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INVARIANT THEORY FOR LINEAR DIFFERENTIAL SYSTEMS MODELED AFTER THE GRASSMANNIAN $\mathrm{Gr}(n,2n)$

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Abstract. We find invariants for the differential systems of rank $2n$ in n^2 variables with n unknowns under the linear changes of the unknowns with variable coefficients. We look for a set of coefficients that determines the other coefficients, and give transformation rules under the linear changes above and coordinate changes. These can be considered as a generalization of the Schwarzian derivative, which is the invariant for second order ordinary differential equations under the change of the unknown by multiplying a non-zero function. Special treatment is done when $n = 2$: the conformal structure obtained through the Plücker embedding is studied, and a relation with line congruences is discussed.

§1. Introduction

In order to help understand our result, we recall the prototype. Let us consider linear ordinary differential equations

$$u'' + \alpha u' + \beta u = 0$$

($u' = du/dx$) together with changes of unknown $u \to ku$ ($k \not\equiv 0$). Two such equations are said to be equivalent if one of such changes of unknown takes one into the other, that is, the ratio of any two linearly independent solutions of one equation relates projectively to that of the other. For a given equation $u'' + \alpha u' + \beta u = 0$, we can find a suitable nonzero function k so that the equation changes into an equation of the form

$$u'' + \bar{\beta} u = 0;$$

the new coefficient $\bar{\beta}$ is a rational function of α, β, and their derivatives: actually we have $\bar{\beta} = \beta - \alpha'/2 - \alpha^2/4$. For any equation equivalent to this equation, the Schwarzian derivative $\{r; x\} = \frac{3}{4}\left(r''/r\right)^2 - \frac{1}{2}r'''/r'$ of the ratio

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r of any two linearly independent solutions is equal to $\bar{\beta}$. The Schwarzian derivative satisfies the following chain rule for coordinate change $x \leftrightarrow y$:

$$\{z; x\}(dx)^2 = \{y; x\}(dx)^2 + \{z; y\}(dy)^2.$$

This prototype treats linear equations in 1 variable of rank (= dimension of local solutions around a nonsingular point) 2 with 1 unknown. Our hope is to generalize the above theory of Schwarzian derivatives to systems of linear equations in m variables of rank r with n unknowns. The corresponding Schwarz map is defined as

$$x = (x^1, \dots, x^m) \longmapsto [u_{(1)}, \dots, u_{(r)}] \in \mathrm{Gr}(n, r),$$

where $u_{(j)}$ are linearly independent n-column solutions and $\mathrm{Gr}(n, r)$ is the (n, r)-Grassmannian manifold; for example, $\mathrm{Gr}(1, r)$ is the $(r-1)$-dimensional projective space $\mathbf{P}^{r-1}$. We are not so optimistic to believe the existence of a sufficiently nice theory of Schwarzian derivatives for general (m, r, n).

When ($m \geq 2$, $r = m + 1$, $n = 1$), it is the well-known theory of projective connections (see e.g. [5]). We treated in [3] the case ($m \geq 2$, $r = m+2$, $n = 1$), and studied the conformal connections when the image of the Schwarz map is a quadratic hypersurface in $\mathbf{P}^{m+1}$. Several differential-geometric studies were made when ($m = 1$, $r = 2n$, $n \geq 2$) in [2] and [4]. In the paper [1] we encountered a system with ($m = 4$, $r = 6$, $n = 1$) as the uniformizing equation of a 4-parameter family of K3 surfaces; the geometry appeared there strongly suggests that the target of the Schwarz map should be $\mathrm{Gr}(2, 4)$ rather than a quadratic hypersurface in $\mathbf{P}^5$, that is, the system should be tranformed into a system with ($m = 4$, $r = 4$, $n = 2$). In this way, we are led to the study of systems in 4 variables of rank 4 with 2 unknowns. Meanwhile we realized that the study of systems with ($m = n^2$, $r = 2n$, $n \geq 2$) is not more difficult (or rather more transparent) than that of the restricted system with $n = 2$.

So, in this paper, we treat systems of linear differential equations in n^2 variables x^{ij} ($1 \leq i, j \leq n$) of rank (= dimension of local solutions around a nonsingular point) $2n$ with n unknowns u^k ($1 \leq k \leq n$). We consider the transformation K of unknowns

$$(u^k) \longrightarrow \left(\sum_l K^k_l u^l\right), \quad \det(K^k_l) \not\equiv 0;$$

two systems related under such changes are said to be *equivalent*. Our Schwarz map is defined on the n^2-dimensional affine space with the coordinates $x = (x^{ij})$ and the target is the Grassmannian manifold $\mathrm{Gr}(n, 2n)$; two equivalent systems define the same Schwarz map. We assume that $n \geq 2$ and the Schwarz map is nondegenerate.

To explain the result of this paper, we write down our system as

$$(E) = E_n(a, b, \alpha, \beta) \begin{cases} u^k_{:11:11} = \sum_l \alpha^k_l \, u^l_{:11} + \sum_l \beta^k_l \, u^l, \\ u^k_{:ij} \quad\;\; = \sum_l a^k_{ijl} \, u^l_{:11} + \sum_l b^k_{ijl} \, u^l, \end{cases}$$

$1 \leq k, l, i, j \leq n$, where $f_{:ij}$ stands for $\partial f / \partial x^{ij}$, and

$$a^k_{11l} = \delta^k_l, \quad b^k_{11l} = 0.$$

We prove that two systems $E_n(a, b, \alpha, \beta)$ and $E_n(\bar{a}, \bar{b}, \bar{\alpha}, \bar{\beta})$ are equivalent if and only if there is an invertible $n \times n$ matrix $K = (K^k_l)$ such that

$$\bar{\mathbf{A}} = K \mathbf{A} K^{-1},$$

where

$$\mathbf{A} = (a^k_l), \quad a^k_l = \sum_{i,j} a^k_{ijl} \, dx^{ij}.$$

That is, $\{a^k_{ijl}\}$ form the essential part of the coefficients. Though there is no natural representative in an equivalence class, and so no counterpart of the Schwarzian derivative either, the matrix 1-form $\mathbf{A}$ will play for it; we call $\mathbf{A}$ the *essential* coefficients of the system. We also give transformation formulas for $\mathbf{A}$ under coordinate changes.

The annoying fact, which we always encounter when treating systems in several variables, is that there are no canonical way to write such systems. In this paper, we also treat such systems expressed in the following form

$$(\underline{E}) \begin{cases} u^k_{:kk:kk} = \sum_l \underline{\alpha}^k_l \, u^l_{:ll} + \sum_l \underline{\beta}^k_l \, u^l, \\ u^k_{:ij} \qquad = \sum_l \underline{a}^k_{ijl} \, u^l_{:ll} + \sum_l \underline{b}^k_{ijl} \, u^l, \quad 1 \leq i, j, k \leq n. \end{cases}$$

When we discuss the associated conformal structure in the case $n = 2$, this form will be convenient.

When $n = 2$, as we mentioned above, the Plücker image of $\mathrm{Gr}(2, 4)$ is a quadratic hypersurface, which naturally carries a conformal structure. We express the pull-back of the conformal structure in terms of the essential coefficients. In order to get a converse expression, we define two differential 1-forms associated with the system, and compute the covariance of these forms relative to linear change of unknowns and relative to coordinate change. In view of the covariance, we give a procedure of deriving the essential coefficients from the conformal structure.

