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THE CENTERS OF A RADICAL RING

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ABSTRACT. It is shown that the nth center of a radical ring coincides with that of its adjoint group, from which a result of Jennings is sharpened and a conjecture of his is confirmed.

Jennings [1] proved that the associated Lie ring of a radical ring is nilpotent if and only if its adjoint group is nilpotent, and he conjectured that the nilpotent classes of them are the same in this case. The conjecture was verified partially by Laue [2]. In this note, we prove that the nth center of a radical ring coincides with that of its adjoint group. This theorem has been conjectured and proved for $n = 2$ by Laue [2]. As a corollary of our result, Jennings' conjecture is proved and his result is improved.

Let R be a Jacobson radical ring. Then $(R, \circ)$ is a group, called the *adjoint group of* R, with respect to the composition $a \circ b = a + b - ab$ for $a, b \in R$. Also, $(R, +, [\,,\,])$ is a Lie ring, called the *associated Lie ring* of R where $[a, b] = ab - ba$ for $a, b \in R$. The inverse of $a \in R$ in $(R, \circ)$ will be denoted by a'. The nth center Z_n of the ring R (respectively, Y_n of the group $(R, \circ)$) is defined inductively as follows,

$$Z_0 = 0, \quad Z_n = \{a \in R \mid [a, x] \in Z_{n-1} \text{ for all } x \in R\}, \quad n \geq 1.$$

(respectively, $Y_0 = 0, \quad Y_n = \{a \in R \mid a' \circ x' \circ a \circ x \in Y_{n-1} \text{ for all } x \in R\}, \quad n \geq 1$).

For brevity, we shall write $[x_1, x_2, \ldots, x_n]$ for $\big[\cdots[x_1, x_2], \ldots, x_n\big], n \geq 2$, and use the formal identity 1. The identities of commutators such as $[xy, z] = x[y, z] + [x, z]y$ and $[xy, z] + [yz, x] + [zx, y] = 0$ will be used freely.

The main result of this paper is the following theorem.

THEOREM. *Let R be a radical ring. Then $Z_n = Y_n$ for any natural number n.*

To prove the theorem, the following lemmas due to Laue [2] will be required.

LEMMA 1. *For all $a \in Z_n$ and $x, y \in R$ with $[x, y] = 0$, we have $y[a, x] \in Z_{n-1}$ and $(1 - y)[a, x] \in Z_{n-1}$.*

PROOF. See the proof of [2, Lemma 2].

LEMMA 2. *Z_n is a radical subring of R and $(1 - x)Z_n(1 - x') \subset Z_n$ for all $x \in R$.*

PROOF. It is clear from [2, Lemma 1, 2].

We proceed with a sequence of lemmas for our further work.

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LEMMA 3. $Z_n \subset Y_n$.

PROOF. The proof is by induction on n. There is nothing to prove for $n = 1$. Let $n > 1$ and assume $Z_{n-1} \subset Y_{n-1}$. For $a \in Z_n$, $x, y \in R$, easy calculations yield

$$\begin{aligned}[a' \circ x' \circ a \circ x, y] &= -\big[(1-a')(1-x')[a,x], y\big] \\ &= [a, x, (1-a')y(1-x')] \\ &\quad - [a, x(1-a'), y(1-x')] \\ &\quad - \big[(1-x')[a,x], (1-a')y\big] \\ &\quad - \big[(1-x')[a,x], a', y\big].\end{aligned}$$

By Lemma 1, $(1-x')[a,x] \in Z_{n-1}$. Thus $[a' \circ x' \circ a \circ x, y] \in Z_{n-2}$ and then $a' \circ x' \circ a \circ x \in Z_{n-1}$. By the inductive hypothesis, we have $a' \circ x' \circ a \circ x \in Y_{n-1}$. Hence $a \in Y_n$ and $Z_n \subset Y_n$, as desired.

We shall prove the inclusion $Y_n \subset Z_n$. We begin with

LEMMA 4. *If $Y_{n-1} \subset Z_{n-1}$, then $2Y_n \subset Z_n$.*

PROOF. For $a \in Y_n$, we have

$$(1) \qquad (1 - x' \circ a')[a, x] \in Z_{n-1} \text{ for all } x \in R,$$

since

$$(1 - x' \circ a')[a,x] = (a' \circ x' \circ a \circ x)' \in Y_{n-1} \subset Z_{n-1} \text{ for all } x \in R.$$

Now, for $y \in R$, by Lemma 1, we obtain

$$y(1-x)\big[(1-x' \circ a')[a,x], y(1-x)\big] \in Z_{n-2},$$
$$(1 - y \circ x)\big[(1 - x' \circ a')[a,x], y \circ x\big] \in Z_{n-2},$$

or,

$$\big[y[a, (1-a')x], y(1-x)\big] \in Z_{n-2},$$
$$\big[(1-y)[a, (1-a')x], y \circ x\big] \in Z_{n-2}.$$

