BULL. AUSTRAL. MATH. SOC. 34C11, 34D05
VOL. 40 (1989) [25-35]

DECAY ESTIMATES FOR SOME NONLINEAR SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

MITSUHIRO NAKAO

Precise decay estimates as $t \to \infty$ are derived for a class of nonlinear second order ordinary differential equations of the form

$$\frac{d}{dt}\{h\left(\frac{dx}{dt}\right)\} + g\left(t, \frac{dx}{dt}\right) + f(x) = 0 \text{ on } (0, \infty)$$

where h, g and f are functions like

$$h(u) = |u|^\alpha u, \; g(t, u) = (1+t)^\theta |u|^\beta u, \; f(u) = |u|^\gamma u$$

with $\alpha > -1$, $\beta > -1$ and $\gamma > -1$.

1. INTRODUCTION

In this paper we shall be concerned with the decay property of solutions of the ordinary differential equations

$$\frac{d}{dt}\{h\left(\frac{dx}{dt}\right)\} + g\left(t, \frac{dx}{dt}\right) + f(x) = 0 \text{ on } (0, \infty) \tag{1.1}$$

where h, g, f are continuous functions defined on $\mathbb{R}$ or $\mathbb{R}^+ \times \mathbb{R}$ ($\mathbb{R}^+ \equiv [0, \infty)$) satisfying specific conditions described below (see Section 2).

A typical example is

$$h(u) = |u|^\alpha u, \; g(t,u) = (1+t)^\theta |u|^\beta u, \; f(u) = |u|^\gamma u \tag{1.2}$$

for some $\alpha > -1$, $\beta > -1$ and $\gamma > -1$.

For the moment let us consider the case (1.2). As is easily seen, if $\alpha = \beta = \gamma = 0$ and $\theta = 0$ the solutions of (1.1) decay exponentially as $t \to \infty$. Moreover, if $\alpha = 0$, $\beta \geqslant 0$, $\gamma \geqslant 0$ and $-1 \leqslant \theta \leqslant \beta + 1$ we know the following result (see [**1**, **2**, **3**, **6**])

(i) If $\theta = -1$ or $\theta = \beta + 1$ and $0 < \beta < \gamma$, then

$$E(t) \equiv \frac{1}{2}|\dot{x}(t)|^2 + \frac{1}{\gamma+2}|x(t)|^{\gamma+2} \leqslant C(E(0))\{\log(2+t)\}^{-\nu}$$

Received 7 September 1988

https://doi.org/10.1017/S0004972700003464 Published online by Cambridge University Press

with $\nu = (\gamma+2)/(\beta\gamma+\beta+\gamma)$.

(ii) If $-1 < \theta < \beta + 1$ and $\beta + \gamma > 0$, then

$$E(t) \leqslant C(E(0))(1+t)^{-\nu}$$

with

$$\nu = (\gamma+2)\min\left(1+\theta, \frac{\beta+1-\theta}{\beta+1}\right)/(\beta\gamma+\beta+\gamma).$$

(iii) If $\theta < 1$ and $\beta = \gamma = 0$, then

$$E(t) \leqslant C(E(0))e^{-kt^{1-|\theta|}}$$

with some $k = k(E(0)) > 0$.

The object of this paper is to extend these results to a class of more general equations including the case (1.2) with $\alpha > -1$, $\beta > -1$, and $\gamma > -1$. As a particular case we shall show that if $-1 < \theta < \beta + 1$ and $\alpha > \beta > \gamma > -1$, the solutions of (1.1) decay much faster than exponentially, that is,

$$E(t) \equiv \frac{1}{\alpha+2}|\dot{x}(t)|^{\alpha+1} + \frac{1}{\gamma+2}|x(t)|^{\gamma+2} \leqslant C(E(0))e^{-ke^{\nu t}} \tag{1.3}$$

with some $k = k(E(0)) > 0$ and a certain $\nu > 0$.

An estimate like (1.3), which seems at a glance to be very curious, is already known for a semilinear wave equation with singular nonlinearities

$$u_{tt} - u_{xx} + |u|^{\alpha} u_t + |u|^{\beta} u = 0,\ 0 < x < 1,\ 0 < t < \infty$$

with $0 > \alpha > \beta > -1$ (see [4]). Our result tells us that such rapid decay is rather common in second order nonlinear equations.