When $n = 2$, the system (E) has a nice geometric interpretation. Since each component of the unknowns is a vector in $\mathbf{P}^3$, the pair of fundamental solutions defines a line that depends on the four variables x. Thus the system can be seen as defining a 2-parameter family of line congruences; here a line congruence is a 2-parameter family of lines in $\mathbf{P}^3$. With this geometrical interpretation, we introduce a normal form of the system (E). Relying on this normalization, we give a non-trivial example of 2-parameter families of line congruences such that both associated focal surfaces are quadratic surfaces.

§2. Non-degeneracy

Let us consider a system $(E) = E_n(a, b, \alpha, \beta)$. Since we can easily see that every derivative of u^k can be expressed in terms of u^l and $u^l_{:11}$, and so that any system of this form is of rank at most $2n$. We *assume* that our system is of rank $2n$. In other words, the corresponding matrix system $dU = \Omega U$ with respect to the unknown $2n$-vector

$$U = {}^t(u^1_{:11}, \dots, u^n_{:11}, u^1, \dots, u^n)$$

admits $2n$ linearly independent solutions. Let

$$u_{(j)} = {}^t(u^1_{(j)}, \dots, u^n_{(j)}), \quad j = 1, \dots, 2n$$

be a basis of the solutions. We *assume* also that the Schwarz map

$$\mathcal{S} : (x^{ij}) \longmapsto [u_{(1)}, \dots, u_{(2n)}] \in \mathrm{Gr}(n, 2n)$$

from the x-space to the $(n, 2n)$-Grassmannian manifold

$$\mathrm{Gr}(n, 2n) = \mathrm{GL}(n) \setminus \{n \times 2n \text{ matrices of rank } n\}$$

is nondegenerate. Let us paraphrase this assumption.

Proposition 1. *The Schwarz map of the system* (E) *is nondegenerate if and only if the* $n^2 \times n^2$*-determinant*

$$W = \det(a^k_{ijl})_{(i,j),(k,l)}$$

does not vanish identically.

A straightforward computation leads to

Lemma 1. *The transformation*

$$u^k \longrightarrow \sum_l K^k_l\, u^l, \quad \det K^k_l \neq 0$$

changes the coefficients a *as*

$$a^k_{ijl} \longrightarrow \sum K^k_p\, a^p_{ijq}\, (K^{-1})^q_l,$$

in other words,

$$\mathbf{A} = (a^k_l) \longrightarrow K\mathbf{A}K^{-1}, \quad a^k_l = \sum a^k_{ijl}\, dx^{ij},$$

and α *as*

$$\mathcal{A} = (\alpha^k_l) \longrightarrow (2K_{:11} + K\mathcal{A})K^{-1}.$$

The identity

$$\det\bigl(K^k_p a^p_{ijq}(K^{-1})^q_l\bigr)_{(i,j),(k,l)} = (\det K)^n \det(a^p_{ijq})_{(i,j),(p,q)}(\det K^{-1})^n$$

implies that W is invariant under the transformation K. Now take K the $n \times n$ matrix consisting of n linearly independent solutions of the system. Then the new system admits the n solutions

$$e_{(1)} = {}^t(1, 0, \ldots, 0), \ldots, e_{(n)} = {}^t(0, \ldots, 0, 1);$$

this implies $b^k_{ijl} = 0$. Let

$$v_{(1)} = {}^t(v^1_{(1)}, \ldots, v^n_{(1)}), \ldots, v_{(n)} = {}^t(v^1_{(n)}, \ldots, v^n_{(n)})$$

be n solutions which together with $e_{(1)}, \ldots, e_{(n)}$ form a basis of the solutions. We have

$$\frac{\partial v^k_{(l)}}{\partial x^{ij}} = \sum_p a^k_{ijp}\, v^p_{(l):11}, \quad 1 \leq k, l \leq n$$

so that the jacobian of the Schwarz map is given by

$$\det\left(\frac{\partial v^k_{(l)}}{\partial x^{ij}}\right)_{(i,j),(k,l)} = W(\det v^p_{(l):11})^n.$$

Since a fundamental solution of the corresponding matrix system $dU = \Omega U$ can be given by

$$\begin{pmatrix} e_{(1):11} & \cdots & e_{(n):11} & v_{(1):11} & \cdots & v_{(n):11} \\ e_{(1)} & \cdots & e_{(n)} & v_{(1)} & \cdots & v_{(n)} \end{pmatrix} = \begin{pmatrix} 0 & v_{:11} \\ I_n & v \end{pmatrix},$$

where $v = (v_{(1)}, \dots, v_{(n)})$, we conclude that $\det v_{:11} \neq 0$. Thus the jacobian vanishes if and only if W does; this completes the proof of Proposition 1.

§3. The model equation

Let us consider a system of linear homogeneous differential equations in n^2 independent variables x^{ij} with n unknowns u^k

$$(\underline{E}) \begin{cases} u^k_{:kk:kk} = \sum_l \underline{\alpha}^k_l\, u^l_{:ll} + \sum_l \underline{\beta}^k_l\, u^l, \\ u^k_{:ij} \quad\ = \sum_l \underline{a}^k_{ijl}\, u^l_{:ll} + \sum_l \underline{b}^k_{ijl}\, u^l, \quad 1 \le i, j, k \le n. \end{cases}$$

3.1. (E) versus $(\underline{E})$

Here we compare the coefficients of the two expressions (E) and $(\underline{E})$. The equations

$$u^k_{:kk} = \sum_l a^k_{kkl}\, u^l_{:11}, \quad u^k_{:11} = \sum_l \underline{a}^k_{11l}\, u^l_{:ll} \mod (u^1, \dots, u^n)$$

lead to

Proposition 2. *(a^k_{ijl}) and $(\underline{a}^k_{ijl})$ as well as (a^k_{ijl}, b^k_{ijl}) and $(\underline{a}^k_{ijl}, \underline{b}^k_{ijl})$ are birationally related. $(\underline{\alpha}^k_l, \underline{\beta}^k_l)$ can be expressed as rational functions of (a, b, α, β) and their derivatives, and vice versa. The denominators are $\det(a^k_{kkl})_{k,l}$ and $\det(\underline{a}^k_{11l})_{k,l}$, respectively.*

3.2. The model equation

The system with a fundamental set of solutions

$$\begin{pmatrix} u^1 \\ \vdots \\ u^n \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}, \begin{pmatrix} x^{11} \\ \vdots \\ x^{n1} \end{pmatrix}, \dots, \begin{pmatrix} x^{1n} \\ \vdots \\ x^{nn} \end{pmatrix},$$

can be written as

$$\begin{cases} u^k_{:kk:kk} = 0, \\ u^k_{:ij} = \delta^k_i u^j_{:jj}, \quad 1 \leq i, j, k \leq n. \end{cases}$$

This system is the model equation of the system $(\underline{E})$ above, which means that every system of the form $(\underline{E})$ satisfying

$$\det(\underline{a}^k_{ijl})_{(i,j),(k,l)} \neq 0$$

is equivalent to the model system through a transformation K and a coordinate change. In fact, if we take K the $n \times n$ matrix consisting of n linearly independent solutions, and transform the system, then the new system has linearly independent solutions $e_{(1)}, \ldots, e_{(n)}$ and, say, $v_{(1)}, \ldots, v_{(n)}$. Now we have only to change the coordinates as $x^{ij} \to v^i_{(j)}$.

