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PULLBACK-FLAT ACTS ARE STRONGLY FLAT

Dedicated to the memory of Professor Kenneth P. McDowell

SYDNEY BULMAN-FLEMING

ABSTRACT. Let S be a monoid. A right S-system A is called *strongly flat* if the functor $A \otimes -$ (from the category of left S-systems into the category of sets) preserves pullbacks and equalizers. (This concept arises in B. Stenström, Math. Nachr. **48**(1971), 315–334 under the name *weak flatness*). The main result of the present paper is a proof that for A to be strongly flat it is in fact sufficient that $A \otimes -$ preserve only pullbacks. The approach taken is to develop an "interpolation" condition for pullback-preservation, and then to show its equivalence to Stenström's conditions for strong flatness.

Introduction. In 1971 B. Stenström [5] studied right S-systems A having the property that the functor $A \otimes -$ preserves pullbacks and equalizers: in the terminology of P. Normak [3] such an S-system is called pullback-flat and equalizer-flat or, equivalently, strongly flat.

In this paper we demonstrate that every pullback-flat S-system is in fact strongly flat, resolving a question left unanswered in [3]. The approach taken is to first show that A is pullback-flat if and only if it satisfies the condition

(PF) If $as = a's'$ and $at = a't'$ for $a, a' \in A$, $s, s', t, t' \in S$, then $a = a''u$, $a' = a''v$, $us = vs'$, $ut = vt'$ for some $a'' \in A$ and $u, v \in S$.

We then show that (PF) is equivalent to the conjunction of Stenström's conditions (called (P) and (E) in [3]) which characterize strong flatness.

1. **Preliminaries.** Let S be a monoid. A right S-system is a set A equipped with a mapping $A \times S \longrightarrow A$, $(a, s) \longmapsto as$, such that $(as)t = a(st)$ and $a1 = a$ for all $a \in A$ and $s, t \in S$. Left S-systems are defined dually, and the category of right (left) S-systems is denoted $\mathrm{Ens}-S$ ($S-\mathrm{Ens}$). In both cases the morphisms are the obvious "S-maps". For $A \in \mathrm{Ens}-S$ and $B \in S-\mathrm{Ens}$ the tensor product $A \otimes B$ is the quotient set $(A \times B)/\tau$, where τ is the smallest equivalence relation on $A \times B$ containing all pairs $\big((as, b), (a, sb)\big)$ for $a \in A$, $b \in B$, and $s \in S$. The τ-class containing (a, b) is denoted $a \otimes b$ and the canonical map $(a, b) \longmapsto a \otimes b$ from $A \times B$ into $A \otimes B$ has the customary universal property with respect to balanced maps from $A \times B$ into sets.

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For any fixed $A \in \text{Ens}-S$, $A \otimes -$ is a functor from $S - \text{Ens}$ into Ens (the category of sets). When this functor preserves monomorphisms A is called *flat*; when it preserves pullbacks (equalizers) A is called *pullback-flat* (*equalizer-flat*); and A is called *strongly flat* when it is both pullback-flat and equalizer-flat. Strongly flat S-systems were first investigated in [5] (where they were called *weakly flat*, a term which has since assumed a different meaning; see e.g. [2]). The following result appears in [5]:

THEOREM 1.1. *$A \in \text{Ens}-S$ is strongly flat if and only if A satisfies conditions (P) and (E) below:*

(P) $as = a's'$ for $a, a' \in A$, $s, s' \in S$ implies $a = a''u$, $a' = a''v$, $us = vs'$ for some $a'' \in A$, $u, v \in S$.

(E) $as = as'$ for $a \in A$, $s, s' \in S$ implies $a = a''u$, $us = us'$ for some $a'' \in A$, $u \in S$.

It has long been known that strongly flat S-systems are flat. We list briefly some of the results given in [3]. In $\text{Ens}-S$:

— pullback-flat implies (P), but not conversely
— equalizer-flat implies (E), but not conversely
— (P) implies flat, but not conversely
— equalizer-flat implies flat, but not conversely
— (E) does not imply flat.

Normak and Stenström left open the possibility that pullback-flatness alone characterizes strong flatness. The main purpose of the present paper is to prove that this is in fact so.

