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Lip α APPROXIMATION ON CLOSED SETS WITH lip α EXTENSION

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ABSTRACT. Let F be a relatively closed subset of a domain G in the complex plane. Let f be a function that is the limit, in the Lip α norm on F, of functions which are holomorphic or meromorphic on G $(0 < \alpha < 1)$. We prove that, under the same conditions that give Lip α-approximation $(0 < \alpha < 1)$ on relatively closed subsets of G, it is possible to choose the approximating function m in such a way that $f - m$ can be extended to a function belonging to $\mathrm{lip}(\alpha, \bar{F})$.

1. **Introduction.** If G is an arbitrary open subset of $\mathbb{C}$, H(G) (respectively $M(G)$) will represent the class of all holomorphic (respectively meromorphic) functions in G. If F is a relatively closed subset of G, then $H(F)$ will denote the set of holomorphic functions on a neighborhood of F; $A(F)$ is the algebra of continuous functions on F and holomorphic in the interior F^o of F; and $M_F(G)$ will be the set of all functions in $M(G)$ without poles on F.

Alice Roth in [5] proved that if f is a uniform limit on F of functions belonging to $H(G)$ or $M(G)$, then it is possible to select the approximating function m in such a way that the difference function $m - f$ can be extended continuously into the closure of F, including the points of $\partial F \cap \partial G$ for which f itself has no continuous extension. In this work this result is improved in the following direction. Suppose f is a Lip α limit of functions belonging to $H(G)$ or $M(G)(0 < \alpha < 1)$. Then we will prove that it is possible to choose the approximating function m in such a way that $m - f$ can be extended to a function belonging to $\mathrm{lip}(\alpha, \bar{F})$. In this case we will say that f is LE-approximable by meromorphic or holomorphic functions. So, in Sections 2 and 3 we characterize the LE-approximation by means of the same conditions that characterize the Lip α-approximation $(0 < \alpha < 1)$ on relatively closed subsets of G [1].

In the final section, the LE-approximation by holomorphic functions is also used to obtain some results which we can consider as theorems of decomposition of approximable functions. These results give a description of $[H(G)]^*_{\alpha,F}$, the class of all functions which are lip α limits on F of sequences with elements from $H(G)$. Thus some results of A. Stray [6] are generalized for the Lip α-norm.

2. **Preliminaries.** Let F be a relatively closed subset of a domain G in the complex plane $\mathbb{C}$ and f be a bounded complex function on F. Define the modulus of continuity w_f

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by setting

$$w_f(r) = \sup\{|f(z) - f(w)| : z, w \in F|z - w| \leq r\}$$

For $0 < \alpha < 1$, denote

$$\|f\|_{\alpha,F} = \sup\left\{\frac{w_f(r)}{r^\alpha} : r > 0\right\},$$

$$\mathrm{Lip}(\alpha, F) = \{f : F \longrightarrow \mathbb{C} : \|f\|_{\alpha,F} < \infty\}$$

and

$$\mathrm{lip}(\alpha, F) = \{f \in \mathrm{Lip}(\alpha, F) : r^{-\alpha} w_f(r) \longrightarrow 0, \text{ as } r \longrightarrow 0^+\}.$$

If f is defined on F, we will say that f belongs to $\mathrm{lip}_{\mathrm{loc}}(\alpha, F)$ if $f \in \mathrm{lip}(\alpha, K)$, for every compact subset K of F. The Lip α norm is defined by

$$\|f\|^*_{\alpha,F} = \|f\|_{\alpha,F} + \|f\|_{\infty,F}$$

where $\|f\|_{\infty,F}$ is the supremum norm. With this norm $\mathrm{Lip}(\alpha, F)$ is a Banach algebra.

We define

$$A_\alpha(F) = \{f \in \mathrm{lip}_{\mathrm{loc}}(\alpha, F) : f \in H(F^o)\}$$

and

$$A_{\alpha u}(F) = \{f \in A_\alpha(F) : \exists g \in \mathrm{lip}(\alpha, \bar{F}) \text{ and } g_{|F} = f\}.$$

If A is a class of complex functions on F we will denote $[A]^*_{\alpha,F}$ the set of all functions which are limits in Lip α norm on F of functions belonging to A.

