Noether operator such that $\kappa(S^n) = n\kappa(S)$.*
(ii) *For every strictly singular operator $A\colon X \to X$, the operator $S + A$ is a Noether operator such that $\kappa(S + A) = \kappa(S)$.*

Let $1 < p < \infty$. Let λ denote Lebesgue measure in the open interval $\Omega =]-1, 1[$. By $\mathcal{L}^p(\Omega)$ we denote the usual Banach space of functions f on Ω (strictly speaking, equivalence classes of functions modulo λ-null functions) such that $f\,|f|^{p-1}$ is λ-integrable. The *finite Hilbert transform* $T_p\colon \mathcal{L}^p(\Omega) \to \mathcal{L}^p(\Omega)$ is defined by the Cauchy principal value

$$(T_p f)(t) = \lim_{\varepsilon \downarrow 0} \left[\int_{-1}^{t-\varepsilon} + \int_{t+\varepsilon}^{1}\right] \frac{f(\tau)}{\pi(\tau - t)} d\lambda(\tau), \quad t \in \Omega,$$

for every $f \in \mathcal{L}^p(\Omega)$. Then T_p is a continuous linear operator by the M. Riesz theorem; the details can be found in [**2**, Section 13], for example.

Lemma 2.

(i) *If $1 < p < 2$, then T_p is a Noether operator such that $\kappa(T_p) = 1$.*
(ii) *The operator T_2 is not a Noether operator; its range $\mathcal{R}(T_2)$ is a proper dense subspace of $\mathcal{L}^2(\Omega)$.*
(iii) *If $2 < p < \infty$, then T_p is a Noether operator such that $\kappa(T_p) = -1$.*

Proof: Statements (i) and (iii) are due to Söhngen [**7**]. See also [**2**, Section 13] and [**5**, Propositions 2.4 and 2.6] for alternative proofs.

The fact that T_2 is not a Noether operator has been proved in the general setting; see, for example, [**1**, Theorem IX.5.3] or [**4**, Theorem IV.5.1]. Alternatively that fact can easily be derived from the observation that $\mathcal{R}(T_2)$ does not contain the constant function 1. For a characterisation of $\mathcal{R}(T_2)$, see [5, Theorem 3.2]. ☐

For every $p \in]1, \infty[$, the identity operator $I_p \colon \mathcal{L}^p(\Omega) \to \mathcal{L}^p(\Omega)$ is clearly a Noether operator such that

$$\kappa(I_p) = 0. \tag{1}$$

We now present the main result.

THEOREM 3. *Let $1 < p < \infty$. Then the linear operator $T_p^n + I_p \colon \mathcal{L}^p(\Omega) \to \mathcal{L}^p(\Omega)$ is not strictly singular, especially it is not compact, for any positive integer n.*

PROOF: Fix a positive integer n and let $A_p = T_p^n + I_p$.

Firstly assume that $1 < p < 2$. Then, by Lemmas 1 and 2, the operator T_p^n is a Noether operator such that

$$\kappa(T_p^n) = n. \tag{2}$$

If A_p were strictly singular, then by Lemma 1(ii), the Noether operator $T_p^n = (-I_p) + A_p$ would have index 0 because of (1). This contradicts (2), so that A_p is not strictly singular.

Secondly, if A_2 were strictly singular, then $T_2^n = (-I_2) + A_2$ would be a Noether operator. However, this is not the case because the range of T_2^n is a proper dense subspace of $\mathcal{L}^2(\Omega)$ by Lemma 2(ii).

In the case when $2 < p < \infty$, the operator T_p^n is a Noether operator whose index is $-n$ by Lemmas 1 and 2. So A_p is not strictly singular because of (1) as in the first case. ☐

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