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A TAUBERIAN THEOREM FOR A SCALE OF LOGARITHMIC METHODS OF SUMMATION

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1. Introduction. We suppose throughout that p is a non-negative integer, and use the following notations:

$$\pi_p(x) = \begin{cases} \dfrac{1}{\log_0 x \,.\, \log_1 x \ldots \log_p x}, & \text{for } x \geqq e_p, \\ 0, & \text{otherwise,} \end{cases}$$

where $\log_0 x = x$ for $x \geqq e_0 = 1$, $\log_{n+1} x = \log(\log_n x)$ for $x \geqq e_{n+1} = e^{e_n}$ $(n = 0, 1, 2, \ldots)$;

$$\sigma_p(x) = \sum_{n=0}^{\infty} \pi_p(n) x^n \quad (-1 < x < 1);$$

$$s_n = \sum_{k=0}^{n} a_k \quad (n = 0, 1, 2, \ldots);$$

$$t_n = \frac{1}{\log_{p+1} n} \sum_{k=0}^{n} \pi_p(k) s_k \quad (n \geqq e_{p+1}).$$

We shall say that $\sum_{n=0}^{\infty} a_n$ is summable L_p to s and write

$$\sum_{n=0}^{\infty} a_n = s(L_p) \quad \text{or} \quad s_n \to s(L_p),$$

if

$$\lim_{x \to 1^-} \frac{1}{\sigma_p(x)} \sum_{n=0}^{\infty} \pi_p(n) s_n x^n = s.$$

We shall say that $\sum_{n=0}^{\infty} a_n$ is summable l_p to s, and write

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if $t_n \to s$ as $n \to \infty$.

Since $\sum_{n=0}^{\infty} \pi_p(n) = \infty$ the L_p method is regular [**3**, Theorem 1], i.e., every convergent series is summable to its natural sum. It is easily seen that the l_p method is equivalent to a $(\bar{N}, q_n)$ method with $q_n = \pi_p(n)$ and hence is regular [**4**, p. 57].

It follows from a known result that the L_0 method is equivalent to a standard logarithmic method L (see, for example, [**2**]). Using a standard result on $\bar{N}$

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methods (see, for example, [**4**, Theorem 14]) the l_0 method can easily be shown to be equivalent to the l method which has been considered by a number of authors (see, for example, [**4**; **5**; **7**]).

The aim of this paper is to establish the following Tauberian theorem.

THEOREM. *If* $\sum_{n=0}^{\infty} a_n = s(L_p)$ *and if the following Tauberian condition holds:*

(T_p) $\liminf (s_n - s_m) \geqq 0$ *when* $n > m \to \infty$ *and* $\log_{p+2} n - \log_{p+2} m \to 0$, *then* $\sum_{n=0}^{\infty} a_n$ *converges.*

The case $p = 0$ of this result is due to Kwee [**7**]. An immediate consequence of the above theorem and Lemmas 3 and 6 (below) is the following corollary, the case $p = 0$ of which includes "$_0$" Tauberian results established by Ishiguro [**5**].

COROLLARY. *If a series* $\sum_{n=0}^{\infty} a_n$ *is* L_p *or* l_p *summable and if for* H *a positive constant* $a_n \geqq -H\pi_{p+1}(n)$ *for* $n \geqq e_{p+1}$, *then the series converges.*

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2. Preliminary results.

We require the following lemmas.

LEMMA 1. *If*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = c > 0 \quad \textit{and if} \quad \lim_{x\to a} g(x) = \infty$$

where $-\infty \leqq a \leqq \infty$, *then*

$$\lim_{x\to a} \frac{\log_{p+1} f(x)}{\log_{p+1} g(x)} = 1.$$

Proof. Since

$$\lim_{x\to a} \frac{\log f(x)}{\log g(x)} = \lim_{x\to a} \frac{\log\{f(x)/g(x)\}}{\log g(x)} + 1 = 1,$$

the result holds for the case $p = 0$, and the general case can be established by induction.