For any subset S of the complex plane $\mathbb{C}$, the interior, the closure, and the boundary of S will be represented by S^o, $\bar{S}$ and ∂S, respectively. Also, $G^* = G \cup \{*\}$ will denote the one-point compactification of G.

Let $h(r)$ be a nonnegative increasing function defined on $[0, \infty)$. For any $E \subset \mathbb{C}$, the Hausdorff content $M_h(E)$ is the infimum of all sums

$$\sum_{D \in S} h(\operatorname{diam} D)$$

where S runs over all countable coverings of E by closed (or open) balls. In case $h(r) = r^\beta$, $0 < \beta < 1$, we write $M_h = M^\beta$, and we will say that $M_h(E)$ is the β-dimensional Hausdorff content of E.

The lower β-dimensional Hausdorff content $M^\beta_*(E)$ is defined by the sup $M_h(E)$, where h runs over all the functions that satisfy

i) $h : [0, \infty) \longrightarrow [0, \infty)$,
ii) h increasing,
iii) $h(r) \leq r^\beta, \forall r \geq 0$,
iv) $h(r) = o(r^\beta)$, $r \longrightarrow 0^+$.

DEFINITION 2.1. The function $f : F \longrightarrow \mathbb{C}^*$ is said to be LE-*approximable* on F by functions from $M(G)\big(H(G)\big)$ if, given $\varepsilon > 0$, there are functions m and e with the following properties:

i) $m \in M(G)\big(H(G)\big)$, $e \in H(F^o) \cap \mathrm{lip}(\alpha, \bar{F})$,
ii) $f(z) - m(z) = e(z)$ $(z \in F)$,
iii) $\|e\|^*_{\alpha,\bar{F}} < \varepsilon$.

Finally, we introduce the notation

$$\|f\|_{\mathrm{Lip}\,1,E} = \sup\left\{\frac{|f(z) - f(w)|}{|z - w|} : z, w \in E\right\}.$$

3. **LE-approximation by meromorphic functions** In this section we shall prove that the necessary and sufficient conditions obtained by J. C. Fariña [1] to hold for the Lip α-approximation of a function f on F by functions from $M_F(G)$ are sufficient for the LE-approximation by meromorphic functions.

THEOREM 3.1. *Let K be a compact subset of $\mathbb{C}^*$, and let g be any function meromorphic in a neighbourhood of K. Then, given any $\varepsilon > 0$, there exists a rational function r such that*

$$\|r - g\|^*_{\alpha,K} < \varepsilon$$

and

$$\|r - g\|_{\mathrm{Lip}\,1,K} < \varepsilon.$$

Moreover, if we fix $u \in K$, we can choose r such that $r(u) = g(u)$.

PROOF. Since that the Lip α contains the uniform norm, this results follows from ([5], Lemma A) and Lip α-Runge theorem ([1], §3) where it was shown that any functions meromorphic in a neighbourhood of K can be approximated by rational functions in the Lip α-norm. Observe that if the function $d(g; z_1, z_2)$ in [5] verifies that $|d(r-g; z_1, z_2)| < \varepsilon$ with $z_1, z_2 \in K$ then $\|r-g\|_{\mathrm{Lip}\,1,K} < \varepsilon$. Moreover, r can be chosen such that $(r-g)(u) = 0$, for u a fixed point in K, replacing r by $r + g(u) - r(u)$.

THEOREM 3.2. *Suppose K_1, K_2 and K are compact sets in the extended complex plane $\mathbb{C}^*$ such that $K_1 \cap K_2 = \emptyset$. If r_1 and r_2 are rational functions with $\|r_1 - r_2\|^*_{\alpha,K} < \varepsilon$, then there exists a constant M, depending only on K_1 and K_2, and a rational function r such that,*

$$\|r - r_i\|^*_{\alpha,K\cup K_i} < M\varepsilon \quad (i = 1, 2) \tag{3.1}$$

and

$$\|r - r_i\|_{\mathrm{Lip}\,1,K_i} < M\varepsilon, \quad (i = 1, 2). \tag{3.2}$$

