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COMMUTATIVITY PRESERVING MAPS OF FACTORS

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1. Introduction. By a von Neumann algebra M we mean a weakly closed, self-adjoint algebra of operators on a Hilbert space $\mathscr{H}$ which contains I, the identity operator. A factor is a von Neumann algebra whose centre consists of scalar multiples of I.

In all that follows $\phi:M \rightarrow N$ will be a one to one, *-linear map from the von Neumann factor M onto the von Neumann algebra N such that both ϕ and ϕ^{-1} preserve commutativity. Our main result states that if M is not of type I_2 then $\phi = c\widetilde{\theta} + \lambda$ where $\widetilde{\theta}$ is an isomorphism or an anti-isomorphism, c is a non-zero scalar, and λ is a *-linear map from M into Z_N, the centre of N.

Our interest in this problem was aroused by several recent results. In [**1**], Choi, Jafarian, and Radjavi proved that if S is the real linear space of $n \times n$ matrices over any algebraically closed field, $n \geqq 3$, and ψ a linear operator on S which preserves commuting pairs of matrices, then either $\psi(S)$ is commutative or there exists a unitary matrix U such that

$$\psi(A) = cU^*AU + f(A)I \quad \text{or} \quad \psi(A) = cU^*A^tU + f(A)I$$

for all A in S. They proved an analogous result for the collection of all bounded self-adjoint operators on an infinite dimensional Hilbert space when ψ is one to one. Subsequently, Omladic [**7**] proved that if $\psi:L(X) \rightarrow L(X)$ is a bijective linear operator preserving commuting pairs of operators where X is a non-trivial Banach space, then

$$\psi(A) = cUAU^{-1} + f(A)I \quad \text{or} \quad \psi(A) = UA'U^{-1} + f(A)I$$

where U is a bounded invertible operator on X and A' is the adjoint of A.

We viewed this problem as one involving mappings between the Lie algebras M and N which preserve the zero brackets. Our technique is to show, as in [**6**] where bracket preserving maps were studied, that on projections P in M,

$$\phi(P) = \theta(P) + \lambda(P)I \quad \text{or} \quad \phi(P) = -\theta(P) + \lambda(P)I$$

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where θ is a projection orthoisomorphism. This representation is harder to achieve than in [**6**], but once having it the techniques of [**6**] are applied together with results concerning the linear span of projections in a factor to give the result. A key tool used in [**6**] is a theorem of Dye [**3**] relating projection orthoisomorphisms to C^*-isomorphisms.

The techniques of this paper give the result as long as the dimension of the underlying Hilbert space is >4. However, since the Choi, Jafarian, Radjavi theorem implies our theorem for all type I_n factors, $n > 2$, and since we would have to invoke their theorem for $n = 3, 4$, we shall assume that M is not a finite factor of type I. We use [**2**] as a general reference for the theory of von Neumann algebras.

2. The decomposition $\phi = \theta + \lambda$.

LEMMA 1. *N is a factor.*

Proof. Let Z_M, Z_N be the centers of M, N respectively. Since $\phi(Z_M) = Z_N$ and Z_M is 1-dimensional, Z_N is 1-dimensional.

LEMMA 2. *We can assume, by dividing by an appropriate constant, that $\phi(I) = I$.*

Proof. Since $Z_N = \mathbf{C}I$ and since ϕ is one to one, $\phi(I) = \beta I$ for $\beta \neq 0$. Replace ϕ by $(1/\beta)\phi$.