§4. The transformation formula under changes of unknowns

4.1. A set of essential coefficients

We assumed that the rank of the system E is $2n$; this implies that the coefficients of the system must satisfy the so-called *integrability condition*. We analyze this condition and see that the coefficients a^k_{ij} almost determine the remaining ones. Thanks to Lemma 1, we may assume $\mathcal{A} = (\alpha^k_l) = 0$. Note that we still have a freedom of transformations K satisfying $K_{:11} = 0$.

Define 1-forms as

$$a^k_p = \sum_{i,j} a^k_{ijp}\, dx^{ij}, \quad b^k_p = \sum_{i,j} b^k_{ijp}\, dx^{ij}.$$

Then we have

$$\begin{aligned} du^k &= \sum a^k_p\, u^p_{:11} + \sum b^k_p\, u^p, \\ du^k_{:11} &= \sum a^k_{p:11}\, u^p_{:11} + \sum a^k_p\, u^p_{:11:11} + \sum b^k_{p:11}\, u^p + \sum b^k_p\, u^p_{:11} \\ &= \sum (a^k_{p:11} + b^k_p) u^p_{:11} + \sum (a^k_l\, \beta^l_p + b^k_{p:11}) u^p. \end{aligned}$$

Thus the matrix 1-form Ω defined by $dU = \Omega U$ can be expressed as

$$\Omega = \begin{pmatrix} a^k_{p:11} + b^k_p & \sum a^k_l\, \beta^l_p + b^k_{p:11} \\ a^k_p & b^k_p \end{pmatrix}.$$

The integrability condition is given by

$$d\Omega = \Omega \wedge \Omega;$$

let us check its entries: left-bottom, left-top and right-bottom.

The left-bottom reads

$$da^k_p = \sum a^k_q \wedge a^q_{p:11} + \sum a^k_q \wedge b^q_p + \sum b^k_q \wedge a^q_p,$$

which implies

$$\sum a^k_q \wedge b^q_p + \sum b^k_q \wedge a^q_p = c^k_p \quad (:= da^k_p - \sum a^k_q \wedge a^q_{p:11}).$$

To show the above, we need the following lemma which will be proved later.

Lemma 2. *Let*

$$A^k_p = \sum A^k_{ijp}\, dx^{ij} \quad (k, p = 1, \dots, n)$$

be 1*-forms in variables* x^{ij} $(i, j = 1, \dots, n)$ *satisfying* $\det(A^k_{ijp})_{(i,j),(k,p)} \neq 0$, *and*

$$C^k_p \quad (k, p = 1, \dots, n)$$

2*-forms satisfying* $\sum C^k_k = 0$. *Then equations*

$$E^k_p : A^k_q \wedge B^q_p + B^k_q \wedge A^q_p = C^k_p, \quad k, p = 1, \dots, n$$

for the unknown 1*-forms* B^q_p *in* x *determine*

$$B^k_q \ (k \neq q) \quad \textit{and} \quad B^k_k - B^p_p.$$

That is, they can be expressed in terms of A *and* C.

The left-top reads

$$da^k_{p:11} + db^k_p = \sum (a^k_{q:11} + b^k_q) \wedge (a^q_{p:11} + b^q_p) + \sum (a^k_l \beta^l_q + b^k_{q:11}) \wedge a^q_p;$$

in particular, their coefficients of $dx^{ij} \wedge dx^{11}$ imply

$$-a^k_{ijp:11:11} - b^k_{ijp:11} = \sum a^k_{ijl}\, \beta^l_p + b^k_{ijp:11} - \sum \beta^k_q\, a^q_{ijp}.$$

When $k \neq p$, since b^k_p are already expressed in terms of a (and their derivatives), these identities can be regarded as equations for β^k_p. A scalar version of the lemma above says that

$$\beta^k_p \ \ (k \neq p) \quad \text{and} \quad \beta^k_k - \beta^p_p$$

can be expressed in terms of a.

When $k = p$, since

$$\sum a^k_{ijl}\,\beta^l_p - \sum \beta^k_q\,a^q_{ijp} = \sum_{l \neq k} a^k_{ijl}\,\beta^l_p - \sum_{q \neq k} \beta^k_q\,a^q_{ijp},$$

(so β^k_k do not appear,) the identities above give expressions of $2b^k_{k:11}$ in terms of a.

The right-bottom reads

$$db^k_p = \sum a^k_q \wedge (a^q_l \beta^l_p + b^q_{p:11}) + \sum b^k_q \wedge b^q_p.$$

When $k \neq p$, since

$$\sum b^k_q \wedge b^q_p = \sum_{q \neq k,p} b^k_q \wedge b^q_p + b^k_p \wedge (b^p_p - b^k_k),$$

the equality determines β^p_p, if $a^k_q \wedge a^q_p \neq 0$ for some k. Note that, since $\beta^k_k - \beta^p_p$ are expressed in terms of a, this condition is equivalent to $\mathbf{A} \wedge \mathbf{A} \neq 0$, where $\mathbf{A} = (a^k_l)$ is the matrix of essential coefficients.

When $k = p$, since the right hand-side is already determined, this gives an expression of db^k_k.

In this way, the coefficients

$$b^k_p \ \ (k \neq p), \ \ b^k_k - b^p_p, \ \ \beta^l_p \ \ (l \neq p), \ \ b^k_{k:11}, \ \ \beta^k_k, \ \ db^k_k$$

are determined, that is, expressed in terms of a, in this order. Thus we get

PROPOSITION 3. *Under the assumptions* $\det(a^k_{ijp})_{(i,j),(k,p)} \neq 0$ *and* $\mathbf{A} \wedge \mathbf{A} \neq 0$, *where* $A = (a^k_l)$, $a^k_l = \sum a^k_{ijl}\,dx^{ij}$, *the coefficients a determine the other coefficients b and β up to adding an exact 1-form $dk(x)$, where $k(x)$ is independent of x^{11}, to b^k_k $(k = 1, \dots, n)$. This ambiguity is caused by the scalar transformation $K = k(x)I_n$.*

Hence we have the following main theorem.

Theorem 1. *Two systems* $E_n(a,b,\alpha,\beta)$ *and* $E_n(\bar{a},\bar{b},\bar{\alpha},\bar{\beta})$ *are equivalent if only if there is an invertible* $n\times n$ *matrix* K *such that*

$$\bar{\mathbf{A}} = K^{-1}\mathbf{A}K$$

provided that $\det(a^k_{ijp})_{(i,j),(k,p)} \neq 0$ *and* $\mathbf{A}\wedge\mathbf{A}\neq 0$. *Here* $\mathbf{A}$ *and* $\bar{\mathbf{A}}$ *are the matrices of the essential coefficients of the systems* $E_n(a,b,\alpha,\beta)$ *and* $E_n(\bar{a},\bar{b},\bar{\alpha},\bar{\beta})$, *respectively.*

4.2. Sketch of the proof of Lemma 2

Note that

$$\bigwedge_{k,p} A^k_p = \det(A^k_{ijp})_{(i,j),(k,p)}\,dx, \quad dx = \bigwedge_{i,j} dx^{ij}.$$

For each unknown 1-form $B := B^k_p$, we derive from the equations E^k_p in Lemma 2

$$X : \Big(\bigwedge_{(p,q)\neq(i,j)} A^p_q\Big)\wedge B = X_{ij}\,dx,$$

for every i,j, where X_{ij} is a function expressible in terms of A and C. These will determine B.