LEMMA 2.

$$\sigma_p(x) \sim \log_{p+1} \frac{1}{1-x} \quad as \quad x \to 1^-.$$

Proof. For $p = 0$ the result is obvious, and for $p > 0$ since

$$\limsup_{n\to\infty} n\pi_p(n) = 0$$

we obtain from a theorem due to Agnew [**1**, Theorem 1.1] that

$$\limsup_{x\to 1^-} \left| \sum_{k=0}^{\infty} x^k \pi_p(k) - \sum_{k=0}^{[1/\log x^{-1}]} \pi_p(k) \right| \leqq 0,$$

where $[y]$ denotes the largest integer not exceeding y. Also, it is familiar that

(1) $$\sum_{k=0}^{n} \pi_p(k) \sim \log_{p+1} n \quad as \quad n \to \infty.$$

The lemma now follows by Lemma 1. Notice in particular that

(2) $$\sigma_p(e^{-1/x}) \sim \log_{p+1} x \quad \text{as} \quad x \to \infty.$$

LEMMA 3. *If* $\sum_{n=0}^{\infty} a_n = s(l_p)$ *and condition* (T_p) *holds, then* $\sum_{n=0}^{\infty} a_n$ *converges.*

Proof. The proof is modelled on Kwee's proof of the case $p = 0$ [**7**, Lemma 3]. Assume, without loss of generality, that $s = 0$, and let N be the integer such that $N - 1 < e_{p+2} \leqq N$. Then for $n > m \geqq N$

$$t_n \log_{p+1} n - t_m \log_{p+1} m = s_{m+1}\pi_p(m + 1) + \ldots + s_n \pi_p(n).$$

Let ϵ be an arbitrary positive number. By condition (T_p) there are numbers

$$M = M(\epsilon) \geqq N \quad \text{and} \quad \delta = \delta(\epsilon) > 0$$

such that: if $n > m \geqq M$ and $\log_{p+2} n - \log_{p+2} m \leqq \delta$, then $s_n - s_m \geqq -\epsilon$ and hence

$$(s_m - \epsilon) \sum_{k=m+1}^{n} \pi_p(k) \leqq t_n \log_{p+1} n - t_m \log_{p+1} m \leqq (s_n + \epsilon) \sum_{k=m+1}^{n} \pi_p(k);$$

i.e.,

$$s_m - \epsilon \leqq \left[t_n \frac{\log_{p+1} n}{\log_{p+1} m} - t_m \right] \left[\log_{p+1} m \Big/ \sum_{k=m+1}^{n} \pi_p(k) \right],$$

and

$$s_n + \epsilon \geqq \left[t_n \frac{\log_{p+1} n}{\log_{p+1} m} - t_m \right] \left[\log_{p+1} m \Big/ \sum_{k=m+1}^{n} \pi_p(k) \right].$$

Keeping ϵ fixed and letting $n > m \to \infty$ subject to

$$\tfrac{1}{2}\delta \leqq \log_{p+2} n - \log_{p+2} m \leqq \delta,$$

we get

$$\limsup_{m\to\infty} s_m \leqq \epsilon \quad \text{and} \quad \liminf_{n\to\infty} s_n \geqq -\epsilon;$$

since $t_n \to 0$,

$$e^{\delta} \geqq \frac{\log_{p+1} n}{\log_{p+1} m} \geqq e^{\frac{1}{2}\delta} > 1 + \tfrac{1}{2}\delta$$

and

$$\left[\log_{p+1} m \Big/ \sum_{k=m+1}^{n} \pi_p(k) \right] \sim \frac{\log_{p+1} m}{\log_{p+1} n - \log_{p+1} m}$$

$$= \frac{1}{(\log_{p+1} n / \log_{p+1} m - 1)} = O(1).$$

It follows that $\lim_{n\to\infty} s_n = 0$.

