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THREE-GROUPS WITH CYCLIC CENTRE AND CENTRAL QUOTIENT OF MAXIMAL CLASS

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Abstract

All three-groups with cyclic centre and such that the quotient by the centre has maximal class are listed and a presentation with generators and relations for each is given.

Let G be a p-group with cyclic centre Z and such that G/Z has maximal class. It is known (though not in the literature) and not difficult to verify that the analysis of p-groups of maximal class as set out in Blackburn (1958) or Chapter III, Section 14 of Huppert (1967) carries through for such a group G; one works with the upper central series instead of the lower central series in the maximal class case. Let

$$(1) = G_{a+1} < Z = G_a < G_{a-1} < \ldots < G_2 < G$$

be the upper central series of G, where a is the class of G. When $a = 2$ or 3, G has an abelian maximal subgroup and is readily described (Conlon (1976)). For $a \geqslant 4$, we have that

$$C_G(G_2/G_4) = \ldots = C_G(G_{a-2}/G_a)$$

and this is a maximal subgroup G_1 of G. The subgroup $C_G(G_{a-1})$ is also maximal and G is called *exceptional* if $G_1 \neq C_G(G_{a-1})$. The quotient G/Z is always a nonexceptional maximal class group. G is not exceptional if a is even. Further if $s \in G - G_1 - C_G(G_{a-1})$, then $s^p \in Z$. If $s_1 \in G_1 - G_2$, and if $s_{i+1} = [s_i, s]$ $(i = 1, \ldots, a-1)$, then $G_i = \langle G_{i+1}, s_i \rangle$. All maximal subgroups of G which contain Z (and so G_2) and which are not equal to G_1 or $C_G(G_{a-1})$ are also nonexceptional groups with centre Z. If $p = 2$, then G_1 is abelian and G is described in Conlon (1976).

When $p = 3$ and $|Z| = 3$ (G has maximal class a), then $G_1' = [G_2, G_1] \leqslant Z = G_a$, G is nonexceptional and such groups are described by Blackburn (1958). But if $|Z| \geqslant 3^2$, then G can be exceptional. However if $\bar{G} = G/Z$, then $\bar{G}$ is of maximal class $a-1$ and so $[\bar{G}_2, \bar{G}_1] \leqslant \bar{G}_{a-1}$, whence $[G_2, G_1] \leqslant G_{a-1}$. Thus $[s_2, s_1] = s_{a-1}^f z$ for $0 \leqslant f < 3$ and $z \in Z$. Thus G is certainly a quotient of a group with a presentation:

(1) $$G(a) = \langle s, s_1, \ldots, s_a, Z \mid Z \text{ is a central cyclic 3-group}, s_{i+1} = [s_i, s]\ (i = 1, \ldots, a-1),\ s^3 = z_1,\ (ss_1^{-1})^3 = z_2, [s_2, s_1] = s_{a-1}^f z, \text{ where } s_a, z_1, z_2 \text{ and } z \in Z \rangle.$$

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When $a = 2$ *or* 3, one can further suppose that $f = 0$ and $z = 1$. We have the following consequences (2), …, (9) of the relations in (1):

(2) $$[s_3, s_1] = s_a^f z^{-3}.$$

Other than (2) and the relation $[s_2, s_1] = s_{a-1}^f z$, we have the following for $1 \leqslant i < j \leqslant a$:

(3) $$[s_j, s_i] = z^{m(j-i).3^{[(i+j-2)/2]}},$$

where $m(l+12) = m(l)$ for all $l \geqslant 0$ and for $0 \leqslant l < 12$ we have:

l	0	1	2	3	4	5	6	7	8	9	10	11
$m(l)$	0	1	−1	2	−1	1	0	−1	1	−2	1	−1

In particular $m(2l+1) \equiv (-1)^l \pmod 3$, for $l > 0$.

(4) $$z^{3^{[(a-1)/2]}} = 1.$$

(5) If $[s_{a-1}, s_1] \neq 1$, then a is odd and z has order exactly $3^{(a-1)/2}$. Also in this case we have:

$$[s_{a-1}, s_1] = z^{(-3)^{(a-3)/2}}.$$

(6) If $f \neq 0$, then $[s_{a-1}, s_1] = s_a^{-1}$.

(7) $$[s_{a-1}, s_{a-2}] = \dots = [s_{a-1}, s_2] = 1.$$

(8) $$s_1^3 = (s_3^{-1})^{s^{-1}} s_{a-1}^{-f} z^2 z_1 z_2^{-1} = s_4 s_6 \dots s_5^{-1} s_3^{-1} s_{a-1}^{-f} z^2 z_1 z_2^{-1}.$$

For $i > 1$,

$$s_i^3 = (s_{i+2}^{-1})^{s^{-1}} z^{2.3^{i-1}} = s_{i+3} s_{i+5} \dots s_{i+4}^{-1} s_{i+2}^{-1} z^{2.3^{i-1}}.$$

For $a \geqslant 3$,

$$s_{a-1}^3 = 1.$$

(9) $$s_a^3 = 1.$$

By passing to the quotient $G(a)/Z$ and using induction on a, it is readily seen that $G(a)$ is a 3-group and that $|G(a)| \leqslant |Z| . 3^a$.