Moreover, if $u \in K_2$ is a fixed point, we can choose r such that

$$(r - r_2)(u) = 0. \tag{3.3}$$

PROOF. Since the Lip α norm contains the uniform norm, by following the proof of Lip α-Fusion Lemma ([1], Theorem 3) we may obtain the estimations (4′) and (8′) of [5] for the function g used in [1]. Note that this function is the same in the two papers, as well as the function f that is denoted by F in [1]. Hence we must just to apply Theorem 3.1 to get a rational function satisfying (3.1) and (3.2). Moreover to obtain (3.3), we use the next estimations that were obtained in [1] and [5] respectively

$$\|F - r_i\|^*_{\alpha,K\cup K_i} \leq C\varepsilon, \quad i = 1, 2,$$
$$\|F - r_i\|_{\mathrm{Lip}\,1,K_i} < C\varepsilon \quad i = 1, 2,$$

where $F \in M(K_1 \cup K_2 \cup K)$ and C depends only on K_1 and K_2. So, if we make $F_1 = F - (F - r_2)(u)$, then $F_1 \in M(K_1 \cup K_2 \cup K)$,

$$\|F_1 - r_i\|^*_{\alpha,K\cup K_i} \leq 2C\varepsilon, \quad i = 1, 2,$$
$$\|F_1 - r_i\|_{\mathrm{Lip}\,1,K_i} \leq 2C\varepsilon, \quad i = 1, 2,$$

and by means of Theorem 3.1, there exists a rational function r such that

$$\|F_1 - r\|^*_{\alpha,K\cup K_1\cup K_2} \leq \varepsilon,$$
$$\|F_1 - r\|_{\mathrm{Lip}\,1,K\cup K_1\cup K_2} \leq \varepsilon$$

and $r(u) = F_1(u) = r_2(u)$. Hence this function r satisfies (3.1), (3.2) and (3.3) with $M = 2C + 1$ and Theorem 3.2 is proved.

The main result of this section is the next theorem where it is proved that Lip α-approximation by meromorphic functions is equivalent to LE-approximation by meromorphic functions.

THEOREM 3.3. *Let F be a relatively closed subset of a domain G in $\mathbb{C}$. Then the following statements are equivalent:*

(a) f can be approximated in Lip α*-norm on F by functions in $M_F(G)$.*

(b) If K is a compact subset of F then $f_{|K} \in [R(K)]^*_{\alpha,K}$.

(c) f is LE *-approximable on F by functions in $M_F(G)$.*

PROOF. (a) $\Rightarrow$ (b) if f can be approximated on F by functions $g \in M_F(G)$ in Lip α-norm, and if K is a compact subset of F, then each such g is analytic on K and, by Lip α-Runge theorem, f can be approximated on K in Lip α-norm by rational functions.

(c) $\Rightarrow$ (a) is trivial.

(b) $\Rightarrow$ (c) We may suppose that F is bounded because in the general case, let K be a compact subset of F and consider the transformation $w = T(z) = \frac{1}{z - z_o}$ with $z_o \in G \setminus F$. If we define the function $\tilde{f}(w) = f(z)$ in $T(F)$ one has that

$$\tilde{f}_{|T(K)} \in \Big[R\big(T(K)\big)\Big]_{\alpha,T(K)} \tag{3.4}$$

since given $\varepsilon > 0$ and suppose that (b) is satisfied, then there exists a rational function r such that

$$\|f - r\|_{\alpha,K} < \varepsilon. \tag{3.5}$$

By defining $h(z) = f(z) - r(z)$ and $\tilde{h}(w) = \tilde{f}(w) - \tilde{r}(w)$ in the same way as $\tilde{f}(w)$ we have

$$\frac{|\tilde{h}(w_1) - \tilde{h}(w_2)|}{|w_1 - w_2|^\alpha} = \frac{|h(z_1) - h(z_2)|}{|w_1 - w_2|^\alpha}$$
$$= |z_1 - z_0|^\alpha |z_2 - z_0|^\alpha \frac{|h(z_1) - h(z_2)|}{|z_1 - z_2|^\alpha} \leq \Delta_K^{2\alpha} \|h\|_{\alpha,K}$$

where $\Delta_K = \sup\{|z - z_0|, z \in K\}$ and $w_i = \frac{1}{z_i - z_0}$, for $i = 1, 2$. Since Δ_K only depends on K, (3.4) is proved and from the localization theorem for Lip α norms ([1], Theorem 4) we may deduce that

(3.6) $$\tilde{f} \in \left[M_{T(F)}\big(T(G)\big)\right]_{\alpha,T(F)}.$$

Moreover if we have LE-approximation to $\tilde{f}$ in $T(F)$ we also have LE-approximation to f in F.