Although the class of equations we consider is somewhat artificial it is a very convenient model for understanding how the nonlinearities influence the solutions quantitatively.

2. Assumptions and result

Concerning the functions f, g and h appearing in (1.1) we make the following assumptions

A_1. $h(\cdot)$ belongs to $C(\mathbf{R}) \cap C^1(\mathbf{R} - \{0\})$ and moreover satisfies

https://doi.org/10.1017/S0004972700003464 Published online by Cambridge University Press

$$k_0 |u|^\alpha \leqslant h'(u) \leqslant k_1 |u|^\alpha \ (u \neq 0) \text{ and } h(0) = 0 \tag{2.1}$$

for some $\alpha > -1$ and positive constants k_0, k_1.

A$_2$. $g(t,u)$ is a continuous function on $\mathrm{R}^+ \times \mathrm{R}$ and satisfies

$$k_0 a(t) |u|^{\beta+2} \leqslant g(t,u)u \leqslant k_1 b(t) |u|^{\beta+2} \tag{2.2}$$

for some $\beta > -1$, k_0, $k_1 > 0$. Here $a(t)$ and $b(t)$ are nonnegative functions on R^+ satisfying

$$a(t) > 0 \ a.e. \text{ and } \{\int_t^{t+1} a(s)^{-p} dx\}^{1/p} \leqslant d_0(1+t)^{-\theta} \tag{2.3}$$

and

$$\int_t^{t+1} b(s)^{\beta+2} a(s)^{-\beta-1} ds \leqslant d_1(1+t)^\theta \tag{2.4}$$

for some $0 < p < \infty$ and d_0, $d_1 > 0$.

A$_3$. $f(\cdot)$ belongs to $C(\mathrm{R})$ and satisfies

$$k_0 |u|^{\gamma+2} \leqslant f(u)u \leqslant k_1 |u|^{\gamma+1} \tag{2.5}$$

for some $\gamma > -1$ and k_0, $k_1 > 0$.

We could weaken a little the assumptions above, for example, we could employ, instead of (2.2),

$$k_0 a(t) |u|^{\beta_0+2} \leqslant g(t,u)u \leqslant k_1\{b_1(t) |u|^{\beta_1+2} + b_2(t) |u|^{\beta_2+2}\}.$$

To make the essential features clear, however, we restrict ourselves to the typical case $A_1 - A_3$.

Since $h(u)$ may have a singularity at $u = 0$ we employ the following definition of solution.

DEFINITION 1: A function $x(\cdot)$ defined on $[0,T)$, $0 < T < \infty$, is said to be a solution of the equation (1.1) on $[0,T)$ with the initial value $(x_0, x_1) \in \mathrm{R}^2$ if $x(\cdot) \in C^1([0,T))$, $h(\dot{x}(\cdot)) \in C^1([0,T))$ and equation (1.1) is satisfied on $(0,T)$ together with the initial condition $x(0) = x_0$, $\dot{x}(0) = x_1$.

Concerning the global existence of solution we have:

THEOREM 1. *For each* $(x_0, x_1) \in \mathbf{R}^2$ *the problem (1.1) with* $x(0) = x_0$, $\dot{x}(0) = x_1$ *admits a global solution* $x(\cdot)$, *that is, a solution on* $[0, \infty)$.

PROOF: Setting $y_1 = h(\dot{x})$ and $y_2 = x$ the problem is equivalent to the system

$$\text{(2.6)} \qquad \begin{cases} \dot{y}_1 = -g\big(t, h^{-1}(y_1)\big) - f(y_2) \\ \dot{y}_2 = h^{-1}(y_1) \end{cases}$$

with $y_1(0) = h(x_1)$ and $y_2(0) = x_0$.

Setting also

$$\text{(2.7)} \qquad V(y_1, y_2) = \int_0^{h^{-1}/(y_1)} h'(u)u\,du + \int_0^{y_2} f(u)\,du$$

we have easily

$$V(y_1, y_2) \geqslant C\Big(|y_1|^{\alpha+2} + |y_2|^{\gamma+2}\Big) \text{ and } \dot{V}(y_1, y_2) \leqslant 0.$$

Thus, V is a Lyapunov function for the system (2.1). The result follows immediately from this fact. ∎

Our result on the decay property of the solutions of (1.1) reads as follows.

