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DERIVED FUNCTORS OF THE TORSION FUNCTOR AND LOCAL COHOMOLOGY OF NONCOMMUTATIVE RINGS

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Abstract

Let R be an associative ring which is not necessarily commutative. For any torsion theory τ on the category of left R-modules and for any nonnegative integer n we define and study the notion of the nth local cohomology functor with respect to τ. For suitably nice rings a bound for the nonvanishing of these functors is given in terms of the τ-dimension of the modules.

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The right derived functors of the torsion functor determined by an arbitrary torsion theory on a module category were first studied by Dickson [7]. The relation between torsion theories and local cohomology was first considered by Suominen [21] for the special case of categories of sheaves. For module categories over a commutative ring the basic results were obtained by Cahen [6] and these have recently been extended by Albu and Nastasescu [1, 2] and by Bijan-Zadeh [5]. Our purpose here is to show how similar results can be obtained for categories of modules over noncommutative rings.

Throughout the following, R will denote an arbitrary associative (but not necessarily commutative) ring with unit element 1. The category of unitary left R-modules will be denoted by R-mod. Morphisms in R-mod will be written as acting on the right. All other functions will be written as acting on the left. If M is a left R-module then the injective hull of M will be denoted by $E(M)$.

The complete brouwerian lattice of all hereditary torsion theories defined on R-mod will be denoted by R-tors. Notation and terminology concerning such

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theories will follow [8]. In particular, if $\tau \in R$-tors we denote the τ-torsion endofunctor of R-mod by $T_\tau(-)$ and the τ-localization endofunctor of R-mod by $Q_\tau(-)$. If M is a left R-module then the canonical R-homomorphism from M to $Q_\tau(M)$ will be denoted by λ^τ_M and not, as in [8], by $\hat{\tau}_M$. The localization of the ring R at τ will be denoted by R_τ. The τ-injective hull of a left R-module M will be denoted by $E_\tau(M)$. A submodule N of a left R-module M is said to be *τ-dense* in M if and only if M/N is a τ-torsion left R-module.

If M is a left R-module then the meet of all torsion theories relative to which M is torsion will be denoted by $\xi(M)$ and the join of all torsion theories relative to which M is torsionfree will be denoted by $\chi(M)$. Then $\xi = \xi(0)$ is the unique minimal element of R-tors and $\chi = \chi(0)$ is the unique maximal element of R-tors.

A nonzero τ-torsionfree left R-module M is said to be *τ-cocritical* if and only if every nonzero submodule of M is τ-dense in it. Such modules are necessarily uniform. A left R-module is said to be *cocritical* if and only if it is τ-cocritical for some torsion theory τ. A torsion theory of the form $\chi(M)$ for some cocritical left R-module M is said to be *prime*. The set of all prime torsion theories in R-tors is denoted by R-sp. Any theory theory $\tau \in R$-tors partitions R-sp into two disjoint parts:

$$\mathbf{P}(\tau) = \{\pi \in R\text{-sp} \mid \pi \geqslant \tau\} \quad \text{and} \quad \mathbf{V}(\tau) = \{\pi \in R\text{-sp} \mid \pi \ngeqslant \tau\}.$$

If M is a left R-module then the set of associated primes of M, denoted by ass(M), is the set of all primes in R-sp of the form $\chi(N)$, where N is a cocritical submodule of M. The ring R is said to be *left definite* if and only if ass$(M) \neq \varnothing$ for any nonzero left R-module M. Left noetherian rings are easily seen to be left definite. If R is left definite then $\tau = \wedge\mathbf{P}(\tau)$ for any torsion theory τ in R-tors other than χ. (In fact, this relation holds for an even larger class of rings, which need not concern us here.)

For any nonempty subset U of R-sp we can define a torsion theory $\delta(U)$ in R-tors by saying that a left R-module M is $\delta(U)$-torsion if and only if the following conditions hold:

(i) every nonzero homomorphic image of M has a cocritical submodule; and

(ii) if N is a cocritical submodule of a nonzero homomorphic image of M then $\chi(N) \in U$.

If $U \subseteq U'$ are nonempty subsets of R-sp then it is clear that $\delta(U) \leqslant \delta(U')$. Then the ring R is left definite if and only if for every torsion theory τ in R-tors there exists a subset U of R-sp for which $\tau = \delta(U)$ [10, Proposition 2]. Indeed, if R is left definite then for any subset U of R-sp we also have $\delta(U) = \wedge[R\text{-sp} \setminus U]$ [17].

The *support* of a left R-module M, denoted by supp(M), consists of all those elements of R-sp relative to which M is not torsion. If R is a left definite ring and if U is a nonempty subset of R-sp then a left R-module M is $\delta(U)$-torsion if and only if $\text{supp}(M) \subseteq U$.

Finally, a torsion theory τ in R-tors is said to be *perfect* if and only if every left R_τ-module is τ-torsionfree when considered as a left R-module.

1. Local cohomology functors

Let $\tau \in R$-tors. For any nonnegative integer n we define the *nth local cohomology functor* with respect to τ to be the nth right derived functor $R^nT_\tau(-)$ of the τ-torsion endofunctor of R-mod. In particular, we note that $R^nT_\tau(M)$ is a τ-torsion left R-module for any left R-module M and that $R^0T_\tau(-)$ equals $T_\tau(-)$ since the latter functor is always left exact. Moreover, the proof of Proposition 2.1 of [1] carries over to the noncommutative case and so we see that for any left R-module M and for each nonegative integer n there exists a natural isomorphism in the category of abelian groups between $R^nT_\tau(M)$ and $\varinjlim \text{Ext}^n_R(R/I, M)$, where the limit is taken over the idempotent filter of all τ-dense left ideals of R. Moreover, if the ring R is left noetherian and if $\{M_i \mid i \in \Omega\}$ is a directed system of left R-modules then for each nonnegative integer n we have $\varinjlim_{i\in\Omega} R^nT(M_i) \cong R^nT_\tau(\varinjlim M_i)$.

