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CONGRUENCES AND GREEN'S RELATIONS ON EVENTUALLY REGULAR SEMIGROUPS

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Abstract

A semigroup is eventually regular if each of its elements has some power that is regular. Let $\mathscr{K}$ be one of Green's relations and let ρ be a congruence on an eventually regular semigroup S. It is shown for $\mathscr{K} = \mathscr{L}$, $\mathscr{R}$ and $\mathscr{D}$ that if A and B are regular elements of S/ρ that are $\mathscr{K}$-related in S/ρ then there exist elements $a \in A$, $b \in B$ such that a and b are $\mathscr{K}$-related in S. The result is not true for $\mathscr{H}$ or $\mathscr{J}$.

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1. Introduction

It is of interest to know what information can be obtained about a semigroup from its homomorphic images. Thus this paper was motivated by the work of Hall [3]. Sometimes it is possible to reconstruct a semigroup from its homomorphic image, often because of the similar properties of the semigroup and its image. A semigroup is eventually regular if each of its elements has some power that is regular [2]. The class of all eventually regular semigroups contains all regular semigroups and all group-bound (whence all finite) semigroups. Let $\mathscr{K}$ be one of Green's relations and let ρ be a congruence on a semigroup S. The following question will be considered:

QUESTION. *If A and B are regular elements of S/ρ that are $\mathscr{K}$-related in S/ρ, then are there elements $a \in A$, $b \in B$ such that a and b are $\mathscr{K}$-related in S?*

This question was studied for the case of S a regular semigroup by Hall in [3]. He found that the answer is always yes for $\mathscr{L}$, $\mathscr{R}$ and $\mathscr{D}$. There are regular semigroups for which the answer is no for $\mathscr{H}$ and for $\mathscr{J}$. Rhodes [4] shows that for finite semigroups (so $\mathscr{D} = \mathscr{J}$) and for all elements of S/ρ, the answer is always yes for $\mathscr{L}$, $\mathscr{R}$ and $\mathscr{D}$ and for finite regular semigroups the answer is always yes for $\mathscr{H}$.

In [3] there is a sequence of theorems, corollaries and lemmas, numbered consecutively from 1 to 15. Let us denote these by R1, R2, R3, ..., R15, with the number being the same as in [3]. In Section 3 of this paper R1, R2, R3, R4, R5, R6, R9, R10 and R11 are extended from regular to eventually regular semigroups. Result R15 has already been extended from regular to eventually regular semigroups, essentially as Theorem 3 of [2]. Results R7, R8 and R12 are false for the two element null semigroup and R13 and R14 are statements about arbitrary semigroups. The methods used in Section 3 are basically those of [3] with eventual regularity being used to inject sufficient regularity into the situation to enable modified proofs to hold.

2. Preliminaries

Whenever possible the notations and conventions of Clifford and Preston [1] will be used. If x is an element of a semigroup S then $V(x) = \{y \in S \mid xyx = x$ and $yxy = y\}$ will denote the set of inverses of x in S. A semigroup S is called idempotent-surjective if, for all congruences on S, every idempotent congruence class contains an idempotent [2]. It is well known that regular semigroups are idempotent-surjective (Lallement's Lemma) and it follows from [2, Corollary 2] that eventually regular semigroups are idempotent-surjective.

The following result is well known and will be used in Section 3.

RESULT 1. *Let a and b be elements of a semigroup S such that a is a regular element. Then $L_b \leqslant L_a$ if and only if $b = ba'a$ for each inverse a' of a.*

3. Green's relations and congruences

The next lemma is a slight modification of [2, Theorem 1] and is the foundation upon which most of the results of this section are based. It shows that regularity in S/ρ can be "pulled back" to S and from this it is shown (Theorems 5 and 8) that being $\mathscr{L}$ or $\mathscr{D}$-related in S/ρ can also be "pulled back" to S.

Lemma 1. *Let ρ be a congruence on an eventually regular semigroup S and let A, B be elements of S/ρ such that $B \in V(A)$ in S/ρ. Then there exist elements a, b in S such that $a \in A$, $b \in B$ and $b \in V(a)$ in S. Furthermore, if A is an idempotent of S/ρ then a can be chosen to be an idempotent also.*

Proof. Take any elements $x \in A$, $y \in B$. Since S is eventually regular there exists an integer $n > 1$ such that $((xy)^2)^n$ is regular. Let $z \in V((xy)^{2n})$, let $Z = z\rho$ and put $a = (xy)^{2n-1}zxyx$ and $b = y(xy)^{2n-2}zxy$. It is easy to verify that $b \in V(a)$ and that $Z \in V(AB)$. Thus $a \in (AB)^{2n-1}ZABA = ABZABA = ABA = A$ and $b \in B(AB)^{2n-2}ZAB = BABZAB = BAB = B$. Finally, if A is an idempotent of S/ρ then by [2, Corollary 2], x can be chosen to be idempotent and then $a = a^2$.