THEOREM 2. *Let* $x(t)$ *be a solution of (1.1) on* $[0, \infty)$ *and set*

$$E(t) = \int_0^{\dot{x}(t)} h'(u)u\,du + \int_0^{x(t)} f(u)\,du$$
$$\Big(\geqslant C\Big(|\dot{x}(t)|^{\alpha+2} + |x(t)|^{\gamma+2}\Big)\Big).$$

We set also

$$\eta = \max\{-\theta, \frac{\theta}{\beta+1}\} \text{ and } \sigma = \min\{\frac{\alpha+2}{\beta+2}, \frac{(\gamma+2)(\alpha+1)}{(\gamma+1)(\beta+2)}, \frac{(\gamma+2)(\beta+1)}{(\gamma+1)(\beta+2)}\}.$$

(I) *Assume that* $\theta = -1$ *or* $\beta + 1$ *and let* $\sigma < 1$. *Then we have*

$$\text{(2.8)} \qquad E(t) \leqslant C(E(0))\{\log(2+t)\}^{-\nu},$$

with $\nu = \sigma/(1-\sigma)$.

(II) *Assume that* $-1 < \theta < \beta + 1$ *and* $\sigma < 1$. *Then*

$$\text{(2.9)} \qquad E(t) \leqslant C(E(0))(1+t)^{-\nu},$$

with $\nu = (1-\eta)\sigma/(1-\sigma)$.

(III) *Assume that* $-1 < \theta < \beta + 1$ *and let* $\alpha = \beta > \gamma$ *or* $\alpha > \beta = \gamma$. *Then*

$$E(t) \leqslant C(E(0))e^{-kt^{1-\eta}} \tag{2.10}$$

where k is a positive constant depending on $E(0)$ and other known constants.

(IV) *Assume that $-1 < \theta < \beta + 1$ and $\sigma > 1$, that is, $\alpha > \beta > \gamma$. Then*

$$E(t) \leqslant C(E(0))e^{-ke^{(\nu-\varepsilon)t}}, \quad 0 < \varepsilon < \nu, \tag{2.11}$$

with $\nu = \log \sigma (> 0)$, where k is a positive constant depending on $E(0)$ and ε. We can take $\varepsilon = 0$ if $\theta = 0$.

Remark. When $\theta = -1$ or $\beta + 1$ and $\sigma = 1$ we can show, instead of (2.11),

$$E(T) \leqslant C(E(0))(1+t)^{-\nu}$$

for some $\nu > 0$ depending on $E(0)$.

3. Some Lemmas

The following lemma is essential for precise estimation of the solutions.

Lemma 1. *Let $\phi(t)$ be a nonnegative function on $\mathbb{R}^+ = [0, \infty)$, satisfying the difference inequality*

$$\sup_{t \leqslant s \leqslant t+1} \phi(s)^{1+r} \leqslant C_0(1+t)^{\theta}(\phi(t) - \phi(t+1)) + \delta(t) \tag{3.1}$$

for some $C_0 > 0$, $\theta < 1$, $r \geqslant 0$ and $\delta(t)$ a bounded function on $\mathbb{R}^+$. Then, $\phi(t)$ has the following decay property

(i) *if $\theta = 1$, $r > 0$ and $\delta(t) = 0\left((\log t)^{-1-1/r}\right)$ as $t \to \infty$. then*

$$\phi(t) \leqslant C(\phi(0))\{\log(2+t)\}^{-1/r};$$

(ii) *if $\theta < 1$, $r > 0$ and $\delta(t) = 0\left(t^{-(1-\theta)(1+1/r)}\right)$ as $t \to \infty$, then*

$$\phi(t) \leqslant C(\phi(0))(1+t)^{-(1-\theta)/r};$$

(iii) *if $\theta < 1$, $r = 0$ and $\delta(t) = 0\left(e^{-t^{1-\theta}}\right)$ as $t \to \infty$, then*

$$\phi(t) \leqslant C(\phi(0))e^{-kt^{1-\theta}}$$

for some $k = k(\phi(0)) > 0$.