Let us work on the unknown 1-form B^k_p $(k\neq p)$. The equations E^k_k, E^p_p, E^k_p, and E^p_k read

$$\begin{aligned}
E^k_k &: \sum_{q\neq k,p}(A^k_q\wedge B^q_k - A^q_k\wedge B^k_q) + A^k_p\wedge B^p_k - A^p_k\wedge B^k_p &&= C^k_k,\\
E^p_p &: \sum_{q\neq p,k}(A^p_q\wedge B^q_p - A^q_p\wedge B^p_q) + A^p_k\wedge B^k_p - A^k_p\wedge B^p_k &&= C^p_p,\\
E^k_p &: \sum_{q\neq k,p}(A^k_q\wedge B^q_p - A^q_p\wedge B^k_q) + (A^k_k - A^p_p)\wedge B^k_p + A^k_p\wedge(B^p_p - B^k_k) &&= C^k_p.
\end{aligned}$$

We multiply some 1-forms A^*_* to each equation to kill terms containing B^*_* except the multiple of the B^k_p, and we get the equation of the form

$$F : \Big(\bigwedge A^*_*\Big)\wedge B^k_p = \text{a form expressed in terms of } A \text{ and } C.$$

The coefficients of B^k_p in the three equations thus obtained, call them F^k_k, F^p_p, and F^k_p, have the unique factor A^k_p in common. To get such an equation that the coefficients of B^k_p does not have A^k_p as a factor, we make use of the equation

$$E^p_k : \sum_{q\neq p,k} (A^p_q \wedge B^q_k - A^q_k \wedge B^p_q) + (A^p_p - A^k_k) \wedge B^p_k + A^p_k \wedge (B^k_k - B^p_p) = C^p_k,$$

which does not contain the term B^k_p. To eliminate the last terms in the left hand-sides of E^k_p and E^p_k, we form $A^p_k E^k_p - A^k_p E^p_k$:

$$\begin{aligned} A^p_k \wedge \sum_{q\neq k,p} (A^k_q \wedge B^q_p - A^q_p \wedge B^k_q) - A^k_p \wedge \sum_{q\neq k,p} (A^p_q \wedge B^q_k - A^q_k \wedge B^p_q) \\ + A^p_k \wedge (A^k_k - A^p_p) \wedge B^k_p - A^k_p \wedge (A^p_p - A^k_k) \wedge B^p_k \\ = A^p_k \wedge C^k_p - A^k_p \wedge C^p_k. \end{aligned}$$

To eliminate the last term of the left hand-side of this equation, we add $(A^p_p - A^k_k) \wedge E^p_p$ and get

$$\begin{aligned} (A^p_p - A^k_k) \wedge \sum_{q\neq k,p} (A^p_q \wedge B^q_p - A^q_p \wedge B^p_q) + A^p_k \wedge \sum_{q\neq k,p} (A^k_q \wedge B^q_p - A^q_p \wedge B^k_q) \\ - A^k_p \wedge \sum_{q\neq k,p} (A^p_q \wedge B^q_k - A^q_k \wedge B^p_q) + A^p_k \wedge (A^k_k - A^p_p) \wedge B^k_p \\ = (A^p_p - A^k_k) \wedge C^p_p + A^p_k \wedge C^k_p - A^k_p \wedge C^p_k. \end{aligned}$$

We multiply some 1-forms A^*_* to this equation to kill terms containing B^*_* except the multiple of the B^k_p and we get the equation of the form F. The coefficient of B^k_p in this equation and those of F^k_k, F^p_p, and F^k_p have no factor in common. Thus by multiplying some 1-forms A^*_* to these four equations, we can get a system of the form X.

§5. Coordinate changes

Let us consider a coordinate transformation from $x = (x^{ij})$ to $y = (y^{ij})$. Put $\bar{u}^k(y) = u^k(x(y))$, and

$$a^k_l = \sum a^k_{ijl}\, dx^{ij}, \quad \bar{a}^k_l = \sum \bar{a}^k_{ijl}\, dy^{ij}.$$

Then the equations of the first order of (E) can be written as

$$\sum a^k_l u^l_{:11} = du^k = \sum \bar{a}^k_l \bar{u}^l_{:11}.$$

Substituting

$$\bar{u}^l_{:11} = \sum_{i,j} u^l_{:ij} \frac{\partial x^{ij}}{\partial y^{11}} = \sum_{i,j,p} a^l_{ijp} u^p_{11} \frac{\partial x^{ij}}{\partial y^{11}} \quad \bmod (u^1, \dots, u^n)$$

into the above identity, we have

$$\sum_l a^k_l u^l_{:11} = \sum_{p,l} \sum_{i,j} \bar{a}^k_p a^p_{ijl} \frac{\partial x^{ij}}{\partial y^{11}} u^l_{:11}.$$

Thus we get the following theorem giving the transformation formula.

THEOREM 2. *Let* $\mathbf{A}$ *be the matrix of the essential coefficients of a system* (E) *in* x*-coordinates. If* $\bar{\mathbf{A}}$ *denotes the matrix in* y*-coordinates, then they are related as*

$$\mathbf{A} = \bar{\mathbf{A}} L, \quad \text{where} \quad L = \left(\sum_{i,j} a^p_{ijl} \frac{\partial x^{ij}}{\partial y^{11}} \right)_{p,l}.$$

§6. Conformal structures through the Plčker embeddings

From now on up to the end of this paper, assume $n = 2$.

6.1. Plücker embedding

When $n = 2$, the target space of the Schwarz map $\mathcal{S}$ is the Grassmannian $\mathrm{Gr}(2,4)$, which can be embedded (the so-called Plücker embedding) into the 5-dimensional projective space as a quadratic hypersurface. The pull-back of the natural conformal structure on the quadratic hypersurface defines a conformal structure on the source space, the $x = (x^{ij})$-space. In this section, we see how this conformal structure on the x-space can be expressed in terms of the coefficients a of the system.