Let M be a nonzero left R-module having a minimal injective resolution

$$0 \to M \to E_0 \to E_1 \to \cdots$$

and for each $k \geqslant 0$ let $\chi_k(M) = \chi(E_0 \oplus \cdots \oplus E_k) = \wedge_{i=0}^k \chi(E_i)$. Then a left R-module N is $\chi_k(M)$-torsion if and only if $\text{Ext}^i_R(N', M) = 0$ for any [cyclic] submodule N' of N and any $i \leqslant k$. See page 149 of [19] for details. For notational simplicity, we set $\chi_{-1}(M) = \chi$ for any left R-module M.

If M is a left R-module, if n is a nonnegative integer, and if $\tau \in R$-tors then we say that M has *τ-dominant dimension* equal to n if and only if $\chi_{n-1}(M) \leqslant \tau$ and $\chi_n(M) \nleqslant \tau$. In terms of the above minimal injective resolution of M, this is equivalent to saying that E_i is τ-torsionfree for all $i < n$, while E_n is not τ-torsionfree. We denote the τ-dominant dimension of M by τ-dom.dim(M). If τ-dom.dim$(M) \neq n$ for any nonnegative integer n, we write τ-dom.dim$(M) = \infty$. Dominant dimension has been extensively studied. See, for example, [14, 16, 20].

(1.1) Example. A ring R is said to be *left local* if and only if all simple left R-modules are isomorphic. Let R be a left local ring and let N be a simple left R-module. For any left R-module M, we see that $\xi(N)$-dom.dim$(M) = 0$ if and only if $E(M)$ is not $\xi(N)$-torsionfree, that is, if and only if $\text{Hom}_R(N, E(M)) \neq 0$.

But this condition is equivalent to the condition that $E(M)$ (and hence M) have a submodule isomorphic to N. Thus we see that $\xi(N)$-dom.dim$(M) = 0$ if and only if soc$(M) \neq 0$.

(1.2) PROPOSITION. *If $\tau \in R$-tors and if n is a natural number then the following conditions on a left R-module M are equivalent*:

(1) τ-dom.dim$(M) \geqslant n$.

(2) $R^iT_\tau(M) = 0$ *for all* $i < n$.

PROOF. We will proceed by induction on n. In particular, we note that τ-dom.dim$(M) \geqslant 1 \Leftrightarrow M$ is τ-torsionfree $\Leftrightarrow R^0T_\tau(M) = 0$. Now assume inductively that $n > 1$ and that whenever $k < n$ we have τ-dom.dim$(M') \geqslant k \Leftrightarrow R^iT_\tau(M') = 0$ for all $i < k$, this holding for any left R-module M'. In particular, let $\overline{M} = E(M)/M$. Then

$$\begin{aligned} R^iT_\tau(M) = 0 \quad \text{for all } i < n &\Leftrightarrow M \text{ is } \tau\text{-torsion free and } R^iT_\tau(\overline{M}) = 0 \\ &\qquad \text{for all } i < n-1 \\ &\Leftrightarrow M \text{ is } \tau\text{-torsionfree and} \\ &\qquad \tau\text{-dom.dim}(\overline{M}) \geqslant n-1 \\ &\Leftrightarrow \tau\text{-dom.dim}(M) \geqslant n, \end{aligned}$$

and so we are done.

The commutative version of this theorem was proven in [6].

(1.3) COROLLARY. *If $\tau \in R$-tors and if M is a left R-module satisfying τ-dom.dim$(M) \geqslant n$ then for any R-monomorphism $\alpha: M \to M$ we have τ-dom.dim$(M/M\alpha) \geqslant n - 1$.*

PROOF. By hypothesis we have an exact sequence $0 \to M \xrightarrow{\alpha} M \to M/M\alpha \to 0$ of left R-modules which induces a long exact sequence

$$\cdots \to R^iT_\tau(M) \to R^iT_\tau(M/M\alpha) \to R^{i+1}T_\tau(M) \to \cdots.$$

Since $R^iT_\tau(M) = 0$ for all $i < n$ by Proposition 1.2, we have $R^iT_\tau(M/M\alpha) = 0$ for all $i < n - 1$ and so, by Proposition 1.2, τ-dom.dim$(M/M\alpha) \geqslant n - 1$.

We would now like to calculate $R^iT_\tau(M)$ for certain types of torsion theories τ and left R-modules M. Recall that a torsion theory $\tau \in R$-tors is *stable* if and only if the class of all τ-torsion left R-modules is closed under taking injective hulls. The basic properties of stable torsion theories are summarized in [8]. In particular, if R is a commutative noetherian ring then every element of R-tors is stable.

For any torsion theory $\tau \in R$-tors and for any τ-torsion left R-module M we have $R^1T_\tau(M) = 0$ [7, Lemma 2]. For stable torsion theories this result can be further extended.

(1.4) Proposition. *If $\tau \in R$-tors is stable and if M is a τ-torsion left R-module then $R^iT_\tau(M) = 0$ for all $i > 0$.*

Proof. Let $0 \to M \to E_0 \to E_1 \to \cdots$ be a minimal injective resolution of M. Since M is τ-torsion and since τ is stable, we see that each E_i is τ-torsion and so the complex $0 \to T_\tau(E_0) \to T_\tau(E_1) \to \cdots$ is exact at $T_\tau(E_i)$ for all $i > 0$, which is what we need to show.

(1.5) Corollary. *If $\tau \in R$-tors is stable and if M is a left R-module then $R^iT_\tau(M) \cong R^iT_\tau(M/T_\tau(M))$ for all $i > 0$.*

Proof. The exact sequence $0 \to T_\tau(M) \to M \to M/T_\tau(M) \to 0$ induces a long exact sequence

$$0 \to T_\tau(T_\tau(M)) \to T_\tau(M) \to T_\tau(M/T_\tau(M)) \to R^1T_\tau(T_\tau(M)) \to \cdots$$

in which, by Proposition 1.4, we know that $R^iT_\tau(T_\tau(M)) = 0$ for all $i > 0$. From this the result follows immediately.