For the proof of Lemma 1 see [**2**] or Redheffer and Walter [**5**]. Using Lemma 1 we can obtain

https://doi.org/10.1017/S0004972700003464 Published online by Cambridge University Press

LEMMA 2. *Let $\phi(t)$ be a decreasing function on $\mathbb{R}^+$, satisfying*

$$\phi(t) \leqslant C_0 \sum_{i=1}^{n} (1+t)^{\theta_i} (\phi(t) - \phi(t+1))^{\alpha_i}, \tag{3.2}$$

for some $C_0 > 0$. Then $\phi(t)$ has the following decay property

(i) *if $0 < \min_{1 \leqslant i \leqslant n} \{\alpha_i\} \equiv \sigma < 1$ and $\max_{1 \leqslant i \leqslant n} \{\theta_i/\alpha_i\} \equiv \eta = 1$, then*

$$\phi(t) \leqslant C(\phi(0))\{\log(2+t)\}^{-\nu}$$

with $\nu = \sigma/(1-\sigma)$;

(ii) *if $0 < \sigma < 1$ and $\eta < 1$, then $\phi(t) \leqslant C(\phi(0))(1+t)^{-\nu}$ with $\nu = (1-\eta)\sigma/(1-\sigma)$;*

(iii) *if $\sigma = 1$ and $\eta < 1$, then $\phi(t) \leqslant C(\phi(0))\exp\{-kt^{1-\eta}\}$ for some $k = k(\phi(0)) > 0$.*

PROOF: All cases can be proved similarly, and we give the proof only for case (ii). First, note that if $\alpha_i \leqslant \alpha_j$ and $\theta_i \leqslant \theta_j$ for some i, j we can remove the term $(1+t)^{\theta_j}(\phi(t) - \phi(t+1))^{\theta_j}$ from the right-hand side of (3.2). Therefore, without loss of generality, we may assume

$$\alpha_1 > \alpha_2 > \ldots > \alpha_n \text{ and } \theta_1 > \theta_2 > \ldots > \theta_n.$$

Then, from (3.2) we have

$$\min_{1 \leqslant i \leqslant n} \phi(t)^{1/\alpha_i} \leqslant C_0 n \max_{1 \leqslant i \leqslant n} (1+t)^{\theta_i/\alpha_i} (\phi(t) - \phi(t+1))$$

and hence

$$\begin{aligned} \phi(t)^{1/\alpha_n} &= \{\frac{\phi(t)}{\sup_s \phi(s)}\}^{1/\alpha_n} \sup_s \phi(s)^{1/\alpha_n} \\ &\leqslant \{\phi(t)/\phi(0)\}^{1/\alpha_i} \phi(0)^{1/\alpha_n} (\forall i) \\ &\leqslant C(\phi(0))(1+t)^{\eta}(\phi(t) - \phi(t+1)) \end{aligned} \tag{3.3}$$

with $\eta = \max_{1 \leqslant i \leqslant n} \theta_i/\alpha_i$.

Thus, applying Lemma 1 (ii) to (3.3) we have the desired estimate. ∎

LEMMA 3. *Let $\phi(t)$ be a nonnegative decreasing function on $\mathbb{R}^+$, satisfying*

$$\phi(t) \leqslant C_0 e^{-k_0 t^{1-\theta}} \text{ for } t \in \mathbb{R}^+$$

https://doi.org/10.1017/S0004972700003464 Published online by Cambridge University Press

with some C_0, $k_0 > 0$ *and* $\theta < 1$, *and the difference inequality*

$$\phi(t+1) \leqslant C_1 \sum_{i=1}^{n} (1+t)^{\theta_i} \phi(t)^{\alpha_i} \tag{3.4}$$

with $C_1 > 0$ *and* θ_i, α_i *such that*

$$\sigma \equiv \min_{1 \leqslant i \leqslant n} \{\alpha_i\} > 1 \text{ and } \eta \equiv \max_{1 \leqslant i \leqslant n} \{\alpha_i/\theta_i\} < 1.$$

Then, for any $0 < \varepsilon << 1$ *there exist* $C_\varepsilon = C(\varepsilon, \phi(0))$ *and* $k = k(\varepsilon, \phi(0))$ *such that*

$$\phi(t) \leqslant C_\varepsilon e^{-ke^{(\nu-\varepsilon)t}} \text{ for } t \geqslant 0 \tag{3.5}$$

where we set $\nu = \log \sigma (> 0)$. *When* $\eta = 0$ *we can take* $\varepsilon = 0$ *in (3.5).*

PROOF: It follows from (3.4) that

$$\min_{1 \leqslant i \leqslant n} \phi(t+1)^{1/\alpha_i} \leqslant C_1 n \max_{1 \leqslant i \leqslant n} (1+t)^{\theta_i/\alpha_i} \phi(t)$$

and hence

$$\phi(t+1)^{1/\sigma} \leqslant C(1+t)^{\eta} \phi(t) \tag{3.6}$$

for some $C > 0$.