LEMMA 4. *If* $0 < d < 1$ *and if* $g(x)$ *is a real valued function, continuous on each of the intervals* $[0, d)$, $[d, 1]$, *which tends to a finite limit as* $x \to d^-$, *and if* $s_n \geqq 0$ *and* $s_n \to s(L_p)$, *then*

$$\lim_{x \to 1^-} \frac{1}{\sigma_p(x)} \sum_{n=0}^{\infty} \pi_p(n) s_n x^n g(x^n) = s \cdot g(1).$$

Proof. By Lemmas 1 and 2, we have, for $c \geqq 0$,

$$\begin{aligned} \lim_{x \to 1^-} \frac{1}{\sigma_p(x)} \sum_{n=0}^{\infty} \pi_p(n) s_n x^n \cdot x^{cn} &= \lim_{x \to 1^-} \frac{\sigma_p(x^{c+1})}{\sigma_p(x)} \frac{1}{\sigma_p(x^{c+1})} \sum_{n=0}^{\infty} \pi_p(n) s_n x^{(c+1)n} \\ &= s \lim_{x \to 1^-} \frac{\sigma_p(x^{c+1})}{\sigma_p(x)} \\ &= s \lim_{x \to 1^-} \frac{\log_{p+1}(1 - x^{c+1})^{-1}}{\log_{p+1}(1 - x)^{-1}} \\ &= s. \end{aligned}$$

Thus the lemma holds for $g(x) = x^c$, and the full result follows by an argument similar to that used by Ishiguro [**6**, Lemma 2].

LEMMA 5. *If* $s_n \to s(L_p)$ *and* $s_n \geqq -M$, *then* $s_n \to s(l_p)$.

Proof. The proof is similar to Ishiguro's proof of the case $p = 0$ [**6**, Theorem 2]. Let

$$g(x) = \begin{cases} 0 & \text{for } 0 \leqq x < 1/e, \\ 1/x & \text{for } 1/e \leqq x \leqq 1, \end{cases}$$

so that $g(1) = 1$ and $g(x^n) = 0$ if $n > 1/\log(1/x)$. Hence, by Lemma 4,

$$\lim_{x \to 1^-} \frac{1}{\sigma_p(x)} \sum_{n \leqq 1/(\log(1/x))} \pi_p(n)(s_n + M) = s + M.$$

Putting $x = e^{-1/n}$, we get, by (1) and (2),

$$\lim_{n \to \infty} \frac{1}{\sigma_p(e^{-1/n})} \sum_{k=0}^{n} \pi_p(k)(s_k + M) = \lim_{n \to \infty} \frac{1}{\log_{p+1} n} \sum_{k=0}^{n} \pi_p(k) s_k + M = s + M.$$

LEMMA 6. *If for* H *a positive constant* $a_n \geqq -H\pi_{p+1}(n)$ *for* $n \geqq e_{p+1}$, *then the condition* (T_p) *is satisfied.*

Proof. There is a positive number H such that

$$a_k \geqq -H\pi_{p+1}(k) \qquad (k \geqq e_{p+1}),$$

so that for $n > m \geqq e_{p+1}$,

$$s_n - s_m = \sum_{k=m+1}^{n} a_k \geqq -H \sum_{k=m+1}^{n} \pi_{p+1}(k) \sim -H(\log_{p+2} n - \log_{p+2} m).$$

Hence, $\liminf(s_n - s_m) \geqq 0$ when $n > m \to \infty$ and $\log_{p+2} n - \log_{p+2} m \to 0$, and condition (T_p) is satisfied.

LEMMA 7. *Let Φ be an increasing continuous non-negative function in* $[0, \infty)$ *such that* $\Phi(u) \to \infty$ *and* $\Phi(u) - \Phi(u-1) \to 0$ *as* $u \to \infty$, *and let*

$$\tau(x) = \sum_{n=0}^{\infty} c_n(x) s_n \text{ for } x > 0.$$

Suppose that the following conditions are satisfied:

(i) (a) $c_n(x) \geqq 0 \quad (x > 0)$,

(b) $c_n(x) \to 0 \quad$ *as* $x \to \infty$,

(c) $\sum_{n=0}^{\infty} c_n(x) = 1 \quad (x > 0)$;

(ii) (a) $\sum_{n=0}^{M} c_n(x) \to 0$ *when* $x > M \to \infty$ *and* $\Phi(x) - \Phi(M) \to \infty$;

(b) $\sum_{n=M}^{\infty} c_n(x)(\Phi(n) - \Phi(M)) \to 0$ *when* $M > x \to \infty$ *and* $\Phi(M) - \Phi(x) \to \infty$;

(iii) $\liminf (s(t) - s(u)) \geqq 0$ *when* $t > u \to \infty$ *and* $\Phi(t) - \Phi(u) \to 0$, *where* $s(t) = s_n$ *for* $n \leqq t < n + 1$;

(iv) $\tau(x)$ *is bounded for* $x > x_0$.