We work on the system $(\underline{E})$, and change notation as follows: The unknowns u^1 and u^2 are denoted by u and v, and the variables are

$$x^1 = x^{11},\ x^2 = x^{12},\ x^3 = x^{21},\ x^4 = x^{22}.$$

We in this section omit colons in differentiation. Thus we write the system as

$$(\underline{E}) \begin{cases} u_{11} = Au_1 + Bv_4 + Cu + Ev, & \\ u_k \ \ = a_k u_1 + b_k v_4 + c_k u + e_k v, & k = 1, \dots, 4 \\ v_k \ \ = p_k u_1 + q_k v_4 + r_k u + s_k v, & k = 1, \dots, 4 \\ v_{44} = Pu_1 + Qv_4 + Ru + Sv, & \end{cases}$$

where

$$a_1 = 1,\ b_1 = 0,\ c_1 = 0,\ e_1 = 0,\ p_4 = 0,\ q_4 = 1,\ r_4 = 0,\ s_4 = 0.$$

The determinant W is now equal to the determinant of the matrix

$$\begin{pmatrix} a_1 & q_1 & p_1 & b_1 \\ a_2 & q_2 & p_2 & b_2 \\ a_3 & q_3 & p_3 & b_3 \\ a_4 & q_4 & p_4 & b_4 \end{pmatrix}.$$

Given a fundamental set of solutions

$$\begin{pmatrix} u^1 & u^2 & u^3 & u^4 \\ v^1 & v^2 & v^3 & v^4 \end{pmatrix},$$

define two vectors $u = [u^1, u^2, u^3, u^4]$ and $v = [v^1, v^2, v^3, v^4]$ and put

$$f = u \wedge v$$

which takes values in $\mathbf{P}^5$. Derivatives of f can be written as linear combinations of six vectors $u \wedge v$, $u_1 \wedge v$, $u \wedge v_4$, $u \wedge u_1$, $v_4 \wedge v$, and $u_1 \wedge v_4$. The coefficients are listed below:

	$u \wedge v$	$u_1 \wedge v$	$u \wedge v_4$	$u \wedge u_1$	$v_4 \wedge v$	$u_1 \wedge v_4$
f	1	0	0	0	0	0
f_1	s_1	1	q_1	p_1	0	0
f_2	$c_2 + s_2$	a_2	q_2	p_2	b_2	0
f_3	$c_3 + s_3$	a_3	q_3	p_3	b_3	0
f_4	c_4	a_4	1	0	b_4	0
f_{14}	σ_1	σ_2	σ_3	σ_4	σ_5	$\sigma_6,$

where

$$\begin{aligned}
\sigma_1 &= U_3 + p_1 U_4 + q_1 S + s_1 c_4 + s_{14}, \\
\sigma_2 &= U_1 + s_1 a_4, \\
\sigma_3 &= p_1 U_2 + q_1 Q + q_1 c_4 + q_{14} + s_1, \\
\sigma_4 &= p_{14} + p_1 c_4 + p_1 U_1 + q_1 P, \\
\sigma_5 &= U_2 - q_1 d_4 + s_1 b_4, \\
\sigma_6 &= 1 - p_1 b_4 + q_1 a_4;
\end{aligned}$$

As usual, the subindex denotes the differentiation relative to x: $f_1 = \partial f/\partial x^1$, $s_{14} = \partial s_1/\partial x^4$, and so on. The list above implies that the vectors f, f_1, f_2, f_3, f_4, and f_{14} can be a basis if and only if

$$(1 - p_1 b_4 + q_1 a_4) W \neq 0.$$

Under this condition, the second derivatives f_{ij} can be expressed as linear combinations of f_k and f:

$$(CE) : f_{ij} = \mathcal{C}_{ij} f_{14} + \sum_k P^k_{ij} f_k + P_{ij} f.$$

Then, the matrix $\mathcal{C} = (\mathcal{C}_{ij})$ represents the conformal tensor induced by the embedding f [3]. We know that the associated metric $\sum \mathcal{C}_{ij}\, dx^i dx^j$ is conformally flat because the image of the Plücker embedding f is in a quadratic hypersurface.

A computation shows the following expression of the matrix $\mathcal{C} =$

$$\begin{pmatrix} 2q_1 & q_2 - p_1 b_2 + q_1 a_2 & q_3 - p_1 b_3 + q_1 a_3 & 1 - p_1 b_4 + q_1 a_4 \\ q_2 - p_1 b_2 + q_1 a_2 & 2(a_2 q_2 - b_2 p_2) & a_2 q_3 + a_3 q_2 - b_2 p_3 - b_3 p_2 & a_2 - p_2 b_4 + q_2 a_4 \\ q_3 - p_1 b_3 + q_1 a_3 & a_2 q_3 + a_3 q_2 - b_2 p_3 - b_3 p_2 & 2(a_3 q_3 - b_3 p_3) & a_3 - p_3 b_4 + q_3 a_4 \\ 1 - p_1 b_4 + q_1 a_4 & a_2 - p_2 b_4 + q_2 a_4 & a_3 - p_3 b_4 + q_3 a_4 & 2a_4 \end{pmatrix}.$$

Note that (ij) component is equal to $a_i q_j + a_j q_i - b_i p_j - b_j p_i$ where $a_1 = 1$, $q_4 = 1$, $p_4 = 0$, and $b_1 = 0$. We can see that $\det \mathcal{C} = W^2$.

Remark 1. For the model system we have

$$\mathcal{C} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

6.2. Invariant differential forms

Using the convention

$$a_1 = 1,\ b_1 = 0,\ c_1 = 0,\ d_1 = 0,\ p_4 = 0,\ q_4 = 1,\ r_4 = 0,\ s_4 = 0,$$

as before, we define 1-forms as

$$\begin{aligned} a &= a_1\, dx^1 + a_2\, dx^2 + a_3\, dx^3 + a_4\, dx^4, \\ &\ \vdots \\ s &= s_1\, dx^1 + s_2\, dx^2 + s_3\, dx^3 + s_4\, dx^4. \end{aligned}$$

We occasionally use the matrices ω and θ defined as

$$\omega = \begin{pmatrix} a & b \\ p & q \end{pmatrix}, \quad \theta = \begin{pmatrix} c & e \\ r & s \end{pmatrix}.$$

With this notation, the equations of the first order of (E) can be written as

$$\begin{aligned} du &= au_1 + bv_4 + cu + ev, \\ dv &= pu_1 + qv_4 + ru + sv. \end{aligned}$$

When u and v denote fundamental vectors of solutions, we have

$$\begin{aligned} du \wedge dv = (a \cdot q - b \cdot p)u_1 \wedge v_4 &+ (a \cdot r - c \cdot p)u_1 \wedge u \\ &+ (a \cdot s - q \cdot c)u_1 \wedge v + (b \cdot r - p \cdot e)v_4 \wedge u \\ &+ (b \cdot s - q \cdot e)v_4 \wedge v + (c \cdot s - e \cdot r)u \wedge v, \end{aligned}$$

where the dot product $\cdot$ means the symmetric product of 1-forms. By definition, the conformal structure is equal to $a \cdot q - b \cdot p$.

We will check the covariance of the forms above relative to linear change of unknowns and to coordinate change. First, consider a transformation K of the unknown (u, v) to (U, V) by

$$U = k_1 u + k_2 v, \quad V = k_3 u + k_4 v.$$

Since

$$\begin{aligned} U_1 &= (k_2 r_1 + k_{11})u + (k_2 s_1 + k_{21})v + (k_1 + k_2 p_1)u_1 + k_2 q_1 v_4, \\ V_4 &= (k_3 c_4 + k_{34})u + (k_3 d_4 + k_{44})v + k_3 a_4 u_1 + (k_3 b_4 + k_4)v_4, \end{aligned}$$

we have the formula of change of the frames as ${}^t(U, V, U_1, V_4) = k\ {}^t(u, v, u_1, v_4)$;

$$k = \begin{pmatrix} k_1 & k_2 & 0 & 0 \\ k_3 & k_4 & 0 & 0 \\ k_2 r_1 + k_{11} & k_2 s_1 + k_{21} & k_1 + k_2 p_1 & k_2 q_1 \\ k_3 c_4 + k_{34} & k_3 d_4 + k_{44} & k_3 a_4 & k_3 b_4 + k_4 \end{pmatrix} =: \begin{pmatrix} K & 0 \\ M & L \end{pmatrix},$$

where K, L, and M are 2×2 matrices. From this expression, the two conditions

$$\det K \neq 0 \quad \text{and} \quad \delta := \det L = (k_1 + k_2 p_1)(k_4 + k_3 b_4) - k_2 k_3 a_4 q_1 \neq 0$$

are necessary for the new system relative to (U, V) to be written in the same form as for (u, v), which we assume in the following. Now, introducing the notation Ω and Θ for U and V in place of ω and θ, we have

$$\Theta = dK \cdot K^{-1} + K(\theta - \omega L^{-1} M)K^{-1}, \quad \Omega = K\omega L^{-1}.$$