Let $\{G_n\}_{n=1}^\infty$ be some exhausting sequence of G by bounded domains G_n such that $\bar{G}_n \subset G_{n+1}$, $\bigcup G_n = G$ and $\operatorname{dist}(\partial G_n, \partial F \cap \partial G) = \frac{1}{n}$, where $\partial F \cap \partial G$ is a compact set. For each $n = 1, 2, \ldots$ we can apply Theorem 3.2 to

$$K_1 = \bar{G}_n, \quad K_2 = \mathbb{C}^* \setminus G_{n+1}, \text{ and } F_n = F \cap \bar{G}_{n+1}.$$

And thus, there exist constants A_n that correspond to the constant M in Theorem 3.2. We may assume that the A_n are increasing and that $A_n > 1$.

Let $\varepsilon > 0$. By hypothesis $f_{|F_n} \in [R(F_n)]_{\alpha,F_n}$, hence we can find a rational functions q_n without poles on F_n such that

(3.7) $$\|f - q_n\|^*_{\alpha,F_n} < \frac{\varepsilon}{2^{n+1}A_n(n+1)}$$

for $n = 1, 2, \ldots$. Since $F_n \subset F_{n+1}, n = 1, 2, \ldots,$ we have:

$$\|q_{n+1} - q_n\|^*_{\alpha,F_n} < \frac{\varepsilon}{2^n A_n(n+1)}.$$

By Theorem 3.2 for each n there exists a rational function r_n which satisfies

(3.8) $$\|r_n - q_n\|^*_{\alpha,F_n \cup K_1} < \frac{\varepsilon}{2^n(n+1)},$$

(3.9) $$\|r_n - q_{n+1}\|^*_{\alpha,F_n \cup K_2} < \frac{\varepsilon}{2^n(n+1)},$$

(3.10) $$\|r_n - q_n\|_{\mathrm{Lip}\,1,K_1} < \frac{\varepsilon}{2^n(n+1)},$$

(3.11) $$\|r_n - q_{n+1}\|_{\mathrm{Lip}\,1,K_2} < \frac{\varepsilon}{2^n(n+1)},$$

and

$$(r_n - q_{n+1})(u) = 0 \text{ for } u \in \partial F \cap \partial G \text{ fixed},$$

Moreover (3.9) implies Lip α convergence, on $\mathbb{C}^* \setminus G$ of

$$g_n(z) = \sum_{k=1}^{n-1} \big(r_k(z) - q_{k+1}(z)\big)$$

as $n \longrightarrow \infty$. On $\partial F \cap \partial G$, since $\mathrm{lip}(\alpha, \partial F \cap \partial G)$ is a closed subalgebra, g_n converges to a function $\phi \in \mathrm{lip}(\alpha, \partial F \cap \partial G)$. On the other hand if $z \in F_n \setminus F_{n-1}$ and $t \in \partial F \cap \partial G$, since $F_n \setminus F_{n-1} \subset \mathbb{C}^* \setminus G_n$ one has

$$\|g_n\|_{\mathrm{Lip}\,1, \mathbb{C}^* \setminus G_n} \leq \sum_{1}^{n-1} \frac{\varepsilon}{2^\nu(\nu+1)} < \varepsilon$$

and hence

$$|g_n(t) - g_n(z)| < |t - z|\varepsilon. \tag{3.12}$$

Now we define

$$m(z) = \sum_{k=1}^{n-1} (r_k - q_{k+1}) + q_n + \sum_{k=n}^{\infty} (r_k - q_k).$$