The following result was first established for commutative rings by Cahen [6].

(1.6) Proposition. *If $\tau \in R$-tors is stable and if M is a left R-module then $R^1T_\tau(M) \cong \operatorname{coker}(\lambda^\tau_M)$.*

Proof. Set $K_\tau = \operatorname{coker}(\lambda^\tau_M)$. Then the short exact sequence

$$0 \to M/T_\tau(M) \to Q_\tau(M) \to K_\tau \to 0$$

gives rise to a long exact sequence

$$0 \to T_\tau(K_\tau) \to R^1T_\tau(M/T_\tau(M)) \to R^1T_\tau(Q_\tau(M)) \to \cdots.$$

Since $Q_\tau(M)$ is τ-torsionfree and τ-injective, we see that τ-dom.dim$(Q_\tau(M)) \geqslant 2$ [20] and so $R^1T_\tau(Q_\tau(M)) = 0$. Moreover, K_τ is τ-torsion by construction of $Q_\tau(M)$ and so $T_\tau(K_\tau) = K_\tau$. Therefore, by Corollary 1.5, $K_\tau \cong R^1T_\tau(M/T_\tau(M)) \cong R^1T_\tau(M)$.

(1.7) Proposition. *If $\tau \in R$-tors is stable and if M is a nonzero τ-dense submodule of its injective hull then $R^iT_\tau(M) = 0$ for all $i > 1$.*

PROOF. The short exact sequence

$$0 \to M \to E(M) \to E(M)/M \to 0$$

gives rise to a long exact sequence

$$\cdots \to R^1T_\tau(E(M)/M) \to R^2T_\tau(M) \\ \to R^2T_\tau(E(M)) \to R^2T_\tau(E(M)/M) \to \cdots.$$

By Proposition 1.4, we know that $R^iT_\tau(E(M)/M) = 0$ for all $i > 0$. Moreover, since, as abelian groups, we have $R^iT_\tau(E(M)) \cong \varinjlim \operatorname{Ext}^i_R(R/I, E(M))$ (where the limit is taken over the filter of all τ-dense left ideals I of R) and since $E(M)$ is injective, we see that $R^iT_\tau(E(M)) = 0$ for all $i > 0$. Therefore $R^iT_\tau(M) = 0$ for all $i > 1$.

(1.8) PROPOSITION. *If $\tau \in R$-tors is stable and if M is a τ-torsionfree left R-module then*

(1) $R^0T_\tau(M) = 0$;

(2) $R^1T_\tau(M) \cong E_\tau(M)/M$;

(3) $R^iT_\tau(M) \cong R^iT_\tau(E_\tau(M))$ *for all* $i > 1$.

PROOF. (1) follows directly from the fact that $R^0T_\tau(M) = T_\tau(M)$. Moreover, the short exact sequence

$$0 \to M \to E_\tau(M) \to E_\tau(M)/M \to 0$$

yields a long exact sequence

$$0 \to R^0T_\tau(M) \to R^0T_\tau(E_\tau(M)) \to R^0T_\tau(E_\tau(M)/M) \to R^1T_\tau(M) \\ \to R^1T_\tau(E_\tau(M)) \to R^1T_\tau(E_\tau(M)/M) \to R^2T_\tau(M) \to \cdots$$

in which $R^0T_\tau(M) = R^0T_\tau(E_\tau(M)) = 0$ by (1) and $R^iT_\tau(E_\tau(M)/M) = 0$ for all $i > 0$ by Proposition 1.4. In particular, this implies (3). Finally, (2) follows directly from Proposition 1.6.

(1.9) PROPOSITION. *Let $\tau \in R$-tors and let M be a left R-module having minimal injective resolution*

$$0 \to M \to E_0 \xrightarrow{\alpha_0} E_1 \xrightarrow{\alpha_1} E_2 \to \cdots.$$

If $M_i = \ker(\alpha_i)$ for all $i \geqslant 0$ then $R^kT_\tau(M_i) \cong R^{k-1}T_\tau(M_{i+1})$ for any $k \geqslant 2$. Moreover, if $R^0T_\tau(M_i) = 0$ then $R^1T_\tau(M_i) \cong R^0T_\tau(M_{i+1})$.

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PROOF. From the exact sequence $0 \to M_i \to E_i \to M_{i+1} \to 0$ we obtain the long exact sequence

$$\begin{aligned} 0 &\to R^0T_\tau(M_i) \to R^0T_\tau(E_i) \to R^0T_\tau(M_{i+1}) \to R^1T_\tau(M_i) \\ &\to R^1T_\tau(E_i) \to R^1T_\tau(M_{i+1}) \to \cdots \to R^{k-1}T_\tau(E_i) \\ &\to R^{k-1}T_\tau(M_{i+1}) \to R^kT_\tau(M_i) \to R^kT_\tau(E_i) \to \cdots \end{aligned}$$

from which we obtain the desired result since for all $k > 0$ we have $R^kT_\tau(E_i) = 0$ by the injectivity of E_i.

As an immediate consequence of Proposition 1.9 we see that if $\tau \in R$-tors and if M is a left R-module then for all positive integers k and h we have $R^{k+h}T_\tau(M) \cong R^hT_\tau(M_k)$, where M_k is defined as in the proof of Proposition 1.9.

(1.10) PROPOSITION. *Let $\tau \leqslant \sigma$ be stable torsion theories in R-*tors. *For any nonnegative integer k and any left R-module M the condition*

(1) $R^iT_\sigma(M) = 0$ *for all* $i \leqslant k$

implies

(2) $R^iT_\tau(M) = 0$ *for all* $i \leqslant k$.

PROOF. If $k = 0$ then for any left R-module M we have $R^0T_\sigma(M) = T_\sigma(M) \supseteq T_\tau(M) = R^0T_\tau(M)$ and so the result is immediate. Next assume that $k = 1$. If M is a left R-module satisfying (1) then, in particular, M is σ-torsionfree and hence τ-torsionfree. Therefore, by Proposition 1.8, we have $R^1T_\tau(M) \cong E_\tau(M)/M \subseteq E_\sigma(M)/M \cong R^1T_\sigma(M)$ and so $R^1T_\sigma(M) = 0$ implies that $R^1T_\tau(M) = 0$.