2. **Pullback-flatness.** It is well known (see [1] for example) that if $A \in \text{Ens}-S$, $B \in S - \text{Ens}$, $a, a' \in A$ and $b, b' \in B$, then $a \otimes b = a' \otimes b'$ in $A \otimes B$ if and only if there exist $a_1, \dots, a_n \in A$, $b_2, \dots, b_n \in B$, and $s_1, \dots, s_n, t_1, \dots, t_n \in S$ such that

$$\begin{array}{rcl} a = a_1 s_1 & & \\ a_1 t_1 = a_2 s_2 & \qquad & s_1 b = t_1 b_2 \\ \vdots & & \vdots \\ a_n t_n = a' & & s_n b_n = t_n b'. \end{array}$$

Such a system of equalities is called a *scheme of length n connecting (a, b) to (a', b')*.

Our first observation in this section is that if $A \in \text{Ens}-S$ satisfies (P), then the description of equality in $A \otimes B$ need involve only schemes of length 1, for any $B \in S-\text{Ens}$:

PROPOSITION 2.1. *For any $A \in \text{Ens}-S$ the following statements are equivalent:*

(1) A satisfies condition (P).

(2) For all $B \in S - \text{Ens}$, $a, a' \in A$, $b, b' \in B$, $a \otimes b = a' \otimes b'$ in $A \otimes B$ if and only if

$$\begin{array}{rcl} a = a_1 s_1 & & \\ a_1 t_1 = a' & \qquad & s_1 b = t_1 b' \end{array}$$

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for some $a_1 \in A$ *and* $s_1, t_1 \in S$.

PROOF. (2) implies (1). Assume (2) holds, and suppose $as = a's'$ for some $a, a' \in A$ and $s, s' \in S$. Then $a \otimes s = a' \otimes s'$ in $A \otimes S$ and so there exist $a_1 \in A$ and $s_1, t_1 \in S$ such that

$$\begin{aligned} a &= a_1 s_1 & & \\ a_1 t_1 &= a' & \qquad s_1 s &= t_1 s' \end{aligned}$$

This visibly gives (P).

(1) implies (2). Clearly, assuming (1), it is just the *only if* part of the equivalence which requires proof. Suppose $a \otimes b = a' \otimes b'$ in $A \otimes B$ via a scheme

$$\begin{aligned} a &= a_1 s_1 & & \\ a_1 t_1 &= a_2 s_2 & \qquad s_1 b &= t_1 b_2 \\ &\vdots & &\vdots \\ a_n t_n &= a' & \qquad s_n b_n &= t_n b'. \end{aligned}$$

of length n, for some $n \geq 2$. Application of condition (P) to the equality $a_1 t_1 = a_2 s_2$ yields $a'' \in A$ and $u, v \in S$ such that $a_1 = a''u$, $a_2 = a''v$, and $ut_1 = vs_2$. The scheme

$$\begin{aligned} a &= a'' u s_1 & & \\ a'' v t_2 &= a_3 s_3 & \qquad u s_1 b &= v t_2 b_3 \\ a_3 t_3 &= a_4 s_4 & \qquad s_3 b_3 &= t_3 b_4 \\ &\vdots & &\vdots \\ a_n t_n &= a' & \qquad s_n b_n &= t_n b' \end{aligned}$$

connects (a, b) to (a', b') and has length $n-1$. The process may be repeated until a scheme of length 1 is obtained. ∎

In the sequel we shall consistently use the convention that if

$$\begin{tikzcd} P \arrow[r] \arrow[d] & B \arrow[d, "\alpha"] \\ C \arrow[r, "\beta"'] & D \end{tikzcd}$$

FIGURE 1

is a pullback diagram (in $S-$Ens or in Ens, for our purposes) then $P = \{(b, c) \in B \times C \mid \alpha(b) = \beta(c)\}$, with first and second coordinate projections as the maps of P into B and C, respectively. (See [3] and [4].)