Remark 1. If both A and B in Lemma 1 above are idempotent then as mentioned in [3] it is not always possible to choose both a and b idempotent and satisfying the conditions of the lemma.

Theorem 2. *Let ρ be a congruence on an eventually regular semigroup S and let n be a positive integer. If $A, B_1, B_2, \ldots, B_n$ are elements of S/ρ such that $B_i \in V(A)$ for $i = 1, 2, \ldots, n$, then there exist elements $a, b_1, b_2, \ldots, b_n$ in S such that $a \in A$, $b_i \in B_i$, and $b_i \in V(a)$, for $i = 1, 2, \ldots, n$. Further, if $A[B_1]$ is an idempotent of S/ρ, then the elements $a[b_1]$ can be chosen to be an idempotent also.*

Proof (by induction). If $n = 1$ the result follows from Lemma 1. Suppose that all of the assertions of the theorem are true for some positive integer $k < n$. Therefore there exist elements $x, y_1, y_2, \ldots, y_k$ in S such that $x \in A$, $y_i \in B_i$, $y_i \in V(x)$ for $i = 1, 2, \ldots, k$. Take any element $y_{k+1} \in B_{k+1}$. Since S is eventually regular there exists an integer $m > 1$ such that $((xy_{k+1})^2)^m$ is regular. Let $z \in V((xy_{k+1})^{2m})$ and put $a = (xy_{k+1})^{2m-1}zxy_{k+1}x$, $b_{k+1} = y_{k+1}(xy_{k+1})^{2m-2}zxy_{k+1}$ and for $i = 1, 2, \ldots, k$ put $b_i = y_ixb_{k+1}xy_i$. It is a routine matter to verify that $a \in A$, $b_i \in B_i$, $b_i \in V(a)$ for $i = 1, 2, \ldots, k+1$. If A is an idempotent of S/ρ, then by the inductive hypothesis x can be taken to be an idempotent and then $a = a^2$. Similarly, if B_1 is idempotent then b_1 can be chosen to be idempotent. Recall from Remark 1 that we cannot always make a and b_1 idempotent simultaneously. The result now follows by induction.

Remark 2 (from [3]). If B_1 and B_2 in Theorem 2 above are idempotents then in general b_1 and b_2 above cannot both be taken to be idempotents. Also if A_1, A_2, B_1, B_2 are elements of S/ρ such that $B_i \in V(A_j)$ for $i, j = 1, 2$ then there are not necessarily $a_j \in A_j$, $b_i \in B_i$ $(i, j = 1, 2)$ such that $b_i \in V(a_j)$ for $i, j = 1, 2$.

COROLLARY 3. *If m is a positive integer and S is an eventually regular semigroup such that each element [idempotent] has at most m inverses, then the same is true for any morphic image of S.*

Thus in particular we have as a corollary the well known result that any morphic image of an inverse semigroup is an inverse semigroup.

LEMMA 4. *Let ρ be a congruence on an eventually regular semigroup S and let $A_1, A_2, \ldots, A_n$ be any regular elements of S/ρ such that $A_1 \mathscr{L} A_2 \mathscr{L} \ldots \mathscr{L} A_n$ in S/ρ. For $i = 1, 2, \ldots, n$ let A_i' be an inverse of A_i in S/ρ. Then there exist elements $a_1, a_2, \ldots, a_n, a_1', a_2', \ldots, a_n'$ in S such that $a_1 \mathscr{L} a_2 \mathscr{L} \ldots \mathscr{L} a_n$ and such that for $i = 1, 2, \ldots, n$, $a_i \in A_i$, $a_i' \in A_i'$ and $a_i' \in V(a_i)$.*