Definition. A von Neumann subalgebra $M_0 \subseteq M$ is *normal* in M if

$$M_0 = (M_0' \cap M)' \cap M$$

where, for any subset $S \subseteq \mathcal{B}(H)$,

$$S' = \{Y \in \mathcal{B}(H) \,|XY = YX \,\forall\, X \in S\}.$$

LEMMA 3. *If M_0 is a normal subalgebra of M, then $N_0 = \phi(M_0)$ is a normal subalgebra of N with the same linear dimension.*

Proof. If S is any subset of M, $\phi(S' \cap M) = \phi(S)' \cap N$. Since M_0 is normal, $M_0 = (M_0' \cap M)' \cap M$ so that

$$\phi(M_0) = (\phi(M_0)' \cap \phi(M)\,)' \cap \phi(M) = (\phi(M_0)' \cap N)' \cap N.$$

Since M_0 is a self-adjoint collection, so is $\phi(M_0)$ which implies that $(\phi(M_0)' \cap N)' \cap N$ is a von Neumann algebra. Hence $N_0 = \phi(M_0)$ is a von Neumann algebra and is normal in N.

LEMMA 4. *If P is a non-central projection in M, then $\phi(P) = \alpha Q + \lambda I$ where Q is a non-central projection in N and $\alpha \neq 0$.*

Proof. By [**5**, Theorems 1 and 4], a finite-dimensional subalgebra of a factor is normal. Let $M_0 = \text{lin.sp.}\{I, P\}$. M_0 is a 2-dimensional subalgebra of M and is thus normal in M. By Lemma 3, $\phi(M_0) = N_0$ is a 2-dimensional von Neumann subalgebra of N, say

$$\phi(M_0) = \text{lin.sp.}\{I, Q\}$$

where Q is a non-central projection. We have $\phi(P) \in \phi(M_0)$ so $\phi(P) = \alpha Q + \lambda I$. If $\alpha = 0$ then P would be central by the commutativity preserving property of ϕ.

LEMMA 5. *If P is a non-central projection and*

$$\phi(P) = \alpha Q + \lambda I = \alpha' Q' + \lambda' I$$

with $\alpha, \alpha' \neq 0$, Q and Q' non-central projections in N, then either (i) $Q = Q'$ *and* $\alpha = \alpha'$, *or* (ii) $Q = I - Q'$ *and* $\alpha = -\alpha'$.

Proof. For an operator $A \in \mathscr{B}(H)$, let $\sigma(A)$ be its spectrum. We have

$$\{\alpha + \lambda, \lambda\} = \sigma(\alpha Q + \lambda I) = \sigma(\alpha' Q' + \lambda' I) = \{\alpha' + \lambda', \lambda'\}.$$

If $\alpha + \lambda = \alpha' + \lambda'$ then $Q = Q'$. If $Q = Q'$ then clearly $\lambda = \lambda'$ so that $\alpha = \alpha'$. If $\alpha + \lambda = \lambda'$ and $\alpha' + \lambda' = \lambda$ then $\alpha = -\alpha'$ and $\lambda \neq \lambda'$ since $\alpha \neq 0$. We would then have

$$Q + Q' = \left(\frac{\lambda - \lambda'}{\alpha}\right) I.$$

This forces

$$\frac{\lambda - \lambda'}{\alpha} = 1.$$

If $Q = I - Q'$ it is easy to see that $\alpha = -\alpha'$.

LEMMA 6. *Let P_1, P_2 be non-central orthogonal projections in M with $P_1 + P_2 \neq I$. There exist orthogonal non-central projections Q_1, Q_2 in N and non-zero scalars α_1, α_2, such that*

$$\phi(P_i) = \alpha_i Q_i + \lambda_i I \quad i = 1, 2.$$

Proof. Let $M_0 = \text{lin.sp. } \{I, P_1, P_2\}$. M_0 is a 3-dimensional abelian subalgebra of M so that $N_0 = \phi(M_0)$ is a 3-dimensional abelian subalgebra of N. We claim that

$$N_0 = \text{lin.sp.}\{I, Q_1, Q_2\}$$

where $\phi(P_i) = \alpha_i Q_i + \lambda_i I$ as in Lemma 4. Clearly $Q_1, Q_2 \in N_0$ since $I \in N_0$, $\phi(P_i) \in N_0$, and $\alpha_i \neq 0$. Suppose

$$\alpha I + \beta Q_1 + \gamma Q_2 = 0.$$

Since

$$\phi(I) = I \quad \text{and} \quad Q_i = \phi\left(\frac{1}{\alpha_i} P_i - \lambda_i I\right), \quad i = 1, 2,$$