Hence, the sum $[a, (1-a')x, y(1-x)] + [(1-y)[a, (1-a')x], x]$ is in Z_{n-2}. However,

$$(2) \qquad [a, (1-a')x, x] \in Z_{n-2} \text{ for all } x \in R,$$

because $(1-x)\big[(1-x' \circ a')[a,x], x\big] \in Z_{n-2}$ by Lemma 1. Therefore,

$$(3) \qquad [a, (1-a')x, y - yx] - \big[y[a, (1-a')x], x\big] \in Z_{n-2} \text{ for all } x, y \in R.$$

Replacing x by $-x$ in (3) gives

$$-[a, (1-a')x, y + yx] - \big[y[a, (1-a')x], x\big] \in Z_{n-2}.$$

which together with (3) implies $2[a, (1-a')x, y] \in Z_{n-2}$; that is, $[2a, (1-a')x, y] \in Z_{n-2}$ for all $x, y \in R$. Thus, we get $2a \in Z_n$, since R is a radical ring. The proof is complete.

REMARK. As pointed out by the referee, by an easy induction on n, Lemma 3 and Lemma 4 give a surprisingly short proof of the theorem in case $2R = R$, or in particular, R is an algebra over a field F with $\operatorname{ch} F \neq 2$.

The following lemma is of independent interest.

LEMMA 5. *$Z_n = \{a \in R \mid (1 - x')[a, x] \in Z_{n-1}$ for all $x \in R\}$.*

PROOF. Set

$$A_n = \{a \in R \mid (1 - x')[a, x] \in Z_{n-1} \text{ for all } x \in R\}.$$

Then the inclusion $Z_n \subset A_n$ follows from [2, Lemma 2] since $(1-x')[a,x] = a - x' \circ a \circ x$. Conversely, we prove $A_n \subset Z_n$ by induction on $n \geq 1$. This is clear for $n = 1$. Let $n > 1$ and assume $A_{n-1} \subset Z_{n-1}$. Then we have to prove $a \in Z_n$ for any $a \in A_n$. Let $a \in A_n$. Since

$$\begin{aligned}[2a, x, y] = {} & y(1 - x)\big[(1 - x')[a, x], y(1 - x)\big] \\ & + (1 - y \circ x)\big[(1 - x')[a, x], y \circ x\big] \\ & - y(1 + x)\big[\big(1 - (-x)'\big)[a, -x], y(1 + x)\big] \\ & - \big(1 - y \circ (-x)\big)\big[\big(1 - (-x)'\big)[a, -x], y \circ (-x)\big],\end{aligned}$$

by Lemma 1, we get $[2a, x, y] \in Z_{n-2}$. Thus

$$2a \in Z_n. \tag{4}$$

By Lemma 1, we have

$$[a, x, x] = (1 - x)\big[(1 - x')[a, x], x\big] \in Z_{n-2},$$

the linearalization of which yields $[a, x, y] + [a, y, x] \in Z_{n-2}$ for all $x, y \in R$, and in particular,

$$[a, x, y(1 - x')] + [a, y(1 - x'), x] \in Z_{n-2} \text{ for all } x, y \in R. \tag{5}$$

It is routine to check

$$\begin{aligned}\big[(1 - x')[a, x], y\big] = {} & (1 - x')\big[[a, y], x\big] + (1 - x')\big([a, x, y(1 - x')] \\ & + [a, y(1 - x'), x]\big)(1 - x) \\ & - (1 - x')[2a, y(1 - x'), x](1 - x),\end{aligned}$$

from which $(1 - x')\big[[a, y], x\big] \in Z_{n-2}$ by (4), (5), Lemma 2 and Lemma 4. Thus, by the inductive hypothesis, we have $[a, y] \in Z_{n-1}$ for all $y \in R$, and so $a \in Z_n$, as desired.

Now we prove the inclusion $Y_n \subset Z_n$, which is recorded as Lemma 6.

LEMMA 6. *$Y_n \subset Z_n$.*

PROOF. The proof is by induction on $n \geq 1$. For $n = 1$ there is nothing to prove. Let $n > 1$ and assume $Y_{n-1} \subset Z_{n-1}$. Then the proof of Lemma 4 is available. Hence for $a \in Y_n$, using (1) and Lemma 1, we have

$$[a, x, a \circ x] = (1 - a \circ x)\big[(1 - x' \circ a')[a, x], a \circ x\big] \in Z_{n-2}.$$

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One sees that

$$[a,x,a] = [a,x-a',a\circ(x-a')] - [a,x,a\circ x],$$
$$[a,x,a'] = -[a,(1-a')x(1-a'),a].$$

Thus we have

(6) $$[a,x,a] \in Z_{n-2}, [a,x,a'] \in Z_{n-2} \text{ for all } x \in R.$$

As $(1-x')[a,x,x] = \big[(1-x'\circ a')[a,x],a\circ x\big] - (1-x')[a,(1-a')x,a](1-x)$, from (1), (6) and Lemma 2, we deduce $(1-x')[a,x,x] \in Z_{n-2}$, and then

$$[a,x,x,x] = (1-x)\big[(1-x')[a,x,x],x\big] \in Z_{n-3}$$

by Lemma 1. Hence

$$[a,x,x,a'] = [a,x+a',x+a',x+a'] - [a,x,x,x]$$
$$- [a,x,a',a'] - [a,x,a',x] \in Z_{n-3}.$$