By the assumption on the decay of $\phi(t)$ as $t \to \infty$ we see that for any $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that

$$C^{\sigma}(1+t)^{\eta\sigma} \phi(t)^{\sigma} \leqslant \phi(t)^{\sigma-\varepsilon} \quad \text{if } t \geqslant T_\varepsilon. \tag{3.7}$$

Therefore we have from (3.6) that

$$\phi(t) \leqslant \phi(t-1)^{\sigma-\varepsilon} \leqslant \phi(t-m)^{(\sigma-\varepsilon)^m} \quad \text{if } t-m \geqslant T_\varepsilon$$

and

$$\phi(t) \leqslant \phi(T_\varepsilon)^{(\sigma-\varepsilon)^{[t-T_\varepsilon]}} \quad \text{if } t \geq T_\varepsilon \tag{3.8}$$

where $[t - T_\varepsilon]$ denotes the integer part of $t - T_\varepsilon$. Since we may assume $\phi(T_\varepsilon) < e^{-1}$ it follows from (3.8) that

$$\begin{aligned} \phi(t) &\leqslant e^{-e^{[t-T_\varepsilon]\log(\sigma-\varepsilon)}} \; (0 < \varepsilon < \sigma) \\ &\leqslant e^{-ke^{\nu_\varepsilon t}} \text{ if } t > T_\varepsilon \end{aligned}$$

with $\nu_\varepsilon = \log(\sigma - \varepsilon) > 0$. Changing the notation yields (3.5).

It is clear that when $\eta = 0$ we can take $\varepsilon = 0$ and $T_\varepsilon = 0$ in (3.7) and the estimate (3.5) holds with $\nu_\varepsilon = \nu = \log \sigma$. ∎

https://doi.org/10.1017/S0004972700003464 Published online by Cambridge University Press

4. Proof of Theorem 2

Let $x(\cdot)$ be a solution of (1.1) (in the sense of Definition 1) and let us recall that

$$E(t) = \int_0^{\dot{x}(t)} h(u)u\,du + \int_0^{x(t)} f(u)\,du.$$

We note again that

$$\begin{aligned} k_0\{\frac{1}{\alpha+2}|\dot{x}(t)|^{\alpha+2} + \frac{1}{\gamma+2}|x(t)|^{\gamma+2}\} \\ \leqslant E(t) \leqslant k_1\{\frac{1}{\alpha+2}|\dot{x}(t)|^{\alpha+2} + \frac{1}{\gamma+2}|x(t)|^{\gamma+2}\} \end{aligned} \tag{4.1}$$

$(k_0, k_1 > 0)$.

Multiplying the equation (1.1) by $\dot{x}(t)$ and integrating over $[t, t+1]$ we have

$$\int_t^{t+1} g(s, \dot{x}(s))\dot{x}(s)ds = E(t) - E(t+1) \equiv D(t)^{\beta+2} \tag{4.2}$$

and by the assumption A_2

$$k_0 \int_t^{t+1} a(s)\,|\dot{x}(s)|^{\beta+2}\,ds \leqslant D(t)^{\beta+2}. \tag{4.3}$$

In what follows C will denote generous positive constants, in particular $C(\phi(0))$ will denote constants depending on $\phi(0)$ continuously.

Now, with the use of the assumption on $a(\cdot)$ we have, for any $r > 0$,

$$\begin{aligned} \int_t^{t+1} |\dot{x}(s)|^r\,ds &= \int_t^{t+1} a(s)^{-p/(p+1)} a(s)^{p/(p+1)}\,|\dot{x}(s)|^r\,ds \\ &\leqslant \{\int_t^{t+1} a(s)^{-p}ds\}^{1/(p+1)}\{\int_t^{t+1} a(s)\,|\dot{x}(s)|^{r(p+1)/p}\,ds\}^{p/(p+1)} \\ &\leqslant C(1+t)^{-p\theta/(p+1)}\{\int_t^{t+1} a(s)\,|\dot{x}(s)|^{\beta+2}\,ds\}^{p/(p+1)} \\ &\qquad \sup_{t\leqslant s\leqslant t+1} |\dot{x}(s)|^{r-p(\beta+2)/(p+1)} \\ &\leqslant C(1+t)^{-p\theta/(p+1)} D(t)^{p(\beta+2)/(p+1)} \\ &\qquad \sup_{t\leqslant s\leqslant t+1} E(s)^{\{r(p+1)-p(\beta+2)\}/(\alpha+2)(p+1)}, \end{aligned} \tag{4.4}$$

where we have assumed that $r(p+1) \geqslant p(\beta+2)$.