Then s_n *is bounded.*

Kwee [**7**], using a result due to Vijayaraghavan (see [**4**, Theorem 238]), has proved this lemma with the additional condition

$$\sum_{n=M}^{\infty} c_n(x) \to 0 \text{ when } M > x \to \infty \text{ and } \Phi(M) - \Phi(x) \to \infty.$$

It has been pointed out that in fact this condition is redundant (see [**8**, Chapter II, Theorem 9]).

3. Proof of the Theorem. The proof is based on Kwee's proof [**7**] of the case $p = 0$. Let

$$\Phi(u) = \begin{cases} \log_{p+2} u & \text{for } u \geqq e_{p+2}, \\ u/e_{p+2} & \text{for } 0 \leqq u < e_{p+2}, \end{cases}$$

and, for $x > 0$, let

$$\tau(x) = \frac{1}{\sigma_p(e^{-1/x})} \sum_{n=0}^{\infty} \pi_p(n) s_n e^{-n/x} = \sum_{n=0}^{\infty} c_n(x) s_n,$$

where

$$c_n(x) = \frac{\pi_p(n) e^{-n/x}}{\sigma_p(e^{-1/x})}.$$

Clearly $\Phi(u)$ is a strictly increasing non-negative continuous function which tends to infinity as u tends to infinity, and by Lemma 1, $\Phi(u) - \Phi(u-1) \to 0$ as $u \to \infty$.

We now show that the other conditions of Lemma 7 are satisfied. Since, by (2),

$$0 \leqq c_n(x) \leqq \frac{\pi_p(n)}{\sigma_p(e^{-1/x})} \sim \frac{\pi_p(n)}{\log_{p+1} x} \qquad (0 < x \to \infty), \tag{3}$$

we obtain (i) (a) and (i) (b); (i) (c) holds by definition of $\sigma_p(e^{-1/x})$. Now using (1) and (3), and letting $x > M \to \infty$ subject to $\log_{p+2}x - \log_{p+2}M \to \infty$, (ii) (a) follows from

$$\sum_{n=0}^{M} \frac{\pi_p(n)}{\log_{p+1}x} \sim \frac{\log_{p+1}M}{\log_{p+1}x} \to 0.$$

For (ii) (b), we will show that $\sum_{n=M}^{\infty} c_n(x)\phi(n) \to 0$ when $M > x \to \infty$, which is more than required by condition (ii) (b). Since $\pi_p(t) \log_{p+2}t$ is a decreasing function of t for $t \geqq e_{p+2}$ we have, for $M \geqq e_{p+2}$,

$$\begin{aligned} 0 \leqq \sum_{n=M}^{\infty} c_n(x)\Phi(n) &\sim [\log_{p+1}x]^{-1} \sum_{n=M}^{\infty} e^{-n/x}\pi_p(n) \log_{p+2}n \\ &\leqq [\log_{p+1}x]^{-1}\pi_p(M) \log_{p+2}M \sum_{n=0}^{\infty} e^{-n/x} \\ &\sim [\log_{p+1}x]^{-1}x\,\pi_p(M) \log_{p+2}M \\ &< M[\log_{p+1}M]^{-1}\pi_p(M) \log_{p+2}M \to 0 \qquad (M > x \to \infty). \end{aligned}$$

Condition (iii) is implied by condition (T_p) of the theorem.

Since $s_n \to s(L_p)$, we have

$$\lim_{x\to\infty} \tau(x) = \lim_{t\to 1-} \frac{1}{\sigma_p(t)} \sum_{n=0}^{\infty} \pi_p(n)s_n t^n = s,$$

and hence condition (iv) is satisfied.

We have thus shown all the conditions of Lemma 7 are satisfied and it follows that s_n is bounded. Hence, by Lemma 5, $s_n \to s(l_p)$ and so, by Lemma 3, $\sum_{n=1}^{\infty} a_n$ converges.

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