PROOF (by induction). If $n = 1$ the result follows from Lemma 1. Suppose that all of the assertions of the lemma are true for some positive integer $k < n$. Therefore there exist elements $x_1, x_2, \ldots, x_k, x_1', x_2', \ldots, x_k'$ in S such that $x_1 \mathscr{L} x_2 \mathscr{L} \ldots \mathscr{L} x_k$ in S and such that for $i = 1, 2, \ldots, k$, $x_i \in A_i$, $x_i' \in A_i'$ and $x_i' \in V(x_i)$. By Lemma 1 there exist x_{k+1}, x_{k+1}' in S such that $x_{k+1} \in A_{k+1}$, $x_{k+1}' \in A_{k+1}'$ and $x_{k+1}' \in V(x_{k+1})$. Since S is eventually regular there exists an integer $m > 1$ such that $(x_1'x_1x_{k+1}'x_{k+1})^m$ is regular. Let $z \in V((x_1'x_1x_{k+1}'x_{k+1})^m)$ and for $i = 1, 2, \ldots, k+1$ put $u_i = x_ix_{k+1}'x_{k+1}z(x_1'x_1x_{k+1}'x_{k+1})^{m-1}x_1'x_1$ and put $u_i' = x_i'u_ix_i'$. By Result 1, $x_ix_j'x_j = x_i$ for $i, j = 1, 2, \ldots, k$. It follows from this that $u_i' \in V(u_i)$ for $i = 1, 2, \ldots, k+1$, that $x_{i-1}x_i'u_i = u_{k-1}$ for $i = 2, 3, \ldots, k+1$ and that $x_{k+1}z(x_1'x_1x_{k+1}'x_{k+1})^{m-1}x_1'u_1 = u_{k+1}$, whence $u_1 \mathscr{L} u_2 \mathscr{L} \ldots \mathscr{L} u_{k+1}$. Let $Z = z\rho$. It is clear that $Z \in V((A_1'A_1A_{k+1}'A_{k+1})^m) = V((A_1'A_1)^m)$ by Result 1, whence $Z \in V(A_1'A_1)$. Then for $i = 1, 2, \ldots, k+1$, by Result 1, we have

$$\begin{aligned} u_i &\in A_iA_{k+1}'A_{k+1}Z\left(A_1'A_1A_{k+1}'A_{k+1}\right)^{m-1}A_1'A_1 \\ &= A_iZA_1'A_1 = A_iA_1'A_1ZA_1'A_1 = A_iA_1'A_1 = A_i \end{aligned}$$

and $u_i' \in A_i'A_iA_i' = A_i'$.

The result follows by induction.

THEOREM 5. *Let ρ be a congruence on an eventually regular semigroup S. Let $A_1, A_2, \ldots, A_n$ be any regular elements of S/ρ such that $A_1 \mathscr{L} A_2 \mathscr{L} \ldots \mathscr{L} A_n$ in S/ρ. Then there exist regular elements $a_1, a_2, \ldots, a_n$ in S such that*

(i) $a_i \in A_i$ $(i = 1, 2, \ldots, n)$,

(ii) $a_1 \mathscr{L} a_2 \mathscr{L} \ldots \mathscr{L} a_n$,

(iii) *a_i is an idempotent of S if A_i is an idempotent of S/ρ $(i = 1, 2, \ldots, n)$.*

Proof. Just use Lemma 4 in Hall's proof of [3, Theorem 5].

The corresponding result to Theorem 5 for $\mathscr{R}$ is immediate by duality.

Corollary 6. *If m is a positive integer and S is an eventually regular semigroup such that each regular $\mathscr{L}$-class of S contains at most m elements [idempotents, $\mathscr{H}$-classes], then the same is true for any morphic image of S.*

Lemma 7. *Let ρ be a congruence on an eventually regular semigroup S. Take any positive integers n, m and for $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$ let A_i, A'_i, B_j, C_j be (regular) elements of S/ρ such that for all i and j, $A_i \mathscr{L} B_j$, $A'_i \in V(A_i)$ and $B_j \mathscr{R} C_j$ in S/ρ. Then for each i and j there exist elements $a_i \in A_i$, $b_j \in B_j$, $a'_i \in A'_i$, $c_j \in C_j$ such that for all i and j, $a_i \mathscr{L} b_j$, $a'_i \in V(a_i)$ and $b_j \mathscr{R} c_j$ in S.*