Now assume inductively that $k > 1$ and that any left R-module M satisfying $R^iT_\sigma(M) = 0$ for all $i \leqslant k - 1$ also satisfies $R^iT_\tau(M) = 0$ for all $i \leqslant k - 1$. By Proposition 1.9 we have $R^iT_\sigma(M) \cong R^{k-1}T_\sigma(E(M)/M)$ and $R^kT_\tau(M) \cong R^{k-1}T_\tau(E(M)/M)$. By assumption, $0 = R^iT_\sigma(M) \cong R^{i-1}T_\sigma(E(M)/M)$ for all $0 < i \leqslant k$ and so, by the induction hypothesis, we see that $R^{i-1}T_\tau(E(M)/M) = 0$ for all $0 < i \leqslant k$. Therefore $R^iT_\tau(M) = 0$ for all $0 < i \leqslant k$. Moreover, we have already seen that $R^0T_\sigma(M) = 0$ implies that $R^0T_\tau(M) = 0$ as well.

A torsion theory $\tau \in R$-tors is *exact* if and only if the localization functor $Q_\tau(-)$: R-mod $\to$ R-mod is exact. See Section 16 of [8] for details about such torsion theories.

(1.11) PROPOSITION. *The following conditions on a torsion theory $\tau \in R$-*tors *are equivalent*:

(1) *τ is exact*;

(2) *If M is a τ-torsionfree τ-injective left R-module then $R^iT_\tau(M) = 0$ for all $i \geqslant 0$.*

PROOF. (1) ⇒ (2): Let

$$(*) \qquad 0 \to M \to E_0 \xrightarrow{\alpha_0} E_1 \xrightarrow{\alpha_1} E_2 \to \cdots$$

be a minimal injective resolution of M. By repeated application of Proposition 16.1 of [8] we see that $E_i/\ker(\alpha_i)$ is τ-torsionfree and τ-injective for all $i \geq 0$ and hence E_i is τ-torsionfree for all such i. This proves that $R^iT_\tau(M) = 0$ for all $i \geq 0$.

(2) ⇒ (1): If M is a left R-module which is τ-torsionfree and τ-injective then $T_\tau(E(M)/M) = E_\tau(M)/M = 0$. Let $(*)$, as above, be a minimal injective resolution of M. Then E_0/M is τ-torsionfree and hence so is E_1. Therefore $R^2T_\tau(M) = \ker(T_\tau(\alpha_2)) = T_\tau(\ker(\alpha_2))$. By hypothesis, $\ker(\alpha_2) \cong E(\ker(\alpha_1))/\ker(\alpha_1)$ is τ-torsionfree. Therefore $\ker(\alpha_1)$ is τ-injective. But it is also τ-torsionfree and so by Proposition 16.1 of [8] we see that $Q_\tau(-)$ is exact and so the torsion theory τ is exact.

(1.12) COROLLARY. *If $\tau \in R$-tors is exact and stable and if M is a τ-torsionfree left R-module then $R^iT_\tau(M) = 0$ for all $i \neq 1$.*

PROOF. This is a direct consequence of Proposition 1.8 and Proposition 1.11.

(1.13) PROPOSITION. *The following conditions on a stable torsion theory $\tau \in R$-tors are equivalent:*

(1) *τ is exact;*

(2) *$R^iT_\tau(M) = 0$ for any left R-module M and for all $i > 1$;*

(3) *$R^2T_\tau(M) = 0$ for any left R-module M.*

PROOF. (1) ⇒ (2): Let M be a left R-module. Set $M' = T_\tau(M)$ and $M'' = M/M'$. Then we have a long exact sequence

$$\begin{aligned} 0 &\to R^0T_\tau(M') \to R^0T_\tau(M) \to R^0T_\tau(M'') \to R^1T_\tau(M') \\ &\to R^1T_\tau(M) \to R^1T_\tau(M'') \to R^2T_\tau(M') \to \cdots. \end{aligned}$$

By Proposition 1.4 we see that $R^iT_\tau(M') = 0$ for all $i > 0$ and by Corollary 1.12 we see that $R^iT_\tau(M'') = 0$ for all $i \neq 1$. Therefore, by exactness, $R^iT_\tau(M) = 0$ for all $i > 1$.

(2) ⇒ (3): This implication is trivial.

(3) ⇒ (1): Let M be τ-torsionfree and τ-injective left R-module. Then $E(M)$ is τ-torsionfree and the short exact sequence

$$0 \to M \to E(M) \to E(M)/M \to 0$$

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induces a long exact sequence

$$\cdots \to R^0T_\tau(E(M)/M) \to R^1T_\tau(M) \to R^1T_\tau(E(M))$$
$$\to R^1T_\tau(E(M)/M) \to R^2T_\tau(M) \to \cdots,$$

where $R^0T_\tau(E(M)/M) = 0$ since $E(M)/M$ is τ-torsionfree by Proposition 5.1 of [8] and where $R^2T_\tau(M) = 0$ by (3). Moreover, by Proposition 1.6 we see that $R^1T_\tau(E(M)) \cong Q_\tau(E(M))/E(M) = 0$. Therefore $R^1T_\tau(E(M)/M) = 0$. By Proposition 1.6 this implies that $E(M)/M$ is τ-torsionfree and τ-injective, which establishes (1) by Proposition 16.1 of [8].

(1.14) Example. Let I be an ideal of a ring R which is finitely-generated as a left ideal of R. Then a left R-module M is $\xi(R/I)$-torsion if and only if every element of M is annihilated by a power of I. Therefore, in this situation, we see that $R^nT_{\xi(R/I)}(M)$ is naturally isomorphic, as an abelian group, to $\varinjlim_{k\geq 0} \mathrm{Ext}^n_R(R/I^k, M)$ for any nonnegative integer n. This shows that, in the case of commutative noetherian rings, the functors $R^nT_{\xi(R/I)}(-)$ coincide with the local cohomology functors studied by Sharp [18]. In the noncommutative noetherian case we obtain the local cohomology functors studied by Barou [3].