From the proof of Localization Theorem for Lip α norm ([1], Theorem 4) $m(z) \in M_F(G)$, and

$$\|m - f\|^*_{\alpha, F} < \varepsilon. \tag{3.13}$$

Finally we will show that if the function $e(z)$ is defined by

$$e(z) = \begin{cases} f(z) - m(z) & \text{if } z \in F \\ \phi(t) & \text{if } z \in \bar{F} \setminus F \end{cases}$$

then $e \in \mathrm{lip}(\alpha, \bar{F})$. For this observe that trivially $e \in \mathrm{Lip}(\alpha, \bar{F}) \cap \mathrm{lip}_{\mathrm{loc}}(\alpha, F \cap G)$, so if K is a compact subset of G, $e \in \mathrm{lip}(\alpha, K \cap F)$. Now, let $\varepsilon_1 > 0$ be an arbitrary constant. There exists $m_0 \in \mathbb{N}$ such that, if $M = |\partial F \cap \partial G|$ is the diameter of $\partial F \cap \partial G$ and $C = \max\{M^\alpha, 1\}$ then

a) $\frac{\varepsilon}{2^{m_0}(m_0+1)} < \frac{\varepsilon_1}{6}$,
b) $\frac{C\varepsilon}{(m_0+1)^{1-\alpha}} < \frac{\varepsilon_1}{6}$.

On the one hand $e \in \mathrm{lip}(\alpha, F \cap \bar{G}_{m_0+2})$ and $e \in \mathrm{lip}(\alpha, \partial F \cap \partial G)$. Thus there are two constants δ_1, δ_2 such that if $z, u \in F \cap G_{m_0+2}$ with $|z - u| < \delta_1$ then

$$\frac{|e(z) - e(u)|}{|z - u|^\alpha} < \epsilon_1.$$

The same holds if $z, u \in \partial F \cap \partial G$ and $|z - u| < \delta_2$.

On the other hand, we can choose δ such that

$$\delta = \min\Big\{\mathrm{dist}(\partial G_{m_0+1}, \partial G_{m_0+2}), \frac{1}{m_0+3}, \Big(\frac{\varepsilon_1}{6\varepsilon}\Big)^{\frac{1}{1-\alpha}}, \delta_1, \delta_2\Big\}$$

and we must show that

$$\frac{|e(z) - e(u)|}{|z - u|^\alpha} < \varepsilon_1$$

whenever $|z-u|<\delta$ with $z,u\in\bar{F}$.

Firstly consider the case $z,u\in F\cap(\mathbb{C}\setminus G_{m_0+2})$, and $|z-u|<\delta$. We can also suppose that $z\in G_{m^*}\setminus G_{m^*-1}$ and $u\in G_{m_0^*}\setminus G_{m_0^*-1}$ with $m^*\geq m_0^*>m_0+2$ and we distinguish three cases:

i) $m^*>m_0^*+1$

$$e(z)=\sum_{k=1}^{m_0^*-2}(r_k-q_{k+1})+\sum_{k=m_0^*-1}^{m^*-2}(r_k-q_{k+1})+(q_{m^*-1}-f)+\sum_{k=m^*-1}^{\infty}(r_k-q_k)$$

so, by (3.11), (3.9), (3.7) and (3.8) respectively

$$\frac{|e(z)-e(u)|}{|z-u|^\alpha}<\varepsilon|z-u|^{1-\alpha}+\frac{\varepsilon_1}{6}+\frac{\varepsilon_1}{6}+\frac{\varepsilon_1}{6}<\varepsilon_1.$$

Analogous estimations hold for the other cases by writing $e(z)$ as follows

ii) $m^*=m_0^*$

$$e(z)=\sum_{k=1}^{m_0^*-2}(r_k-q_{k+1})+(r_{m_0^*-1}-q_{m_0^*})+(q_{m_0^*}-f)+\sum_{k=m_0^*}^{\infty}(r_k-q_k)$$

iii) $m^*=m_0^*+1$

$$e(z)=\sum_{k=1}^{m_0^*-2}(r_k-q_{k+1})+(r_{m_0^*-1}-q_{m_0^*})+(q_{m_0^*}-f)+\sum_{k=m_0^*}^{\infty}(r_k-q_k).$$

Note that if $z\in\mathbb{C}\setminus G_{m_0+2}$ and $u\in G_{m_0+2}$ with $|z-u|<\delta$ then $u\in G_{m_0+2}\setminus G_{m_0+1}$ necessarily, and we can proceed as in ii).