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we have

$$0 = \alpha I + \beta Q_1 + \gamma Q_2 = \phi\left((\alpha - \beta\lambda_1 - \gamma\lambda_2)I + \frac{\beta}{\alpha_1}P_1 + \frac{\gamma}{\alpha_2}P_2\right).$$

This implies

$$\frac{\beta}{\alpha_1}P_1 + \frac{\gamma}{\alpha_2}P_2 \in Z_M$$

since ϕ is one to one. Since $P_1P_2 = 0$ and the P_i are non-central we have $\beta = \gamma = 0$. This forces $\alpha = 0$. Thus $\{I, Q_1, Q_2\}$ is a linearly independent subset of the three-dimensional algebra N_0.

Case (1). $Q_1Q_2 = 0$, and we need do no more.

If $Q_1Q_2 \neq 0$ then, since $Q_1Q_2 \in N_0$ we have

(*) $Q_1Q_2 = \alpha I + \beta Q_1 + \gamma Q_2$ where not all of α, β, γ are zero. Multiplying (*) by Q_1Q_2 we get $\alpha + \beta + \gamma = 1$. Multiplying by Q_1 we see that

$$(1 - \gamma)Q_1Q_2 = (1 - \gamma)Q_1.$$

Case (2). $1 - \gamma \neq 0$. Then $Q_1 = Q_1Q_2$ or $Q_1 \leqq Q_2$. If $Q_1 = Q_2$ then $\{I, Q_1, Q_2\}$ would span a two-dimensional subalgebra so we must have $Q_1 \lneqq Q_2$. In this case we replace Q_2 by $I - Q_2$ and note that

$$\alpha_2Q_2 + \lambda_2I = \alpha_2(I - Q_2) + (\lambda_2 - \alpha_2)I.$$

If $\gamma = 1$ then (*) becomes $Q_1Q_2 = \alpha I + \beta Q_1 + Q_2$ so that

$$(1 - \beta)Q_1Q_2 = (1 + \alpha)Q_2.$$

Case (3). $\beta \neq 1$. Then $1 - \beta = 1 + \alpha$ and $Q_1Q_2 = Q_2$. As in (2), $Q_1 \neq Q_2$, and we replace Q_1 by $I - Q_1$.

Case (4). $\beta = 1$. Then $\alpha = -1$ and $Q_1Q_2 = -I + Q_1 + Q_2$. That is, $I - Q_1 \perp I - Q_2$ so we replace both Q_1 and Q_2 by $I - Q_1$ and $I - Q_2$ respectively.

LEMMA 7. *If P_1, P_2, Q_1, Q_2 and α_1, α_2 are as in Lemma* 6 *then* $\alpha_1 = \alpha_2$.

Proof. Let $M_0 = \text{lin.sp.}\{I, P_1 + P_2\}$. Then M_0 is a 2-dimensional subalgebra of M, so that $\phi(M_0) = N_0$ is a two-dimensional subalgebra of N, say $N_0 = \text{lin.sp.}\{I, Q\}$. Thus

$$\phi(P_1 + P_2) = \alpha_1Q_1 + \alpha_2Q_2 + (\lambda_1 + \lambda_2)I = \alpha Q + \lambda I$$

where the α_i and Q_i, $i = 1, 2$ are as in Lemma 6. Since $\alpha \neq 0$ and Q not central, the spectrum of $\alpha Q + \lambda I$ consists of two points. Thus if

$$A = \alpha_1Q_1 + \alpha_2Q_2 + (\lambda_1 + \lambda_2)I,$$

$\sigma(A)$ consists of two points. Since $Q_1 \perp Q_2$ and $Q_1 + Q_2 \neq I$ we have

$$\sigma(A) = \{\alpha_1 + \lambda_1 + \lambda_2, \alpha_2 + \lambda_1 + \lambda_2, \lambda_1 + \lambda_2\}$$

and so two of these points coincide. Now $\alpha_1, \alpha_2 \neq 0$ so we must have

$$\alpha_1 + \lambda_1 + \lambda_2 = \alpha_2 + \lambda_1 + \lambda_2$$

and so $\alpha_1 = \alpha_2$.