Relations (4), (5), (6) and (9) place restrictions on the choice of z and s_a in Z. Once these are satisfied, then $|G(a)| = |Z| . 3^a$ without any collapse and $G(a)$ gives the correct form of our sought group G of class a. To see this one constructs by induction on a the group $G(a+1)$ from a maximal subgroup $G(a)$ by adding another element s_0 satisfying:

(10) $$[s_0, s] = s_1, \quad [s_1, s_0] = s_{a-1}^{f'} z',$$

and s_0 acts identically on Z. s_0 then gives an automorphism of $G(a)$ provided

(11) $$z^{3^{[(a-2)/2]}} = 1, \quad f = 0, \quad (z')^3 = z \quad \text{and} \quad z_1 = z_2.$$

To assign a value to s_0^3, we choose instead to insist that $(ss_0^{-1})^3 \in Z$ and $(ss_0^{-1})^3$ acts identically on $G(a)$ provided

(12) $$(s_a[s_1, s_{a-2}])^{f'} = 1.$$

The conditions (11) and (12), with s_i replaced by s_{i+1} hold in any case in the maximal subgroup $\langle s, s_2, \ldots, s_{a+1}, Z\rangle$ of a group $G(a+1)$ defined as in (1).

The fact that Z is the centre of $G(a)$ and that $G(a)/Z$ has maximal class $a-1$ is proved by induction on a.

The isomorphism problem is resolved by looking at all possible choices of generators s, s_1 and t and pushing $G = G(a)$ into a standard form; here t is a generator of the centre Z. In separating the isomorphism classes, account may also be taken of the number h of maximal subgroups M of G such that $G_2 \leqslant M$ and $M \neq G_1$ or $C_G(G_{a-1})$ and such that every element in $M - G_2$ has order $3.|Z|$.

a is the class of G, $3^b = |Z|$ and $3^c = |G_1'|$. We can suppose that $z = t^{g.3^{b-c}}$, where $g \equiv \pm 1 \pmod 3$. In the nonexceptional cases we can always make $g = 1$. The exceptional cases occur when $c = [a/2]$ and a is odd and greater than 4; here we can insist that $ss_1 \in C_G(G_{a-1})$ which implies that $[s_{a-1}, s_1] = s_1^{-1}$ and that $g \equiv (-1)^{(a-1)/2} \pmod 3$. G is presented as follows:

(13)
$$\begin{aligned} Gabcdef = \langle s, s_1, \ldots, s_a, t \mid & [s_i, s] = s_{i+1} (i = 1, \ldots, a-1), \\ & [t, s] = [t, s_1] = 1,\ t^{3^{b-1}} = s_a,\ s^3 = t^d, \\ & (ss_1^{-1})^3 = t^e,\ [s_2, s_1] = s_{a-1}^f t^{g.3^{b-c}} \rangle. \end{aligned}$$

The table (14) gives the values of the parameters to obtain a full set of noniso-morphic groups. The values of g and h are also included. The values of d, e, f and g are only significant modulo 3.

TABLE (14)

	a	b	c	d	e	f	g	h
Non-exceptional	$\geqslant 2$	$\geqslant 1$	0	0	0	0	1	0
	$\geqslant 2$	$\geqslant 1$	0	1	1	0	1	3
	$\geqslant 3$	$\geqslant 1$	0	0	1	0	1	2
	odd $\geqslant 3$	$\geqslant 1$	0	0	2	0	1	2
	$\geqslant 4$	$b = c$	$0 < c < [a/2]$	0	0	0	1	1
	$\geqslant 4$	$b = c$	$0 < c < [a/2]$	0	2	0	1	2
	$\geqslant 4$	$b = c$	$0 < c < [a/2]$	1	1	0	1	3
	$\geqslant 4$	$b > c$	$0 < c < [a/2]$	0	0	0	1	0
	$\geqslant 4$	$b > c$	$0 < c < [a/2]$	1	1	0	1	3
	$\geqslant 4$	$b > c$	$0 < c < [a/2]$	2	2	0	1	3
	$\geqslant 4$	$b > c$	$0 < c < [a/2]$	0	1	0	1	2
	odd $\geqslant 5$	$b > c$	$0 < c < [a/2]$	0	2	0	1	2
Exceptional	odd $\geqslant 5$	$b \geqslant c$	$c = [a/2]$	0	0	0	$(-1)^{(a-1)/2}$	0
	odd $\geqslant 5$	$b \geqslant c$	$c = [a/2]$	0	0	1	$(-1)^{(a-1)/2}$	0
	odd $\geqslant 5$	$b \geqslant c$	$c = [a/2]$	0	1	0	$(-1)^{(a-1)/2}$	1
	odd $\geqslant 5$	$b \geqslant c$	$c = [a/2]$	0	2	0	$(-1)^{(a-1)/2}$	1
	odd $\geqslant 5$	$b \geqslant c$	$c = [a/2]$	0	1	1	$(-1)^{)a-1)/2}$	1
	odd $\geqslant 5$	$b \geqslant c$	$c = [a/2]$	0	2	1	$(-1)^{(a-1)/2}$	1
	odd $\geqslant 5$	$b \geqslant c$	$c = [a/2]$	1	1	0	$(-1)^{(a-1)/2}$	2
	odd $\geqslant 5$	$b \geqslant c$	$c = [a/2]$	1	1	1	$(-1)^{(a-1)/2}$	2
	odd $\geqslant 5$	$b \geqslant c$	$c = [a/2]$	1	2	0	$(-1)^{(a-1)/2}$	2
	odd $\geqslant 5$	$b \geqslant c$	$c = [a/2]$	1	2	1	$(-1)^{(a-1)/2}$	2
	odd $\geqslant 5$	$b \geqslant c$	$c = [a/2]$	2	2	0	$(-1)^{(a-1)/2}$	2
	odd $\geqslant 5$	$b \geqslant c$	$c = [a/2]$	2	2	1	$(-1)^{(a-1)/2}$	2

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