Finally, if $t\in\partial F\cap\partial G$, $z\in F\cap G$ and $|z-t|<\delta$ then there exists $m_0^*>m_0+2$ such that $z\in F_{m_0^*}\setminus F_{m_0^*-1}$ and we have

$$\begin{aligned}\frac{|m(z)-f(z)-\phi(t)|}{|z-t|^\alpha}&\leq\frac{|g_{m_0^*}(z)-\phi(t)|}{|z-t|^\alpha}+\frac{|q_{m_0^*}(z)-f(z)|}{|z-t|^\alpha}+\sum_{k=m_0^*}^{\infty}\frac{|r_k(z)-q_k(z)|}{|z-t|^\alpha}\\&\leq\frac{|g_{m_0^*}(z)-g_{m_0^*}(t)|}{|z-t|^\alpha}+\frac{|g_{m_0^*}(t)-\phi(t)|}{|z-t|^\alpha}+\frac{|q_{m_0^*}(z)-f(z)|}{|z-t|^\alpha}\\&\quad+\sum_{k=m_0^*}^{\infty}\frac{|r_k(z)-q_k(z)|}{|z-t|^\alpha}\leq(I)+(II)+(III)+(IV)\end{aligned}$$

From (3.12), (3.7) and (3.9) and by considering that $\operatorname{dist}(\partial G_n,\partial F\cap\partial G)=\frac{1}{n}$

$$(I)<\varepsilon|t-z|^{1-\alpha}<\varepsilon\delta^{1-\alpha}<\frac{\varepsilon_1}{6},$$

$$(III)<\frac{\varepsilon}{2^{m_0^*}}\left(\frac{1}{m_0^*+1}\right)^{1-\alpha}<\frac{\varepsilon_1}{6},$$

$$(IV)<\sum_{k=m_0^*}\frac{\varepsilon}{2^k}\frac{(m_0^*+1)^\alpha}{k+1}<\left(\frac{1}{m_0^*+1}\right)^{1-\alpha}\sum_{k=m_0^*}^{\infty}\frac{\varepsilon}{2^k}<\left(\frac{1}{m_0^*+1}\right)^{1-\alpha}\varepsilon<\frac{\varepsilon_1}{6}.$$

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To estimate (II), recall that $g_m(u_0)$ and hence $\phi(u_0)$ vanishes for u_0 belonging to $\dot{\partial} F \cap \partial G$, so if we denote $g_m - \phi$ by h_m, then

$$(II) = \frac{|h_{m_0^*}(t)|}{|z-t|^\alpha} \leq \|h_{m_0^*}\|_{\alpha,\partial G}\left(\frac{|t-u_0|^\alpha}{|z-t|^\alpha}\right) \leq \|h_{m_0^*}\|_{\alpha,\partial G}\left(|\partial F \cap \partial G|^\alpha (m_0^*+1)^\alpha\right)$$

where

$$\begin{aligned}\|h_{m_0^*}\|_{\alpha,\partial G} &= \|g_{m_0^*} - \phi\|_{\alpha,\partial G} = \Big\|\sum_{k=m_0^*}^{\infty}(r_k - q_{k+1})\Big\|_{\alpha,\partial G} \\ &< \sum_{k=m_0^*}^{\infty}\frac{\varepsilon}{2^k}\frac{1}{k+1} < \frac{1}{m_0^*+1}\sum_{k=m_0^*}\frac{\varepsilon}{2^k}\end{aligned}$$

and

$$(II) < \frac{C}{(m_0^*+1)^{1-\alpha}}\varepsilon < \frac{\varepsilon_1}{6}$$

where C is independent of n, hence

$$\frac{|e(z)-e(t)|}{|z-t|^\alpha} = \frac{|m(z)-f(z)-\phi(t)|}{|z-t|^\alpha} < \varepsilon_1.$$

This completes the proof.