LEMMA 8. *If P_1, P_2 are non-central, orthogonal, equivalent projections in M with $P_1 + P_2 \neq I$ there exist non-central, orthogonal, equivalent projections Q_1, Q_2 in N and $\alpha \neq 0$ such that $\phi(P_i) = \alpha Q_i + \lambda_i I$.*

Proof. Let the Q_i and α be chosen as in Lemma 7, let V be a partial isometry in M such that $V^*V = P_1$, $VV^* = P_2$, and let $\mathscr{A}$ be the non-commutative 5-dimensional von Neumann subalgebra of M generated by $\{I, P_1, P_2, V, V^*\}$. Then $\mathscr{B} = \phi(\mathscr{A})$ is a 5-dimensional von Neumann subalgebra of N generated by

$$\{I, \alpha Q_1 + \lambda_1 I, \alpha Q_2 + \lambda_2 I, X, X^*\}$$

where $X = \phi(V)$. We have that

$$Z_{\mathscr{B}} = \text{lin.sp.}\{I, Q_1 + Q_2\}$$

since

$$Z_{\mathscr{A}} = \text{lin.sp.}\{I, P_1 + P_2\}.$$

Since $\mathscr{B}$ is a non-commutative 5-dimensional von Neumann algebra,

$$\mathscr{B} = M_1 \oplus M_2 \cong \mathbf{C} \oplus M_2(\mathbf{C})$$

where $M_2(\mathbf{C})$ is the algebra of 2×2 matrices over $\mathbf{C}$. Let I_1 and I_2 be the central projections of $\mathscr{B}$ which are the identities of M_1 and M_2 respectively. We have $I_1 + I_2 = I$. Now $Q_1 + Q_2$ is a non-zero central projection in $\mathscr{B}$ so either $Q_1 + Q_2 = I_1$ or $Q_1 + Q_2 = I_2$. But I_1 is not the sum of non-zero orthogonal projections so we have $Q_1 + Q_2 = I_2$. This implies that Q_1 and Q_2 are in M_2 and so are equivalent since they are non-central.

LEMMA 9. *Let M be a factor of type* I_∞, II, *or* III *and let $P \in M$ be a non-central projection. There exists $\alpha \in \mathbf{C}$, $\alpha \neq 0$, independent of P and a non-central projection $Q \in N$ such that $\phi(P) = \alpha Q + \lambda I$.*

Proof. Let $P = P_1$ and let $P_2 \neq P_1$ be any other non-central projection in M. One of $P_1 \vee P_2$, $(I - P_1) \vee P_2$, $P_1 \vee (I - P_2)$ or $(I - P_1) \vee (I - P_2)$ has codimension $\geqq 2$. Suppose it is $P_1 \vee P_2$, the other cases being similar. Thus $I - (P_1 \vee P_2)$ is the sum of two orthogonal projections. (In the type II and III cases we need only that $I - (P_2 \vee P_2) \neq 0$ and then could "halve" $I - (P_1 \vee P_2)$ to get equivalent orthogonal projections. In the type I case the codimension $\geqq 2$ as long as the dimension of $\mathscr{H} \geqq 5$.) Let P_3 be one of them. Then $P_1 \perp P_3$ and $P_1 + P_3 \neq I$. Applying Lemma 7 to P_1 and P_3 we get