If I is an ideal of a left noetherian ring R then the torsion theory $\xi(R/I)$ is stable if and only if I has the Artin-Rees property with respect to every finitely-generated left R-module. That is to say, $\xi(R/I)$ is stable if and only if for every submodule N of a finitely-generated left R-module M and for each natural number n there exists a natural number $h = h(n)$ for which $I^hM \cap N \subseteq I^nN$. [4] This holds, for example, if R is a noetherian ring and if I is generated by a centralizing family of elements (that is, if there exist elements $r_1, \ldots, r_m$ of I such that the image of each r_i is in the center of R modulo the ideal generated by $r_1, \ldots, r_{i-1}$) [3].

2. Various dimensions

Let $\tau \in R$-tors and let M be a left R-module. We define the *τ-dimension* of M, denoted by $\dim_\tau(M)$, as follows:

(1) If $\mathrm{supp}(M) \cap \mathbf{P}(\tau) = \varnothing$ set $\dim_\tau(M) = -1$;

(2) If n is a nonnegative integer satisfying the following conditions:

(i) There exists a chain of the form $\pi_n < \cdots < \pi_0$ in $\mathbf{P}(\tau)$ with $\pi_0 \in \mathrm{supp}(M)$; and

(ii) if $h > n$ there exists no chain of the form $\pi_h < \cdots < \pi_0$ in $\mathbf{P}(\tau)$ with $\pi_0 \in \mathrm{supp}(M)$,

then set $\dim_\tau(M) = n$;

(3) otherwise, set $\dim_\tau(M) = \infty$.

If U is a nonempty subset of R-tors we define $\dim_U(M)$ to be $\sup\{\dim_\tau(M) \mid \tau \in U\}$.

A ring R is *left stable* if and only if every element of R-tors is stable. Left stable left noetherian rings behave very nicely in many ways and they are a convenient generalization of commutative noetherian rings. It is therefore natural to look at them in order to try and calculate that τ-dimension of modules.

Let us recall a construction used in Chapter 12 of [11]. If $\tau \in R$-tors we can define an ascending chain $\tau_0 \leqslant \tau_1 \leqslant \cdots$ in R-tors, called the *Gabriel filtration* of τ, by setting $\tau_0 = \tau$ and $\tau_i = \tau_{i-1} \vee (\bigvee\{\xi(M) \mid M$ is τ_{i-1}-cocritical$\})$ for all positive integers i.

(2.1) PROPOSITION. *Let R be a left stable left noetherian ring and let $\tau \in R$-tors. For a τ-torsionfree cocritical left R-module N and for a positive integer i the following conditions are equivalent*:

(1) $\xi(N) \leqslant \tau_i$;

(2) *If $\pi_h < \cdots < \pi_0 = \chi(N)$ is a chain in $\mathbf{P}(\tau)$ then $h < i$.*

PROOF. We will proceed by induction on i. First let us consider the case of $i = 1$.

Assume (1). Since N is τ-torsionfree and τ_1-torsion, there must exist a τ-cocritical left R-module M such that N is not $\xi(M)$-torsionfree. By stability, this implies that N is $\xi(M)$-torsion and so there exists a nonzero R-homomorphism α from a submodule M' of M to N. Since N is τ-torsionfree, the map α must be monic. Since N is uniform, this implies that M' is isomorphic to a large submodule of N and so $\chi(N) = \chi(M'\alpha) = \chi(M') = \chi(M)$. Thus, by Proposition 2.5.16 of [17] we see that $\chi(N)$ is a minimal element of $\mathbf{P}(\tau)$, proving (2). Conversely, assume (2). If $\chi(N)$ is a minimal element of $\mathbf{P}(\tau)$ then by Proposition 2.5.16 of [17] there exists a τ-cocritical left R-module M satisfying $\chi(N) = \chi(M)$. Hence N is isomorphic to a submodule of $E(M)$ which, by the definition of τ_1 and by stability, is τ_1-torsion. This proves (1).

Now assume that $i > 1$ and that for any $j < i$ we have already established the equivalence of (1) and (2).

Assume that N satisfies (1). If $\xi(N) \leqslant \tau_{i-1}$ then (2) follows by the induction hypothesis. Therefore we can assume that N is not τ_{i-1}-torsion. By stability, this implies that is is τ_{i-1}-torsionfree. As in the proof of the case $i = 1$, this implies that $\chi(N)$ is a minimal element of $\mathbf{P}(\tau_{i-1})$. Therefore, without loss of generality, we can assume that N is in fact τ_{i-1}-cocritical. If $\chi(N) = \pi_0 > \cdots > \pi_h$ is a chain of torsion theories in $\mathbf{P}(\tau)$ then, by stability, π_1 is of the form $\chi(N')$, where N' is a proper homomorphic image of a submodule of N. In particular, N' is τ_{i-1}-torsion and so, by the induction hypothesis, $h \leqslant i - 1$. This proves (2). Conversely, assume (2). If there is no chain in $\mathbf{P}(\tau)$ of the form

$$\pi_{i-1} < \cdots < \pi_0 = \chi(N)$$

then (1) follows by the induction hypothesis. Assume therefore that such a chain exists. Let N' be a proper homomorphic image of N. If M is a cocritical submodule of N' then $\chi(N) > \chi(M)$. Therefore, if $\pi_h < \cdots < \pi_0 = \chi(M)$ is a chain of torsion theories in $\mathbf{P}(\tau)$ we must have $h < i - 1$. By the induction hypothesis, this means that M is τ_{i-1}-torsion and so N' is τ_{i-1}-torsion. Hence N is either τ_{i-1}-torsion or τ_{i-1}-cocritical. In either case, (1) follows.