From this identity, by writing the equations of the first order relative to (U, V) as

$$\begin{aligned} U_i &= A_i U_1 + B_i V_4 + C_i U + D_i V, \\ V_j &= P_j U_1 + Q_j V_4 + R_j U + S_j V, \end{aligned}$$

we have the following formulas:

$$\begin{aligned}
A_1 &= 1, \\
A_2 &= (k_1k_4a_2 + k_1k_3(a_2b_4 - a_4b_2) + k_2k_3(p_2b_4 - q_2a_4) + k_2k_4p_2)/\delta, \\
A_3 &= (k_1k_4a_3 + k_1k_3(a_3b_4 - a_4b_3) + k_2k_3(p_3b_4 - q_3a_4) + k_2k_4p_3)/\delta, \\
A_4 &= a_4(k_1k_4 - k_2k_3)/\delta, \\
B_1 &= 0, \\
B_2 &= (k_1^2b_2 + k_1k_2(q_2 + b_2p_1 - a_2q_1) + k_2^2(q_2p_1 - q_1p_2))/\delta, \\
B_3 &= (k_1^2b_3 + k_1k_2(q_3 + b_3p_1 - a_3q_1) + k_2^2(q_3p_1 - q_1p_3))/\delta, \\
B_4 &= (k_1^2b_4 + k_1k_2(1 - a_4q_1 + b_4p_1) + k_2^2p_1)/\delta, \\
P_1 &= (k_3^2b_4 + k_3k_4(1 - a_4q_1 + b_4p_1) + k_4^2p_1)/\delta, \\
P_2 &= (k_3^2(a_2b_4 - a_4b_2) + k_3k_4(a_2 + p_2b_4 - q_2a_4) + k_4^2p_2)/\delta, \\
P_3 &= (k_3^2(a_3b_4 - a_4b_3) + k_3k_4(a_3 + p_3b_4 - q_3a_4) + k_4^2p_3)/\delta, \\
P_4 &= 0, \\
Q_1 &= q_1(k_1k_4 - k_2k_3)/\delta, \\
Q_2 &= (k_1k_3b_2 + k_2k_3(b_2p_1 - a_2q_1) + k_2k_4(q_2p_1 - q_1p_2) + k_1k_4q_2)/\delta, \\
Q_3 &= (k_1k_3b_3 + k_2k_3(b_3p_1 - a_3q_1) + k_2k_4(q_3p_1 - q_1p_3) + k_1k_4q_3)/\delta, \\
Q_4 &= 1.
\end{aligned}$$

Second, the formulas similar to those in Section 6 relative to a coordinate transformation from $x = (x^1, x^2, x^3, x^4)$ to $y = (y^1, y^2, y^3, y^4)$ is given as follows. Denote by $(y_i^k) = (\partial y^k / \partial x^i)$ the Jacobian matrix and put $\overline{u}(y) =$

$u(x(y))$ and $\overline{v}(y) = v(x(y))$. By a simple calculation, we have

$$\begin{aligned} \frac{\partial \overline{u}}{\partial y^1} &= \frac{c}{dy^1}u + \frac{e}{dy^1}v + \frac{a}{dy^1}u_1 + \frac{b}{dy^1}v_4, \\ \frac{\partial \overline{v}}{\partial y^4} &= \frac{r}{dy^4}u + \frac{s}{dy^4}v + \frac{p}{dy^4}u_1 + \frac{q}{dy^4}v_4, \end{aligned}$$

where we use the notation

$$\frac{\tau}{dy^i} = c_1\frac{\partial x^1}{\partial y^i} + c_2\frac{\partial x^2}{\partial y^i} + c_3\frac{\partial x^3}{\partial y^i} + c_4\frac{\partial x^4}{\partial y^i},$$

for 1-form $\tau = c_1\,dx^1 + c_2\,dx^2 + c_3\,dx^3 + c_4\,dx^4$. Then, the change of frame is written as ${}^t(\overline{u}, \overline{v}, \overline{u}_1, \overline{v}_4) = g\;{}^t(u, v, u_1, v_4)$, where

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ c/dy^1 & e/dy^1 & a/dy^1 & b/dy^1 \\ r/dy^4 & s/dy^4 & p/dy^4 & q/dy^4 \end{pmatrix} = \begin{pmatrix} I & 0 \\ B & A \end{pmatrix}.$$

Letting Ω and Θ denote the matrix 1-forms ω and θ for the coordinate system (y^1, y^2, y^3, y^4), we have

$$\Theta = \theta - \omega A^{-1}B, \quad \Omega = \omega A^{-1}.$$

6.3. How to get a, b, p, and q from $\mathcal{C}_{ij}$

Recall the transformation formulas

$$\begin{aligned} B_4 &= (k_1^2 b_4 + k_1 k_2(1 - a_4 q_1 + b_4 p_1) + k_2^2 p_1)/\delta, \\ P_1 &= (k_3^2 b_4 + k_3 k_4(1 - a_4 q_1 + b_4 p_1) + k_4^2 p_1)/\delta \end{aligned}$$

of the coefficients under K in the previous subsection. We see that if $\delta = \det L \neq 0$ and

$$\mathrm{disc} := (1 - a_4 q_1 + b_4 p_1)^2 - 4 p_1 b_4 \neq 0,$$

then, by solving the quadratic equation in $k_1, \dots, k_4$, we have K such that $\det K \neq 0$ and $B_4 = P_1 = 0$. Note that we still have a transformation K of diagonal form.

Assuming $p_1 = b_4 = 0$, our problem is to solve the following system:

$$2q_1 = \mathcal{C}_{11}, \qquad q_2 + q_1 a_2 = \mathcal{C}_{21}, \qquad 2(a_2 q_2 - b_2 p_2) = \mathcal{C}_{22},$$

$$\begin{aligned} &q_3 + q_1 a_3 = \mathcal{C}_{31}, \quad a_2 q_3 + a_3 q_2 - b_2 p_3 - b_3 p_2 = \mathcal{C}_{32}, \\ &2(a_3 q_3 - b_3 p_3) = \mathcal{C}_{33}, \quad 1 + q_1 a_4 = \mathcal{C}_{41}, \quad a_2 + q_2 a_4 = \mathcal{C}_{42}, \\ &a_3 + q_3 a_4 = \mathcal{C}_{43}, \quad 2a_4 = \mathcal{C}_{44}. \end{aligned}$$