If Figure 1 represents a pullback diagram in $S-$Ens and if $A \in$ Ens $-S$ then tensoring by A produces the outer square in Figure 2: the inner square is the pullback in Ens. Thus,

$$\mathbf{P} = \{(a \otimes b, a' \otimes c') \in (A \otimes B) \times (A \otimes C) \mid a \otimes \alpha(b) = a' \otimes \beta(c')\}$$

$$\begin{tikzcd} A \otimes P \arrow[dr, "\phi"] \arrow[drr] \arrow[ddr] & & \\ & \mathbf{P} \arrow[r] \arrow[d] & A \otimes B \arrow[d, "1 \otimes \alpha"] \\ & A \otimes C \arrow[r, "1 \otimes \beta"'] & A \otimes D \end{tikzcd}$$

FIGURE 2

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and ϕ is given by $\phi\big(a \otimes (b,c)\big) = (a \otimes b, a \otimes c)$ for any $a \in A$ and $(b,c) \in P$. Clearly A is pullback-flat if and only if, for every pullback diagram in $S-$ Ens (Figure 1) the corresponding map ϕ (Figure 2) is bijection.

The next result shows that condition (P) on A is exactly what is needed to make all the maps ϕ surjective:

LEMMA 2.2. *For any $A \in$ Ens $-S$ the following statements are equivalent:*
(1) A satisfies condition (P).
(2) For every pullback diagram in $S-$ Ens (Figure 1) *the mapping ϕ* (Figure 2) *is surjective.*

PROOF. (2) implies (1). Assume (2) holds, and suppose $a, a' \in A$, $s, s' \in S$ are such that $as = a's'$. In Figure 1 take $B = C = D = S$ (where S is considered a left S-system in the natural way), and let α and β denote right multiplication by s and s', respectively. Then $P = \{(u,v) \in S \times S \mid us = vs'\}$, and by (2) the mapping $\phi : A \otimes P \longrightarrow \boldsymbol{P} = \{(a_1,a_2) \in A \times A \mid a_1 s = a_2 s'\}$ given by $\phi\big(a_1 \otimes (u,v)\big) = (a_1 u, a_1 v)$ is surjective. (Free use has been made here of the isomorphism $A \otimes S \cong A$, $a \otimes s \longmapsto as$.) Thus, since $(a,a') \in \boldsymbol{P}$ there exist $a'' \in A$ and $(u,v) \in P$ with $\phi\big(a'' \otimes (u,v)\big) = (a,a')$. In other words, $a''u = a$, $a''v = a'$, and $us = vs'$, as required.

(1) implies (2). Suppose A satisfies condition (P). Consider any pullback diagram in $S-$ Ens (Figure 1) and the corresponding commutative diagram in Ens (Figure 2). To show ϕ is surjective, take any $(a \otimes b, a' \otimes c') \in \boldsymbol{P}$. In view of ($P$), since $a \otimes \alpha(b) = a' \otimes \beta(c')$ in $A \otimes D$, there exist $a'' \in A$ and $u, v \in S$ such that

$$\begin{array}{ll} a = a''u & \\ a''v = a' & \qquad\qquad u\alpha(b) = v\beta(c') \end{array}$$

(Proposition 2.1 was used here). Thus, $(ub, vc') \in P$, and $\phi\big(a'' \otimes (ub, vc')\big) = (a'' \otimes ub, a'' \otimes vc') = (a''u \otimes b, a''v \otimes c') = (a \otimes b, a' \otimes c')$, as required. ∎

We are now ready to be given an "interpolation condition" in the spirit of (P) and (E) which describes pullback-flatness.

THEOREM 2.3. *$A \in$ Ens $-S$ is pullback-flat if and only if A satisfies the condition* (*PF*) *below:*

(*PF*) *If $as = a's'$ and $at = a't'$ for $a, a' \in A$, $s, s', t, t' \in S$, then $a = a''u$, $a''v = a'$, $us = vs'$ and $ut = vt'$ for some $a'' \in A$ and $u, v \in S$.*

PROOF.

NECESSITY OF (PF). Assume A is pullback-flat. Then by Lemma 2.2 A satisfies condition (P) and so the result of Proposition 2.1 is at our disposal.