Since the Lip α-approximation of $A_\alpha(F)$ by $M_F(G)$ can be characterized by means of Hausdorff content ([1], Theorem 4), we have the next corollary

COROLLARY 3.4. *All functions of $A_\alpha(F)$ can be* LE*-approximated by functions in $M_F(G)$ if and only if there exists a constant $C \geq 0$ such that,*

$$M_*^{1+\alpha}(\Delta \setminus F^0) \leq CM^{1+\alpha}(\Delta \setminus F)$$

for every disc $\Delta \subset G$.

4. **LE-approximation by holomorphic functions.** Our goal is now to approximate $f \in A_\alpha(F)$ by functions $g \in H(G)$. The holomorphic approximation in Lip α-norm is characterized by the following theorem.

THEOREM 4.1 (ARAKELYAN'S THEOREM IN Lip α-NORM ([1], THEOREM 13)). *Let F be a relatively closed subset of G. All functions f belonging to $A_\alpha(F)$ can be approximated in* Lip α*-norm on F by holomorphic functions in G if, and only if,*

i) $G^* \setminus F$ is connected and locally connected at $\{*\}$.

ii) There exists a constant $C \geq 0$ such that,

$$M_*^{1+\alpha}(\Delta \setminus F^o) \leq CM^{1+\alpha}(\Delta \setminus F)$$

for every disc $\Delta \subset G$.

Here, just like in the previous section, we will prove that the same conditions i) and ii) characterize the LE-approximation by holomorphic functions. For this observe that ii) is sufficient for the LE-approximation by meromorphic functions. Hence it is enough to move the poles of these functions to the boundary of G, by preserving the approximation itself. These facts are collected in the next lemma.

LEMMA 4.2. *If i) is satisfied and if m is a function in $M_F(G)$, the restriction $m_{|F}$ is* LE*-approximable on F by functions in $H(G)$.*

PROOF. By i), G can be exhausted by domains $G_1, G_2, \ldots$ such that every component of $G \setminus (F \cup \bar{G}_n)$ extends to the boundary of G. So if $m_1 \in M_F(G)$ has a pole at z_1 and z_2 belongs to the same component of $G \setminus (F \cup \bar{G}_n)$ as z_1, then there exists a function $m_2 \in M_F(G)$ which has a pole in z_2 instead of z_1 and the other poles of m_2, different from z_1, are just the remaining poles of m_1. Moreover $m_1 - m_2$ is rational with exactly two poles, z_1 and z_2 ([1], Lemma 9).

Now, by taking into account ([1], Theorem 11), let $\varepsilon > 0$ be given and start with $m_0 = m$ and $G_0 = \emptyset$. Determine successively functions $m_1, m_2, \ldots$ from $M_F(G)$ such that the poles of $m_n \notin F \cup \bar{G}_n$, $m_n - m_{n-1}$ is rational,

$$\|m_n - m_{n-1}\|^*_{\alpha, \bar{F} \cup \bar{G}_{n-1}} < \frac{\varepsilon}{2^n}, \tag{4.14}$$

and $m_n - m_{n-1}$ belongs to $\mathrm{lip}(\alpha, \bar{F})$.

Thus if we define

$$g = \lim_{n \to \infty} m_n = m_N + \sum_{n=N}^{\infty} (m_n - m_{n-1})$$

one has easily that g is holomorphic in G. Furthermore, since $\mathrm{lip}(\alpha, \bar{F})$ is a closed subalgebra of $\mathrm{Lip}(\alpha, \bar{F})$, by (4.14) the difference

$$e = g - m = \sum_{n=1}^{\infty} (m_n - m_{n-1})$$

belongs to $\mathrm{lip}(\alpha, \bar{F})$. Finally we also have from (3.1) that

$$\|e\|_{\alpha, \bar{F}} \leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} < \varepsilon.$$

THEOREM 4.3. *If i) and ii) of Theorem 4.1 hold, then all functions of $A_\alpha(F)$ can be* LE*-approximated on F by functions of $H(G)$.*

PROOF. The condition ii) is sufficient for Corollary 3.4 to hold. So if $f \in A_\alpha(F)$ then f is LE-approximable by meromorphic functions and by (i) we can apply Lemma 4.2 and the theorem is proved.