Let us normalize the conformal tensor $\mathcal{C}_{ij}$ so that $\mathcal{C}_{41} = 1 + \mathcal{C}_{11}\mathcal{C}_{44}/4$ holds; we multiply the tensor by α satisfying the quadratic equation: $\alpha\mathcal{C}_{41} = 1 + \alpha^2\mathcal{C}_{11}\mathcal{C}_{44}/4$. Then we have

$$q_1 = \mathcal{C}_{11}/2, \quad a_4 = \mathcal{C}_{44}/2.$$

The linear equations in a_2, a_3, q_2, q_3:

$$\begin{array}{cccccl} q_1 a_2 & & + q_2 & & = & \mathcal{C}_{21}, \\ & q_1 a_3 & & + q_3 & = & \mathcal{C}_{31}, \\ a_2 & & + a_4 q_2 & & = & \mathcal{C}_{42}, \\ & a_3 & & + a_4 q_3 & = & \mathcal{C}_{43} \end{array}$$

are solved as

$$a_j = (a_4\mathcal{C}_{j1} - \mathcal{C}_{4j})/(q_1 a_4 - 1), \quad q_j = (q_1\mathcal{C}_{4j} - \mathcal{C}_{j1})/(q_1 a_4 - 1), \quad j = 2, 3.$$

Now it remains to solve the quadratic system

$$\begin{aligned} b_2 p_2 \quad &= x := a_2 q_2 - \mathcal{C}_{22}/2, \\ b_3 p_3 \quad &= y := a_3 q_3 - \mathcal{C}_{33}/2, \\ b_2 p_3 + b_3 p_2 &= z := a_2 q_3 + a_3 q_2 - \mathcal{C}_{32}. \end{aligned}$$

We have

$$p_2 = x/b_2, \quad p_3 = y/b_3,$$

and the ratio $\beta := b_2/b_3$ is determined by the quadratic equation

$$y\beta^2 - z\beta + x = 0,$$

of which discriminant can be checked to be a constant times of $\det(\mathcal{C}_{ij})$. Recall that the transformation $k = \mathrm{diag}(k_1, k_4)$ takes b_3 to $b_3 k_4/k_1$; it means that we can normalize $b_3 = 1$.

PROPOSITION 4. *Assume $\delta \neq 0$ and* disc $\neq 0$. *Then the coefficients a, b, p, and q can be derived from $\mathcal{C}_{ij}$ by solving two quadratic equations.*

Remark 2. The uniformizing equation of a 4-dimensional orbifold is obtained in [1]. This equation is given in the form (CE). Thus Proposition 4 gives a method to rewrite it into a system in the form $(\underline{E})$

§7. Families of line congruences defined by (E)

We discuss the relation between our system (E) and a differential geometric object known by the name of *line congruences.*

7.1. A geometric interpretation and a normalization of the system

We give a geometric interpretation to the system written in terms of (u, v, u_1, v_4) as follows. Let u and v be vectors defined by a fundamental set of solutions as in 6.1; then the pair u and v determines a line that combines these points and, by fixing x^2 and x^3, we have a 2-parameter family of lines parameterized by x^1 and x^4, which is usually called a line congruence. Thus, the system we are considering is geometrically a 2-parameter family of line congruences $\mathcal{LC} = \mathcal{LC}(x^2, x^3)$ depending on x^2 and x^3. Each line congruence is described by the subsystem

$$\begin{aligned} u_{11} &= Au_1 + Bv_4 + Cu + Dv, \\ u_4 &= a_4u_1 + b_4v_4 + c_4u + d_4v, \\ v_1 &= p_1u_1 + q_1v_4 + r_1u + s_1v, \\ v_{44} &= Pu_1 + Qv_4 + Ru + Sv; \end{aligned}$$

the remaining equations describe the dependence of the family on x^2 and x^3.

Generally, a line congruence is better understood as a congruence of lines connecting two focal surfaces, which we now explain. Consider a curve $\mathcal{I} : t \to (x^1(t), x^4(t))$ in the parameter space and the corresponding ruled surface $\mathcal{LC}|_{\mathcal{I}}$, the restriction of the congruence onto this curve. This ruled surface $\mathcal{LC}|_{\mathcal{I}}$ is developable only when $u \wedge v \wedge (du/dt) \wedge (dv/dt) = 0$ by definition. This condition is equivalent to

$$q_1\Big(\frac{dx^1}{dt}\Big)^2 + (1 + a_4q_1 - b_4p_1)\frac{dx^1}{dt}\frac{dx^4}{dt} + a_4\Big(\frac{dx^4}{dt}\Big)^2 = 0.$$

Hence, by assuming

$$(1 + a_4q_1 - b_4p_1)^2 - 4a_4q_1 \neq 0,$$

which coincides with the condition disc $\neq 0$ in 6.3, we have two directions at each point on the parameter space called the asymptotic directions, and so the two integral curves passing through the point. Let us consider one of the two integral curves and call it $\mathcal{I}$, and map this curve by u (we may take v instead, of course) then the ruled surface $\mathcal{LC}|_{\mathcal{I}}$ is developable along the image curve $u \circ \mathcal{I}$. By the way, since any developable ruled surface is generally obtained as a family of tangent lines of a certain curve, which is called the directrix curve, we can associate to each line the point where the line is tangent to the directrix curve. Thus, since there are two asymptotic directions at each point, we get two points on each line of the congruence. These two points generate two surfaces, called the focal surfaces. The condition above on coefficients, which we assume in the following, is necessary for the system to define the focal surfaces.

Now choose the coordinates x^1 and x^4 so that the coordinate lines are the integral curves above and let u and v be so chosen, by a linear change of the unknowns if necessary, that they generate the focal surfaces. Then, we must have the expressions

$$u_4 = c_4 u + d_4 v, \quad v_1 = r_1 u + s_1 v;$$

Namely, $a_4 = b_4 = p_1 = q_1 = 0$. Further, by multiplying some factors to u and v separately, we can normalize the system so that $c_4 = 0$ and $s_1 = 0$.

7.2. An example

We have seen that we can generally normalize the system so that

$$u_4 = d_4 v, \quad v_1 = r_1 u.$$

Assuming that $d_4 = 1$ and $r_1 = 1$, we give an example in this subsection.

We start with a seemingly simple system

$$u_4 = v, \quad v_1 = u, \quad u_{11} = v_4, \quad v_{44} = u_1.$$

The focal surface u is described by the induced system

$$u_{11} = u_{44}, \quad u_{14} = u,$$

which admits a fundamental system of solutions defined by

$$\{X = \exp(x^1 + x^4),\ Y = \exp(-x^1 - x^4),\\ Z = \cos(-x^1 + x^4),\ U = \sin(-x^1 + x^4)\}.$$

These solutions define a quadratic surface $XY = Z^2 + U^2$ in the projective space with homogeneous coordinates (X, Y, Z, U). The surface for v is seen to be also a quadratic surface defined by $-XY = Z^2 + U^2$. The induced conformal structure is $(dx^1)^2 + (dx^4)^2$ for both surfaces. See Figure 1 and Figure 2.