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$$\phi(P_1) = \alpha Q_1 + \lambda_1 I, \quad \phi(P_3) = \alpha Q_3 + \lambda_3 I.$$

Applying Lemma 7 to P_2 and P_3 we get

$$\phi(P_2) = \alpha' Q_2' + \lambda_2' I, \quad \phi(P_3) = \alpha' Q_3' + \lambda_3' I.$$

Applying Lemma 5 to the two representations of $\phi(P_3)$ we get $\alpha' = \pm\alpha$. If $\alpha' = -\alpha$, write

$$\phi(P_2) = \alpha(I - Q_2') + (\lambda_2' - \alpha)I.$$

We now replace ϕ by $(1/\alpha)\phi$.

LEMMA 10. *Let M be a factor of type* I_∞, II, *or* III, *and P a non-central projection. Then $\phi(P)$ can be expressed uniquely in one of two ways*

(i) $\phi(P) = \theta(P) + \lambda(P)I$, *or*

(ii) $\phi(P) = -\theta'(P) + \lambda'(P)I$

where $\theta(P)$, $\theta'(P)$ are non-central projections in N, and $\lambda(P)$, $\lambda'(P)$ are scalars.

Proof. With the above normalization

$$\phi(P) = Q + \lambda I = -(I - Q) + (1 + \lambda)I$$

so we let $\theta(P) = Q$, $\lambda(P) = \lambda$, $\theta'(P) = I - Q$, $\lambda'(P) = 1 + \lambda$. If

$$Q + \lambda I = Q' + \lambda' I$$

where Q commutes with Q' then

$$(\lambda - \lambda')^2 I = (Q' - Q)^2 = Q' + Q - {}_2QQ'.$$

This happens if and only if $Q = Q'$.

3. The C^*-isomorphism theorem.

LEMMA 11. $\theta(I - P) = I - \theta(P)$, $\theta'(I - P) = I - \theta'(P)$.

Proof. See [**6**, Lemma 4].

LEMMA 12. *If P and Q are orthogonal projections in M then either*

$$\theta(P) \perp \theta(Q) \quad \text{or} \quad I - \theta(P) \perp I - \theta(Q).$$

Proof. This follows from Lemma 5 and Lemma 9.

Definition. If M is a von Neumann algebra let M_P be the collection of projections in M. A projection orthoisomorphism between von Neumann algebras M and N is a map $\theta: M_P \to N_P$ which is one to one, onto, and such that if $P, Q \in M_P$ with $PQ = 0$ then $\theta(P)\theta(Q) = 0$.

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LEMMA 13. *If $\mathscr{A}$ is an abelian von Neumann subalgebra of M of dimension $\geqq 3$ then either θ or θ' is an orthoisomorphism on $\mathscr{A}_P$ and these possibilities are mutually exclusive. If θ is an orthoisomorphism then both θ and λ are additive on mutually orthogonal projections in $\mathscr{A}_P$. A similar statement holds for θ' and λ'.*

Proof. See [**6**, Lemma 6].

LEMMA 14. *Let $P_1, \dots, P_n$, $n \geqq 3$ be mutually orthogonal equivalent projections in M. If the $\theta(P_i)$ are orthogonal then they are equivalent in N. If the $\theta(P_i)$ are mutually orthogonal then they are equivalent in N.*

Proof. Applying Lemma 8 we have that if $\theta(P_1) \perp \theta(P_2)$ then $\theta(P_1) \sim \theta(P_2)$ in N, etc.

THEOREM 1. *Let $\phi:M \to N$ be a commutativity preserving map of the infinite factor M onto the von Neumann algebra N. Then N is an infinite factor and if $P \in M_P$, $\phi(P) = \theta(P) + \lambda(P)$ where θ is an orthoisomorphism, or*

$$\phi(P) = -\theta'(P) + \lambda'(P)$$

where θ' is an orthoisomorphism. If M is a finite factor, so is N and a similar conclusion holds for ϕ.