(2.2) PROPOSITION. *If R is a left stable left noetherian ring and if $\tau \in R$-*tors *then for a left R-module M and for a nonnegative integer n the following conditions are equivalent*:

(1) $\xi(M) \leqslant \tau_{n+1}$ *and* $\xi(M) \nleqslant \tau_n$.

(2) $\dim_\tau(M) = n$.

PROOF. (1) $\Rightarrow$ (2): By (1), M is not τ_n-torsion and so there exists a cocritical submodule N of M which is not τ_n-torsion and hence is τ_n-torsionfree. On the other hand, M is τ_{n+1}-torsion and hence so is N. Thus, by Proposition 2.1, $\chi(N) \in \operatorname{supp}(M)$ and there exists a chain of the form $\pi_n < \cdots < \pi_0 = \chi(N)$ in $\mathbf{P}(\tau)$. This proves that $\dim_\tau(M) \geqslant n$. Now assume that there exists an element π of $\operatorname{supp}(M)$ and a chain $\pi'_h < \cdots < \pi'_0 = \pi$ in $\mathbf{P}(\tau)$ with $h > n$. Then M is not π-torsion and so there exists a cocritical submodule N' of M which is not π-torsion and hence is π-torsionfree. This implies that $\chi(N') \geqslant \pi$. By Proposition 2.1, this implies that N' is not τ_{n+1}-torsion, and so neither is M. This contradicts (1), proving (2).

(2) $\Rightarrow$ (1): From (2) we deduce that if N is a cocritical submodule of M then N is either τ-torsion or for any chain $\pi_h < \cdots < \pi_0 = \chi(N)$ in $\mathbf{P}(\tau)$ we have $h < n + 1$. Therefore, by Proposition 2.1 we see that every such module N is τ_{n+1}-torsion. By stability, this implies that M is τ_{n+1}-torsion and so $\xi(M) \leqslant \tau_{n+1}$. On the other hand, there exists an element π of $\operatorname{supp}(M)$ and a chain $\pi_n < \cdots < \pi_0 = \pi$. Since M is not π-torsion, there exists a cocritical submodule N' of M which is not π-torsion and hence is π-torsionfree. Therefore $\chi(N') \geqslant \pi$. Indeed, by the condition on the lengths of chains we must in fact have equality here. By Proposition 2.1, this means that $\xi(N') \nleqslant \tau_n$ and so $\xi(M) \nleqslant \tau_n$.

We will say that a ring R is *left effective* if and only if it is left stable, left noetherian, and every element of R-sp is exact. By Proposition 17.1 of [8] we see that, in the presence of the noetherian condition, this last condition is equivalent to the condition that every element of R-sp is perfect. Commutative noetherian rings are clearly left effective. By Example 6.16 of [12] and by Proposition 9 of [22] we see that left noetherian Azumaya algebras are left effective.

(2.3) EXAMPLE. Let R be a prime hereditary noetherian quasi-local ring which is a bounded order in its classical ring of fractions. We claim that R is left effective. Indeed, since R is left hereditary, we know that every element of R-tors is exact by Proposition 16.4 of [8]. Moreover, by Proposition IV.1.7 of [15] we see that R is fully left bounded and left noetherian so the map $P \mapsto \chi(R/P)$ is a bijective correspondence between the set spec(R) of all prime ideals of R and R-sp. See Propositions 6.7 and 6.11 of [11] for details. By Proposition IV.1.1 of [15] we see that the Goldie torsion theory in R-tors is faithful and so it equals the Lambek torsion theory $\chi(R)$. Therefore $\chi(R)$ is stable. If R is not simple then by the quasi-locality of R we see that the Jacobson radical $J(R)$ is the only nonzero prime ideal of R and that $\chi(R/J(R)) = \xi$, since any nonzero ideal of R is a power of $J(R)$. (See pages 50–51 of [15].) Therefore $\chi(R/J(R))$ is also stable, proving that R is left stable and so left effective. Examples of rings of this type can be found in sections I.8 and III.4 of [15].

(2.4) PROPOSITION. *Let R be a left effective ring and let $\tau \in R$-*tors. *If M is a left R-module and if i is a natural number satisfying $R^iT_\tau(M) \neq 0$ then $i \leqslant \dim_\tau(M) + 1$.*

PROOF. Set $k = \dim_\tau(M)$. If $k = \infty$ the result is trivial so we may assume that k is finite. If $k = -1$ the result follows from Proposition 1.4 and so we may assume that k is nonnegative. Since M is the direct union of the directed system of its finitely-generated submodules, it suffices to show that $R^iT_\tau(M') = 0$ for all $i > k + 1$ and for any finitely-generated submodule M' of M. Thus, without loss of generality, we can assume that M itself is finitely-generated and hence noetherian. Since R is left noetherian, it is surely left definite and so every nonzero homomorphic image of M has a nonzero cocritical submodule. Since M is assumed to be noetherian, this means that we can find a chain

$$0 = N_0 \subset N_1 \subset \cdots \subset N_u = M$$

of submodules of M satisfying the condition that $\overline{N}_h = N_h/N_{h-1}$ is cocritical for all $1 \leqslant h \leqslant u$. To prove the proposition, it suffices to show that $R^iT_\tau(\overline{N}_h) = 0$ for all $1 \leqslant h \leqslant u$ and all $i > k + 1$. To do this, we proceed by induction on k.