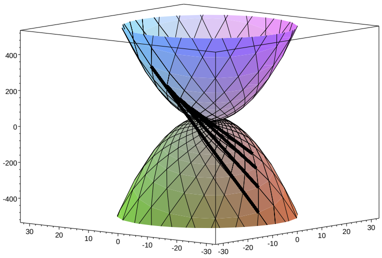

Figure 1

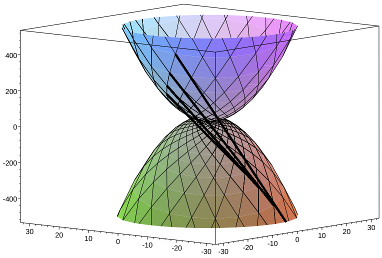

Figure 2

The upper surface in each figure represents the surface $XY = Z^2 + U^2$ and the lower surface represents the surface $-XY = Z^2 + U^2$. The curves drawn on the surfaces are x^1-curves and x^4-curves. The bold linesegments denote linesegments joining two points where the lines belonging to the line congruence are tangent to the focal surfaces. Those in Figure 1 are tangent to x^4-curves of the upper surface and those in Figure 2 are tangent to x^1-curves of the lower surface.

We next try to deform the system above by considering the system

$$\begin{aligned}
u_4 &= v, \qquad v_1 = u,\\
u_{11} &= v_4 + ku, \qquad v_{44} = u_1 - kv,\\
u_j &= a_j u_1 + b_j v_4 + c_j u + d_j v, \quad j = 2, 3,\\
v_j &= p_j u_1 + q_j v_4 + r_j u + s_j v, \quad j = 2, 3.
\end{aligned}$$

The integrability condition of this system has fortunately a fairly simple form, though we do not reproduce it here. Assuming that k is a constant not depending on any of the coordinates, we can see that the following set of coefficients solves the integrability condition.

$$\begin{aligned}
a_2 &= (aS + bC)Eh,\\
b_2 &= -(-2aSf_3 - 2bCf_3 - 2abSh + a^2Ch - b^2Ch)E/2,\\
c_2 &= -(a^2S - 2abC + b^2S)Eh/2\\
&\qquad + (bS - aC - a^3S - 3a^2bC + 3ab^2S + b^3C + kaS + kbC)Ef_3 + g_2,\\
d_2 &= -(-2abS + a^2C - b^2C)Ef_3\\
&\qquad + (a^3S + 3a^2bC - 3ab^2S - b^3C - 2bS + 2aC)hE/2,\\
a_3 &= 0,
\end{aligned}$$

$$\begin{aligned}
b_3 &= (aS+bC)Ef_2,\\
c_3 &= (aC-bS)Ef_2+g_3,\\
d_3 &= -(a^2C-2abS-b^2C)Ef_2,\\
p_2 &= (2aSf_3+2bCf_3-2abSh+a^2Ch-b^2Ch)E/2,\\
q_2 &= -(-bS+aC)hE,\\
r_2 &= -(a^2S+2aCb-Sb^2)Ef_3\\
&\qquad -(-3a^2bS+a^3C-3ab^2C+b^3S-2aS-2bC)hE/2,\\
s_2 &= (bS-aC)Ef_3+g_2+(-a^2S-2abC+b^2S)hE/2,\\
p_3 &= (aS+bC)Ef_2,\\
q_3 &= 0,\\
r_3 &= -(2abC+a^2S-b^2S)Ef_2,\\
s_3 &= (aC-bS+3a^2bC+a^3S-3ab^2S-b^3C-k(bC+aS))Ef_2+g_3,
\end{aligned}$$

where $h=h(x^2)$, $f=f(x^2,x^3)$ and $g=g(x^2,x^3)$ are arbitrary functions satisfying $f_{22}=f_{33}$; E, C, and S denote the functions $\exp(ax^1-bx^4)$, $\cos(bx^1+ax^4)$, and $\sin(bx^1+ax^4)$, respectively; a, b, and k are constant related as $b=-1/a$ and $k=a^2-1/a^2$. The system is nondegenerate.

When

$$k=0,\ a=1,\ f=((x^2)^2+(x^3)^2)/2,\ g=x^2x^3,\ h=1,$$

we see that any solution (u,v) has the form

$$u=PX+QY+RZ+TU,\quad v=PX-QY+TZ-RU,$$

where X, Y, Z, and U are given above, the coefficients are defined by

$$\begin{aligned}
P &= (4b_0(x^2)^2x^3+b_1(x^2+2x^2x^3)+b_2x^2+b_3)\varphi,\\
Q &= b_0\varphi,\\
R &= (2b_0x^2+b_1)\varphi,\\
T &= (b_0(4x^2x^3-2x^2)+b_2)\varphi,
\end{aligned}$$

and b_0, b_1, b_2, and b_3 are constants; φ denotes $\exp(x^2x^3)$. Hence, the following is a set of four independent solutions:

$$\begin{aligned}
u_0 &= (4(x^2)^2x^3X + Y + 2x^2Z + (4x^2x^3 - 2x^2)U)\varphi,\\
v_0 &= (4(x^2)^2x^3X - Y - 2x^2U + (4x^2x^3 - 2x^2)Z)\varphi,\\
u_1 &= ((x^2 + 2x^2x^3)X + Z)\varphi, \quad v_1 = ((x^2 + 2x^2x^3)X - U)\varphi,\\
u_2 &= (x^2X + U)\varphi, \quad v_2 = (x^2X + Z)\varphi,\\
u_3 &= \varphi X, \quad v_3 = \varphi X.
\end{aligned}$$

The surface defined by $u = [u_0, u_1, u_2, u_3]$ for each fixed x^2 and x^3 is a projective transformation of the quadratic surface $XY = Z^2 + U^2$:

$$\begin{pmatrix} u_0\\ u_1\\ u_2\\ u_3 \end{pmatrix} = \varphi \begin{pmatrix} 1 & 2x^2 & 4x^2x^3 - 2x^2 & 4(x^2)^2x^3\\ 0 & 1 & 0 & 2x^2x^3 + x^2\\ 0 & 0 & 1 & x^2\\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} Y\\ Z\\ U\\ X \end{pmatrix}.$$

The surface defined by $v = [v_0, v_1, v_2, v_3]$ is a projective transformation of the quadratic surface $-XY = Z^2 + U^2$:

$$\begin{pmatrix} v_0\\ v_1\\ v_2\\ v_3 \end{pmatrix} = \varphi \begin{pmatrix} -1 & 4x^2x^3 - 2x^2 & -2x^2 & 4(x^2)^2x^3\\ 0 & 0 & -1 & 2x^2x^3 + x^2\\ 0 & 1 & 0 & x^2\\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} Y\\ Z\\ U\\ X \end{pmatrix}.$$

Note that the initial line congruence when $x^2 = x^3 = 0$ is deformed so that the two focal surfaces are transformed by two different projective transformations.

Figure 3 and Figure 4 describe the congruence when $x^2 = 6$ and $x^3 = 1$. Figure 4 is the rotation of Figure 3 by 90 degrees.

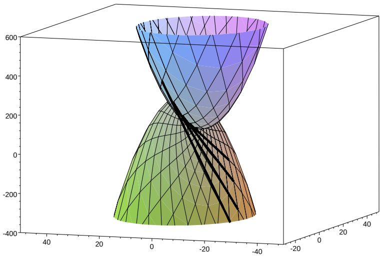

Figure 3

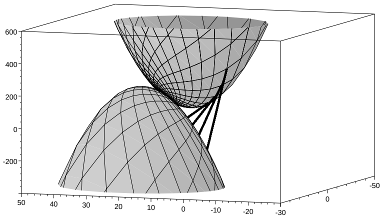

Figure 4

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