https://doi.org/10.1017/S0004972700003464 Published online by Cambridge University Press

Taking $r = p(\beta+2)/(p+1)$ in (4.4) we see that

$$\int_t^{t+1} |\dot{x}(s)|^{p(\beta+2)/(p+1)}\, ds \leqslant C(1+t)^{-p\theta/(p+1)} D(t)^{p(\beta+2)/(p+1)}$$

and hence there exist $t_1 \in [t, t+1/4]$, $t_2 \in [t+3/4, t+1]$ such that

$$(4.5)\qquad |\dot{x}(t_i)| \leqslant C(1+t)^{-\theta/(\beta+2)} D(t),\ i = 1, 2.$$

Next, multiplying the equation (1.1) by $x(t)$ and integrating over $[t_1, t_2]$ we have

$$(4.6)\qquad \begin{aligned} \int_{t_1}^{t_2} f(x(s))x(s)ds &= -\int_{t_1}^{t_2} h(\dot{x}(s))\dot{x}(s)ds - h(\dot{x}(t_2))x(t_2) \\ &\quad + h(\dot{x}(t_1))x(t_1) - \int_{t_1}^{t_2} g(s, \dot{x}(s))x(s)ds \\ &\leqslant C\{\int_{t_1}^{t_2} |\dot{x}(s)|^{\alpha+2}\, ds \\ &\quad + \sum_{i=1}^{2} |\dot{x}(t_i)|^{\alpha+1} \sup_{t\leqslant s\leqslant t+1} |x(s)| \\ &\quad + \int_{t_1}^{t_2} b(s)\, |\dot{x}(s)|^{\beta+1}\, |x(s)|\, ds\}. \end{aligned}$$

Each term of the righthand side of (4.6) is treated as follows.

First, without loss of generality, we may assume p is sufficiently small and we can take $r = \alpha+2$ in (4.4) to get

$$\begin{aligned} \int_{t_1}^{t_2} |\dot{x}(s)|^{\alpha+2}\, ds &\leqslant C(1+t)^{-p\theta/(p+1)} D(t)^{p(\beta+2)/(p+1)} \\ &\quad \times \sup_{t\leqslant s\leqslant t+1} E(s)^{1-p(\beta+2)/(\alpha+2)(p+1)}. \end{aligned}$$

By (4.5),

$$\begin{aligned} \sum_{i=1}^{2} |\dot{x}(t_i)|^{\alpha+1} \sup_{t\leqslant s\leqslant t+1} |x(s)| &\leqslant C(1+t)^{-\theta(\alpha+1)/(\beta+2)} D(t)^{\alpha+1} \\ &\quad \times \sup_{t\leqslant s\leqslant t+1} E(s)^{1/(\gamma+2)}. \end{aligned}$$

https://doi.org/10.1017/S0004972700003464 Published online by Cambridge University Press

Finally,

$$
\begin{aligned}
&\int_{t_1}^{t_2} b(s)\,|\dot{x}(s)|^{\beta+1}\,|x(s)|\,ds \\
&\leqslant \{\int_{t_1}^{t_2} a(s)\,|\dot{x}(s)|^{\beta+2}\,ds\}^{(\beta+1)/(\beta+2)}\{\int_{t_1}^{t_2} b(s)^{\beta+2}a(s)^{-\beta-1}ds\}^{1/(\beta+2)} \\
&\qquad \times \sup_{t\leqslant s\leqslant t+1} |x(s)| \\
&\leqslant C D(t)^{\beta+1}(1+t)^{\theta/(\beta+2)} \sup_{t\leqslant s\leqslant t+1} E(s)^{1/(\gamma+2)}.
\end{aligned}
$$

Thus, we have from (4.6)