First assume that $k = 0$. If $\overline{N}_h$ is τ-torsion the desired result follows from Proposition 1.4. Therefore assume that it is not τ-torsion. By stability, this implies that $\overline{N}_h$ is τ-torsionfree and so $\pi_h = \chi(\overline{N}_h) \in \mathbf{P}(\tau)$. By Proposition 1.11, we know that $R^iT_{\pi_h}(E_{\pi_h}(\overline{N}_h)) = 0$ for all $i \geqslant 0$. By Proposition 1.10, this implies that $R^iT_\tau(E_{\pi_h}(\overline{N}_h)) = 0$ for all $i \geqslant 0$. Set $N'_h = E_{\pi_h}(\overline{N}_h)/\overline{N}_h$. We claim that N'_h is

τ-torsion. Indeed, if $\pi \in \text{supp}(N_h') \cap \mathbf{P}(\tau)$ then $\pi \in \text{supp}(E_{\pi_h}(\overline{N}_h)) = \text{supp}(\overline{N}_h)$ so, by stability and by the uniformity of $\overline{N}_h$, we see that $\overline{N}_h$ must be π-torsionfree. Therefore $\pi_h = \chi(\overline{N}_h) \geqslant \pi$. Since N_h' is in fact π_h-torsion by construction, this inequality must be strict. But this is a contradiction for, by construction, π_h is a minimal element of $\mathbf{P}(\tau) \cap \text{supp}(M)$. Therefore $\pi \notin \mathbf{P}(\tau)$. Thus we see that N_h' is π-torsion for all $\pi \in \mathbf{P}(\tau)$ and so N_h' is τ-torsion, as claimed.

The short exact sequence $0 \to \overline{N}_h \to E_{\pi_h}(\overline{N}_h) \to N_h' \to 0$ induces a long exact sequence

$$0 \to R^0T_\tau(\overline{N}_h) \to R^0T_\tau\left(E_{\pi_h}(\overline{N}_h)\right) \to R^0T_\tau(N_h') \to R^1T_\tau(\overline{N}_h)$$
$$\to R^1T_\tau\left(E_{\pi_h}(\overline{N}_h)\right) \to R^1T_\tau(N_h') \to R^2T_\tau(\overline{N}_h) \to \cdots$$

with respect to which we note the following:

(1) $R^0T_\tau(\overline{N}_h) = R^0T_\tau(E_{\pi_h}(\overline{N}_h)) = 0$ by τ-torsionfreeness;

(2) $R^iT_\tau(E_{\pi_h}(\overline{N}_h)) = 0$ for all $i \geqslant 0$, as remarked above;

(3) N_h' is τ-torsion by the above claim and so $R^iT_\tau(N_h') = 0$ for all $i > 0$ by Proposition 1.4.

Therefore, by exactness, $R^iT_\tau(\overline{N}_h) = 0$ for all $i > 1$, which is what we wanted to show.

Now assume that $k > 0$ and that for any left R-module M'' satisfying $\dim_\tau(M'') < k$ we have $R^iT_\tau(M'') = 0$ for all $i > \dim_\tau(M'') + 1$. In particular, we know that $R^iT_\tau(\overline{N}_h) = 0$ whenever $i > k + 1$ and $\dim(\overline{N}_h) < k$ so we need consider only those indices h for which $\dim_\tau(\overline{N}_h) = k$. Moreover, as before, we can assume that $\overline{N}_h$ is τ-torsionfree.

We claim that in this situation $N_h' = E_{\pi_h}(\overline{N}_h)/\overline{N}_h$ satisfies $\dim_\tau(N_h') < k$. Indeed, since $\dim_\tau(\overline{N}_h) = k$ we see that $\overline{N}_h$ is $\delta(U)$-torsion, where U is the subset of $\mathbf{P}(\tau)$ consisting of those elements π' for which any chain of the form $\pi_t < \cdots < \pi_0 = \pi'$ in $\mathbf{P}(\tau)$ satisfies $t \leqslant k$. By stability, $E_{\pi_h}(\overline{N}_h)$ is also $\delta(U)$-torsion and hence so is N_h'. This is equivalent to the condition that $\varnothing \neq \text{ass}(N_h'/N) \subseteq U$ for every proper submodule N of N_h'. But for each such N we have $\text{ass}(N_h'/N) \subseteq \text{supp}(N_h'/N) \subseteq \text{supp}(N_h') \subseteq \text{supp}(E_{\pi_h}(\overline{N}_h)) = \{\pi' \in R\text{-sp} \mid \pi' \leqslant \pi_h\}$. Thus if $\pi \in \text{ass}(N_h'/N)$ we have $\pi \leqslant \pi_h$ and in fact we cannot have equality here since N_h' is π_h-torsion but not π-torsion.

Let U' be the set of those elements π' in U for which there is no chain of the form $\pi_k < \cdots < \pi_0 = \pi'$ in $\mathbf{P}(\tau)$. Since $\pi_h \in U$, we see by the above that $\pi \in U'$. Thus for any proper submodule N of N_h' we have $\varnothing \neq \text{ass}(N_h'/N) \subseteq U'$. This shows that N_h' is $\delta(U')$-torsion and so $\dim_\tau(N_h') < k$, as claimed. By the induction hypothesis, this means that $R^iT_\tau(N_h') = 0$ for all $i > k$. Again, as before, $R^iT_\tau(E_{\pi_h}(\overline{N}_h)) = 0$ for all $i \geqslant 0$ and so from the long exact sequence

$$\cdots \to R^iT_\tau(N_h') \to R^{i+1}T_\tau(\overline{N}_h) \to R^{i+1}T_\tau\left(E_{\pi_h}(\overline{N}_h)\right) \to \cdots$$

we deduce that $R^iT_\tau(\overline{N}_h) = 0$ for all $i > k + 1$.

(2.5) PROPOSITION. *Let R be a left effective ring and let $\tau \in R$-tors. If M is a left R-module and if i is a natural number satisfying $R^iT_\tau(M) \neq 0$ then $i \leqslant \dim_{V(\tau)}(M)$.*

PROOF. Set $k = \dim_{V(\tau)}(M)$. If k is infinite then we are done trivially and so we can assume that k is finite. Assume $k = -1$. Then $\text{supp}(M) \subseteq \mathbf{P}(\tau)$. If

$$0 \to M \to E_0 \overset{\alpha_0}{\to} E_1 \overset{\alpha_1}{\to} E_2 \to \cdots$$

is a minimal injective resolution of M then for all $i \geqslant 0$ we have $\text{supp}(E_i) \subseteq \text{supp}(M) \subseteq \mathbf{P}(\tau)$. The ring R is left noetherian and so, in particular, left definite. Therefore each E_i is τ-torsionfree and so $R^iT_\tau(M) = 0$ for all $i \geqslant 0$.