Proof. If M is infinite choose mutually orthogonal equivalent projections P_i, $i = 1, 2, 3, 4$ such that

$$\sum_{i=1}^{4} P_i = I$$

and assume the $\theta(P_i)$ are orthogonal. Then the $\theta(P_i)$ are equivalent. Since $P_1 \sim P_3 \sim P_1 + P_2$ we have, from Lemma 8 and the additivity of θ, that $\theta(P_1) \sim \theta(P_1) + \theta(P_2)$ so that N is infinite. Now

$$I = \phi(I) = \sum_{i=1}^{4} \phi(P_i) = \sum_{i=1}^{4} \theta(P_i) + \left(\sum_{i=1}^{4} \lambda(P_i)\right) I$$

which implies

$$\sum_{i=1}^{4} \theta(P_i) = I \quad \text{and} \quad \sum_{i=1}^{4} \lambda(P_i) = 0$$

since the $\theta(P_i)$ are orthogonal. Thus $\theta(I) = I$. In the θ' case, $\theta'(I) = -I$. The proof in the infinite case now follows [**6**, Theorem 2].

If M is finite, and hence of type II_1 since we are ruling out the type I_n case, then so is N since the above reasoning could be applied to ϕ^{-1} if N were infinite. N cannot be of type I_n since ϕ^{-1} preserves linear dimension.

Hence N is also of type II_1. The proof for M and N being II_1-factors now follows [**6**, Theorem 3].

THEOREM 2. *Let $\phi:M \to N$ be a commutativity preserving map from the factor M onto the von Neumann algebra N. Then $\phi = c\tilde{\theta} + \lambda$ where $c \in \mathbf{C}$, $c \neq 0$, $\tilde{\theta}$ is an isomorphism or an anti-isomorphism of M onto N, and λ is a *-linear map from M into $Z_N = \mathbf{C}I$.*

Proof. On projections

$$\phi(P) = \theta(P) + \lambda(P)I \quad \text{or} \quad \phi(P) = -\theta'(P) + \lambda'(P)I$$

as in Theorem 1. Taking the case where θ is an orthoisomorphism there is, by a theorem of Dye [**3**, Theorem 1], a C^*-isomorphism $\tilde{\theta}$ of M on N which agrees with θ on M_P. By [**8**, Theorem 6] every self-adjoint operator in a properly infinite von Neumann algebra is a real linear combination of eight projections, and it was proved in [**4**] that every operator in a II_1-factor is a linear combination of projections. Thus for any factor M, if $A \in M$ then

$$A = \sum_{i=1}^{n} \alpha_i P_i.$$

We have

$$\phi(A) = \phi\left(\sum_{i=1}^{n} \alpha_i P_i\right) = \sum_{i=1}^{n} \alpha_i(\theta(P_i) + \lambda(P_i)I)$$

$$= \sum_{i=1}^{n} \alpha_i\tilde{\theta}(P_i) + \left(\sum_{i=1}^{n} \alpha_i\lambda(P_i)\right)I$$

$$= \tilde{\theta}(A) + (\sum \alpha_i\lambda(P_i))I.$$

That is, $\phi(A) - \tilde{\theta}(A) \in Z_N = \mathbf{C}I$ for each $A \in M$. Setting $\phi(A) - \tilde{\theta}(A) = \lambda(A)$ we see that $\lambda(A)$ is a *-linear map from M into Z_n, and

$$\phi(A) = \tilde{\theta}(A) + \lambda(A).$$

A similar argument applies in the θ' case to give

$$\phi(A) = -\tilde{\theta}(A) + \lambda(A).$$

We recall that ϕ was normalized in Lemma 2 and after Lemma 9 so what we have really proved is

$$\frac{1}{c'}\phi = \pm\tilde{\theta} + \lambda$$

where $\tilde{\theta}$ is a C^*-isomorphism. Since a C^*-isomorphism on a factor is either an isomorphism or an anti-isomorphism we have the result.

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