Proof. From Lemma 4, for $i = 1,2,\ldots,n$ there exist $x_i \in A_i$, $x'_i \in A'_i$ such that $x'_i \in V(x_i)$ and such that $x_1 \mathscr{L} x_2 \mathscr{L} \ldots \mathscr{L} x_n$. From the dual of Lemma 4 there exist, for $j = 1,2,\ldots,m$, $y_j \in B_j$, $w_j \in C_j$ such that $y_j \mathscr{R} w_j$, and y_j is regular. For each $j = 1,2,\ldots,m$ let y'_j be an inverse of y_j, put $y'_j\rho = B'_j$ and let v_j be an element from S^1 such that $w_j v_j = y_j$. Put $d = (x'_1 x_1)(y'_1 y_1)(y'_2 y_2)\cdots(y'_m y_m)$. Since S is eventually regular there exists $k > 1$ such that d^k is regular. Let $z \in V(d^k)$ and for $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$ put

$$a_i = x_i\big(y'_1 y_1\big)\big(y'_2 y_2\big)\ldots\big(y'_m y_m\big)d^{k-1} z x'_1 x_1, \qquad a'_i = x'_i a_i x'_i,$$
$$b_j = y_j\big(y'_{j+1} y_{j+1}\big)\big(y'_{j+2} y_{j+2}\big)\ldots\big(y'_m y_m\big)d^{k-1} z x'_1 x_1,$$
$$c_j = b_j\big(y'_1 y_1\big)\big(y'_2 y_2\big)\ldots\big(y'_{j-1} y_{j-1}\big)y'_j w_j.$$

It is a routine matter to verify (using Result 1) that $a'_i \in V(a_i)$, $y_{j-1} y'_j b_j = b_{j-1}$ $(j = 2,3,\ldots,m)$, $x_n y'_1 b_1 = a_n$, $x_{i-1} x'_i a_i = a_{i-1}$ $(i = 2,3,\ldots,n)$, $y_m d^{k-1} z x'_1 a_1 = b_m$ and $c_j v_j (y'_{j+1} y_{j+1})\ldots(y'_m y_m) d^{k-1} z x'_1 x_1 = b_j$ $(j = 1,2,\ldots,m)$. It now follows that, for each i and j, $a_i \mathscr{L} b_j$ and $b_j \mathscr{R} c_j$. Now $d \in A'_1 A_1$ by Result 1, whence $d^k \in A'_1 A_1$ and so $z\rho = Z$ (say) $\in V(A'_1 A_1)$. Thus using Result 1, we have

$$a_i \in A_i Z A'_1 A_1 = A_i A'_1 A_1 Z A'_1 A_1 = A_i A'_1 A_1 = A_i,$$
$$a'_i \in A'_i A_i A'_i = A'_i,$$
$$b_j \in B_j Z A'_1 A_1 = B_j A'_1 A_1 Z A'_1 A_1 = B_j A'_1 A_1 = B_j,$$

and

$$c_j \in B_j B'_j C_j = C_j, \quad \text{by the dual of Result 1.}$$

The assertions of the lemma have now been proved.

THEOREM 8. *Let ρ be a congruence on an eventually regular semigroup S. Let $A_1, A_2, \ldots, A_n$ be any regular elements of S/ρ such that $A_1 \mathscr{D} A_2 \mathscr{D} \ldots \mathscr{D} A_n$ in S/ρ. Then there exist regular elements $a_1, a_2, \ldots, a_n$ in S such that $a_i \in A_i$ for $i = 1, 2, \ldots, n$ and such that*

(i) $a_1 \mathscr{D} a_2 \mathscr{D} \ldots \mathscr{D} a_n$,

(ii) *a_i is an idempotent of S if A_i is an idempotent of S/ρ $(i = 1, 2, \ldots, n)$.*

PROOF. Just use Lemma 7 in Hall's proof of [3, Theorem 10].

From Theorem 8, Corollary 6 and its dual we have

COROLLARY 9. *If m is a positive integer and S is an eventually regular semigroup such that each regular $\mathscr{D}$-class of S contains at most m elements [idempotents, $\mathscr{L}$-classes, $\mathscr{H}$-classes], then the same is true for any morphic image of S.*

References

[1] A. H. Clifford and G. B. Preston, '*The algebraic theory of semigroups*', (Math. Surveys No. 7, Amer. Math. Soc., Providence, R. I., Vol. I, 1961; Vol. II, 1967).

[2] P. M. Edwards, 'Eventually regular semigroups', *Bull. Austral. Math. Soc.* **28** (1983), 23–38.

[3] T. E. Hall, 'Congruences and Green's relations on regular semigroups', *Glasgow Math. J.* **13** (1972), 167–175.

[4] J. Rhodes, 'Some results on finite semigroups', *J. Algebra* **4** (1966), 471–504.

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