By the Jacobi identity, $\big[[a,x],[a',x]\big] = [a,x,a',x] - [a,x,x,a']$. It follows that

(7) $$\big[[a,x],[a',x]\big] \in Z_{n-3} \text{ for all } x \in R.$$

Now, replacing x by $x+a'$ in (3), one has

$$[a,(1-a')x,y-y(x+a')] - \big[y[a,(1-a')x],x+a'\big] \in Z_{n-2},$$

which shows that

$$[a,(1-a')x,ya'] + \big[y[a,(1-a')x],a'\big] \in Z_{n-2} \text{ for all } x,y \in R.$$

This is equivalent to

(8) $$[a,x,ya'] + \big[y[a,x],a'\big] \in Z_{n-2} \text{ for all } x,y \in R,$$

since R is a radical ring. Oberving that

$$\big[y[a,x],a'\big] = [a,x,a'y] - [a,xa',y],$$

we can see that $[a,x,ya'] + [a,x,a'y] - [a,xa',y] \in Z_{n-2}$; that is,

$$\big[[a,x],[a',y]\big] + [2a,x,ya'] - [a,xa',y] \in Z_{n-2},$$

in which taking $y = x$ and then using (7) and Lemma 4, we have

(9) $$[a,xa',x] \in Z_{n-2} \text{ for all } x \in R.$$

It is easy to see that

$$-(1-a'\circ x')[a,x] = a'\circ x'\circ a\circ x \in Y_{n-1} \subset Z_{n-1}.$$

Thus, $(1-x')[a,x](1-a') \in (1-a)Z_{n-1}(1-a') \in Z_{n-1}$ by Lemma 2, and so, $[a,x(1-a'),x] = (1-x)\big[(1-x')[a,x](1-a'),x\big] \in Z_{n-2}$ by Lemma 1. Clearly,

$$[a,x,x] = [a,x(1-a'),x] + [a,xa',x],$$

whence, by (9),

$$[a,x,x] \in Z_{n-2} \text{ for all } x \in R. \tag{10}$$

Linearizing (2) and (10), we get

$$[a,(1-a')x,y] + [a,(1-a')y,x] \in Z_{n-2} \text{ for all } x,y \in R,$$

$$[a,x,y] + [a,y,x] \in Z_{n-2} \text{ for all } x,y \in R,$$

respectively. From the latter, in particular, it follows that

$$[a,x,(1-a')y] + [a,(1-a')y,x] \in Z_{n-2}.$$

Hence,

$$[a,(1-a')x,y] - [a,x,(1-a')y] \in Z_{n-2} \text{ for all } x,y \in R. \tag{11}$$

One can verify that

$$\begin{aligned}\big[(1-a'\circ x')[a,x],y\big] = \big[&(1-x')[a,x],(1-a')y\big]\\ &+ [a,(1-a')x,y(1-x')]\\ &- [a,x,(1-a')y(1-x')]\\ &+ \big[(1-x')[a,x],a',y\big]\\ &- [a,x,a',y(1-x')].\end{aligned}$$

Now we claim that $\big[(1-x')[a,x],(1-a')y\big] \in Z_{n-2}$. For,

$$\big[(1-a'\circ x')[a,x],y\big] = -[a'\circ x'\circ a\circ x,y] \in Z_{n-2};$$

$$[a,(1-a')x,y(1-x')] - [a,x,(1-a')y(1-x')] \in Z_{n-2},$$

by applying (11) with $y(1-x')$ instead of y;

$$\big[(1-x')[a,x],a',y\big] = \big[(1-a)\big[(1-a'\circ x')[a,x],a'\big],y\big] \in Z_{n-3},$$

by Lemma 1; and $[a,x,a',y(1-x')] \in Z_{n-3}$ by (6). Thus we conclude that

$$(1-x')[a,x] \in Z_{n-1} \text{ for all } x \in R, \tag{12}$$

since R is a radical ring. Now Lemma 5 forces $a \in Z_n$. Therefore, $Y_n \subset Z_n$, completing the proof.

Now, the theorem follows from Lemmas 3 and 6.

The following corollary gives a result of Jennings in a sharper form and confirms a conjecture of his.

COROLLARY. *The associated Lie ring of a radical ring is nilpotent of class n if and only if its adjoint group is nilpotent of class n.*

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REFERENCES

1. S. A. Jennings, *Radical rings with nilpotent associated groups*, Trans. Roy. Soc. Can. ser. III **49**(1955), 31–38.
2. H. Laue, *On the associated Lie ring and the adjoint group of a radical ring*, Canad. Math. Bull. **27**(1984), 215–222.

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