An easy consequence of this theorem gives us the first result of decomposition for the class $[H(G)]^*_{\alpha,F}$.

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COROLLARY 4.4. *Let F be a relatively closed subset of G, verifying i) and ii). Then if $\varepsilon > 0$ and $g \in A_\alpha(F)$, there exist $g_1 \in A_{\alpha u}(F)$ and $g_2 \in H(G)$ such that $\|g_1\|^*_{\alpha,F} < \varepsilon$ and $g = g_1 + g_2$ on F, i.e. $[H(G)]^*_{\alpha,F} = A_{\alpha u}(F) + H(G)$.*

The following question arises immediately. Is the above decomposition true when some condition fails? We don't know if the answer is in general affirmative but if we remove the fact that $G^* \setminus F$ is connected the decomposition is still possible.

Denote $\hat{F} = F \cup \{$ all relatively compact connected component of $G^* \setminus F\}$.

THEOREM 4.5. *Let F be a relatively closed subset of G such that $G^* \setminus \hat{F}$ locally connected at $\{*\}$ and suppose that there exists a constant $C > 0$ such that,*

$$M_*^{1+\alpha}\big(\Delta \setminus (\hat{F})^0\big) \le CM^{1+\alpha}\big(\Delta \setminus (\hat{F})\big) \tag{4.15}$$

for every disc $\Delta \subset G$. Then

$$[H(G)]^*_{\alpha,F} = A_{\alpha u}(\hat{F}) + H(G).$$

PROOF. Let H_i be a relatively compact connected component of $G \setminus F$, and consider $f \in [H(G)]^*_{\alpha,F}$; then given a sequence $\epsilon_n > 0$ with $\epsilon_n \longrightarrow 0$ as $n \longrightarrow \infty$ there exists a holomorphic function f_n in G, such that

$$\|f - f_n\|^*_{\alpha,F} < \varepsilon_n.$$

Therefore

$$\|f - f_n\|_{\infty,\partial H_i} < \varepsilon_n$$

and f_n converges to a function $\bar{f}$ on H_i with the supremum norm. Note also that $\bar{f}$ agrees f in ∂H_i. Moreover, $\bar{f} - f_n \in A(\bar{H}_i)$ where H_i is bounded. Now by taking into account the estimations in [2],

$$\|\bar{f} - f_n\|_{\alpha,H_i} < \|\bar{f} - f_n\|_{\alpha,\partial H_i} < \varepsilon_n.$$

Thus, one has

$$\|\bar{f} - f_n\|_{\alpha,F\cup H_i} < 2\varepsilon_n$$

and

$$\|\bar{f} - f_n\|_{\alpha,\hat{F}} < 3\varepsilon_n.$$

Hence

$$\|\bar{f} - f_n\|^*_{\alpha,\hat{F}} < 4\varepsilon_n$$

and we can affirm that if $f \in [H(G)]^*_{\alpha,\hat{F}}$, f has an extension $\bar{f}$ to $\hat{F}$ such that

$$\bar{f} \in [H(G)]^*_{\alpha,\hat{F}}.$$

Finally note that $\hat{F}$ satisfies (4.15) and $G^* \setminus \hat{F}$ is connected and locally connected, thus from Corollary 4.4 $\bar{f} = g_1 + g_2$ on $\hat{F}$ with $g_1 \in A_{\alpha u}(\hat{F})$ and $g_2 \in H(G)$, and hence $f = g_1 + g_2$ on F.

It is important to observe that the condition ii) in Corollary 4.4 is only used to guarantee that f may be approximated by meromorphic functions, in this case we know already that $\bar{f} \in [H(G)]^*_{\alpha,\hat{F}}$, hence (4.15) is not necessary to prove

$$[H(G)]^*_{\alpha,F} \subset A_{\alpha u}(\hat{F}) + H(G).$$

On the other hand $A_{\alpha u}(\hat{F}) \subset [H(G)]^*_{\alpha,F}$ because $\hat{F}$ satisfies the hypothesis of Theorem 4.1 and $A_{\alpha u}(\hat{F}) \subset A_\alpha(\hat{F})$. This completes the proof.

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