(4.7)

$$
\begin{aligned}
E(t_2) &\leqslant C\int_{t_1}^{t_2}\{\frac{1}{\alpha+2}|\dot{x}(s)|^{\alpha+2}+\frac{1}{\gamma+2}|x(s)|^{\gamma+2}\}ds \\
&\leqslant C\Big\{(1+t)^{-p\theta/(p+1)}D(t)^{p(\beta+2)/(p+1)}\sup_{t\leqslant s\leqslant t+1}E(s)^{1-p(\beta+2)/(\alpha+2)(p+1)} \\
&\qquad +(1+t)^{-\theta(\alpha+1)/(\beta+2)}D(t)^{\alpha+1}\sup_{t\leqslant s\leqslant t+1}E(s)^{(1/\gamma+2)} \\
&\qquad +(1+t)^{\theta/(\beta+2)}D(t)^{\beta+1}\sup_{t\leqslant s\leqslant t+1}E(s)^{1/(\gamma+2)}\Big\} \equiv A(t).
\end{aligned}
$$

Furthermore, by (4.2) and (4.7),

$$
\begin{aligned}
\sup_{t\leqslant s\leqslant t+1}E(s) &\leqslant E(t_2)+\int_t^{t+1} g(s,\dot{x}(s)\dot{x}(s))ds \\
&\leqslant A(t)+D(t)^{\beta+2},
\end{aligned}
$$

and consequently

$$
\begin{aligned}
E(t) = \sup_{t\leqslant s\leqslant t+1}E(s) &\leqslant C\Big\{(1+t)^{-\theta(\alpha+2)/(\beta+2)}D(t)^{\alpha+2} \\
&\quad +(1+t)^{-\theta(\alpha+1)(\gamma+2)/(\gamma+1)(\beta+2)}D(t)^{(\alpha+1)(\gamma+2)/(\gamma+1)} \\
&\quad +(1+t)^{(\gamma+2)\theta/(\beta+2)(\gamma+1)}D(t)^{(\beta+1)(\gamma+2)/(\gamma+1)} \\
&\quad +D(t)^{\beta+2}\Big\}.
\end{aligned}
\tag{4.8}
$$

From (4.7) we obtain also

$$
\begin{aligned}
E(t+1) \leqslant C\Big\{&(1+t)^{-p\theta/(p+1)}E(t)^{1+p(\alpha-\beta)/(\alpha+2)(p+1)} \\
&+(1+t)^{-\theta(\alpha+1)/(\beta+2)}E(t)^{(\alpha+1)/(\beta+2)+1/(\gamma+2)} \\
&+(1+t)^{\theta/(\beta+2)}E(t)^{(\beta+1)/(\beta+2)+1/(\gamma+2)}.
\end{aligned}
\tag{4.9}
$$

Now, we can apply Lemma 2 to the inequality (4.8) to get the estimates (I)-(III) in Theorem 2. When $\sigma > 1$, namely, $\alpha > \beta > \gamma$ it is first verified from (4.8) and Lemma 2 (iii) that

$$E(t) \leqslant C(E(0))e^{-kt} \tag{4.10}$$

with some $k > 0$ under the condition $-1 < \theta < \beta + 1$, and hence application of Lemma 3 to (4.9) yields the estimate (IV) in Theorem 1. The proof of Theorem 2 is now completed. □

References

[1] M. Nakao, 'On bounded, periodic and almost periodic solutions for a system of nonlinear second order ordinary differential equations', *J. Math. Anal. Appl.* **61** (1977), 620–632.

[2] M. Nakao, 'A difference inequality and its application to nonlinear evolution equations', *J. Math. Soc. Japan* **30** (1978), 747–762.

[3] M. Nakao, 'Asymptotic stability for some nonlinear evolution equations of second order with unbounded dissipative term', *J. Differential Equations* **30** (1978), 54–63.

[4] M. Nakao, 'An example of nonlinear wave equation whose solutions decay faster than exponentially', *J. Math. Anal. Appl.* **122** (1987), 260–264.

[5] R. Redheffer and W. Walter, 'A comparison theorem for difference inequalities', *J. Differential Equations* **44** (1982), 111–117.

[6] Y. Yamada, 'On the decay of solutions for some nonlinear evolution equations of second order', *Nagoya Math. J.* **1** (1979), 67–98.

Department of Mathematics
College of General Education Kyushu University
Fukuoka 810
Japan