We are left to consider the case of k nonnegative. As in the proof of Proposition 2.4, we can assume without loss of generality that M is a noetherian left R-module. There therefore exists a chain

$$0 = N_0 \subset N_1 \subset \cdots \subset N_u = M$$

of submodules of M satisfying the condition that $\overline{N}_h = N_h/N_{h-1}$ is cocritical for all $1 \leqslant h \leqslant u$. To prove the proposition, it therefore suffices to show that $R^iT_\tau(\overline{N}_h) = 0$ for all $1 \leqslant h \leqslant u$ and all $i > k$. To do this, we proceed by induction on k.

Assume that $k = 0$ and let $1 \leqslant h \leqslant u$. If $\overline{N}_h$ is not τ-torsionfree then it is τ-torsion and we are done by Proposition 1.4. Therefore assume that $\overline{N}_h$ is τ-torsionfree and consider a minimal injective resolution

$$0 \to \overline{N}_h \to E_0 \overset{\alpha_0}{\to} E_1 \overset{\alpha_1}{\to} E_2 \to \cdots$$

of $\overline{N}_h$. To show that $R^iT_\tau(\overline{N}_h) = 0$ for all $i > 0$ it suffices to show that E_i is τ-torsionfree for each $i > 0$. Set $\pi_h = \chi(\overline{N}_h)$.

We first note that $\text{supp}(\overline{N}_h) \subseteq \mathbf{P}(\tau)$. Indeed, if this were not the case then there would exist a torsion theory π belonging to $\text{supp}(\overline{N}_h) \cap \mathbf{V}(\tau)$. Since $\overline{N}_h$ is not π-torsion, it is π-torsionfree and so $\pi < \pi_h$, contradicting the assumption that $\dim_\pi(M) \leqslant \dim_{V(\tau)}(M) = 0$. We next note that by Proposition 2.7.16, 2.7.4, and 2.6.1 of [17] we have $\text{supp}(\overline{N}_h) = \{\pi \in R\text{-sp} \mid \pi' \leqslant \pi_h\}$. Thus, in particular, we see that $\text{supp}(E_0) = \text{supp}(\overline{N}_h)$ and if $i > 0$ then $\text{supp}(E_i) = \text{supp}(E_{i-1}/\ker(\alpha_{i-1})) \subseteq \text{supp}(E_{i-1})$ and so $\text{supp}(E_i) \subseteq \text{supp}(\overline{N}_h) \subseteq \mathbf{P}(\tau)$ for all $i \geqslant 0$. Since R is left noetherian and so, in particular, left definite, this implies that each E_i is τ-torsionfree and so $R^iT_\tau(\overline{N}_h) = 0$ for all $i > 0$.

Now assume that $k > 0$ and that for any left R-module M'' satisfying $\dim_{V(\tau)}(M'') < k$ we have $R^iT_\tau(M'') = 0$ for all $i > \dim_{V(\tau)}(M'')$. In particular,

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we know that $R^iT_\tau(\bar{N}_h) = 0$ whenever $i > k$ and $\dim_{V(\tau)}(\bar{N}_h) < k$. Assume therefore that $1 \leqslant h \leqslant u$ satisfies $\dim_{V(\tau)}(\bar{N}_h) = k$. If $\bar{N}_h$ is τ-torsion the desired result follows from Proposition 1.4. Hence we can assume without loss of generality that $\bar{N}_h$ is τ-torsionfree.

Consider the exact sequence of left R-modules

$$(*) \qquad 0 \to \bar{N}_h \to E_{\pi_h}(\bar{N}_n) \to N_h' \to 0$$

in which $N_n' = E_{\pi_h}(\bar{N}_h)/\bar{N}_h$. Note that $\operatorname{supp}(N_h') \subseteq \operatorname{supp}(E_{\pi_h}(\bar{N}_h)) = \operatorname{supp}(\bar{N}_h) = \{\pi' \in R\text{-sp} \mid \pi' \leqslant \pi_h\}$. Moreover, N_h' is π_h-torsion so if $\pi \in V(\tau) \cap \operatorname{supp}(N_h')$ then $\dim_\pi(N_h') < \dim_\pi(\bar{N}_h) \leqslant k$ whence $\dim_{V(\tau)}(N_h') < k$. Hence, by the induction hypothesis, we see that $R^iT_\tau(N_h') = 0$ for all $i \geqslant k$. But the short exact sequence $(*)$ induces a long exact sequence

$$\cdots \to R^iT_\tau\big(E_{\pi_h}(\bar{N}_h)\big) \to R^iT_\tau(N_h') \to R^{i+1}T_\tau(\bar{N}_h) \to R^{i+1}T_\tau\big(E_{\pi_h}(\bar{N}_h)\big) \to \cdots.$$

Since $R^iT_\tau(E_{\pi_h}(\bar{N}_h)) = 0$ for all $i \geqslant 0$ by Proposition 1.11 and since $R^iT_\tau(N_h') = 0$ for all $i \geqslant k$, this implies that $R^iT_\tau(\bar{N}_h) = 0$ for all $i > k$, which is what we needed to prove.

Note that, for any left R-module M, $\dim_\xi(M)$ is precisely the *torsion-theoretic Krull dimension* (or *TTK-dimension*) of M as introduced in [9] and [13] and developed in detail in [11]. By Proposition 13.3 of [11] we see that if R is left effective then this coincides with the Gabriel dimension of M. Moreover, it is clear from the definitions that $\dim_U(M) \leqslant \dim_\xi(M)$ for any nonempty subset U of R-tors. We therefore obtain the following immediate corollary of the previous result.

(2.6) Corollary. *Let R be a left effective ring and let $\tau \in R$-tors. If M is a left R-module and if i is a natural number satisfying $R^iT_\tau(M) \neq 0$ then i is no greater than the Gabriel dimension of M.*

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