Now let $\{z\}$ denote a 1-element left S-system, and consider the pullback diagram

$$\begin{array}{ccc} S \times S & \longrightarrow & S \\ \downarrow & & \downarrow \\ S & \longrightarrow & \{z\} \end{array}$$

FIGURE 3

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in $S-\mathrm{Ens}$: as usual, the maps to S are the two coordinate projections. By assumption the mapping ϕ in the diagram

$$\begin{tikzcd}
A \otimes (S \times S) \arrow[dr, "\phi"] \arrow[drr] \arrow[ddr] & & \\
& \boldsymbol{P} \arrow[r] \arrow[d] & A \arrow[d] \\
& A \arrow[r] & A \otimes \{z\}
\end{tikzcd}$$

FIGURE 4

is injective, where $\boldsymbol{P} = \{(a_1, a_2) \in A \times A \mid a_1 \otimes z = a_2 \otimes z\}$ (= the relation of connectedness in A), and $\phi\big(a_1 \otimes (p,q)\big) = (a_1p, a_1q)$ for $a_1 \in A$ and $p, q \in S$.

If $as = a's'$ and $at = a't'$, then $\phi\big(a \otimes (s,t)\big) = \phi\big(a' \otimes (s',t')\big)$ and so $a \otimes (s,t) = a' \otimes (s',t')$ in $A \otimes (S \times S)$. Applying (2) of Proposition 2.1 to the latter equality yields directly elements a'', u and v which are being sought.

SUFFICIENCY OF (PF). First observe that (PF) implies (P) (take $t = s$, $t' = s'$) and so once more (2) of Proposition 2.1 may be used. Now consider any pullback diagram in $S-\mathrm{Ens}$ (Figure 1). By Lemma 2.2 the mapping ϕ in the corresponding Figure 2 diagram is surjective. To see that ϕ is also injective, assume $a, a' \in A$, $b, b' \in B$ and $c, c' \in C$ are such that $(b,c), (b',c') \in P$ and $(a \otimes b, a \otimes c) = (a' \otimes b', a' \otimes c')$ in $\boldsymbol{P}$.

Then by Proposition 2.1 we have

$$\begin{array}{llll} a = a_1u_1 & & a = a_2u_2 & \\ a_1v_1 = a' & u_1b = v_1b' & a_2v_2 = a' & u_2c = v_2c' \end{array}$$

for certain $a_1, a_2 \in A$ and $u_1, v_1, u_2, v_2 \in S$. Applying (PF) to the system of equalities

$$\begin{aligned} a_1u_1 &= a_2u_2 \\ a_1v_1 &= a_2v_2 \end{aligned}$$

yields $a'' \in A$ and $x, y \in S$ for which

$$\begin{array}{lll} a_1 = a''x & & \\ a''y = a_2 & xu_1 = yu_2 & xv_1 = yv_2. \end{array}$$

Since $xu_1b = xv_1b' = yv_2b'$ and $xu_1c = yu_2c = yv_2c'$ we may calculate $a \otimes (b,c) = a_1u_1 \otimes (b,c) = a''xu_1 \otimes (b,c) = a'' \otimes (xu_1b, xu_1c) = a'' \otimes (yv_2b', yv_2c') = a''yv_2 \otimes (b',c') = a' \otimes (b',c')$ in $A \otimes P$, as was to be shown. ∎

It remains to now give the main result, that in fact strongly flat and pullback-flat S-systems are the same.

THEOREM 2.4. *For any* $A \in \text{Ens} - S$ *the following statements are equivalent:*
(1) A is strongly flat.
(2) A satisfies conditions (P) *and* (E) (see Theorem 1.1).
(3) A satisfies condition (PF) (see Theorem 2.3).
(4) A is pullback-flat.

PROOF. The equivalence of (1) and (2), and that of (3) and (4), have already been established. Since by definition (1) implies (4) and since we have already noted that (PF) implies (P), it remains to establish that (PF) implies (E).

Suppose therefore that $as = as'$ for $a \in A$ and $s, s' \in S$. If we apply (PF) to the system of equalities

$$as = as'$$
$$as' = as$$

we obtain $\hat{a} \in A$ and $x, y \in S$ such that

$$a = \hat{a}x$$
$$\hat{a}y = a \quad xs = ys' \quad xs' = ys.$$

Now, apply (PF) to the system

$$a1 = \hat{a}x$$
$$a1 = \hat{a}y$$

to obtain $a'' \in A$ and $u, v \in S$ such that

$$a = a''u$$
$$a''v = \hat{a} \quad u = vx \quad u = vy.$$

We now note that $a = a''u$, and we calculate that $us = vys = vxs' = us'$. Hence, A satisfies condition (E). ∎

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