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The internal-part fragmentation formulas and their relations to the unitarity equations of a field theory; Regge theory

10.1 Introduction

In this chapter we will consider the decay properties of a cluster. We start to derive some results from the internal-part formulas, Eqs. (8.41) and (8.43).

I1 If we sum over all available states in the decay formulas of a cluster of squared mass s we obtain asymptotically, i.e. for large values of s, the behaviour $\sim s^a$. We will consider these state equations both for the case of a single species of flavor and meson and also for the case of many flavors and many hadrons in each flavor channel.

I2 At the same time we will derive the finite-energy version, f_s, of the fragmentation function f in Eqs. (8.16), (8.17). We will show that f_s tends rapidly towards f when s is larger than a few squared hadron masses (just as $H_s \to H$ according to the results of Chapter 9).

The method we will use is to derive a set of integral equations and then to solve them. In that way we will find that there are some necessary relationships between the parameters a, b and the normalisation parameters that constitute a set of eigenvalue equations for the integral equations.

The whole procedure is very similar to that for obtaining the *unitarity conditions for the S-matrix in a quantum field theory*. We will exploit these relationships by showing that the results obtained under I1 are just the same as are obtained for the *multiperipheral ladder equations* in a quantum field theory. Even the methods of constructing the integral equations are the same. One major result is that the parameter a in the fragmentation functions occurs in a similar way to the Regge intercepts in the Reggeon field theory.

It was Gribov who first understood that the unitarity equations of the S-matrix can be used to derive very general relations between the matrix

elements and the cross sections in any field theory and these considerations form the basis of his Reggeon field theory. Due to space limitations we will have to omit it from this book, but it is just as beautiful, simple and general as another part of the work he has initiated, which is presented in terms of the DGLAP equations in Chapter 19.

10.2　The decay properties of a cluster

We start by noting that the internal breakup distribution in Eq. (8.41) contains two factors:

$$dP_{int} = d\tau_n |\mathscr{M}|^2 \tag{10.1}$$

where the n-particle phase space volume, cf. Chapter 4, is given by

$$d\tau_n = \prod_{j=1}^{n} N d^2 p_{oj} \delta^+(p_{oj}^2 - m^2) \delta^2 \left(\sum_{j=1}^{n} p_{oj} - P_{rest} \right) \tag{10.2}$$

and the squared matrix element by the area-law suppression

$$|\mathscr{M}|^2 = \exp(-bA_{rest}) \tag{10.3}$$

The somewhat fancy notation is used in order to make a connection with Fermi's Golden Rule for quantum mechanical transitions (cf. Chapter 3): the probability for a particular transition is obtained by multiplying the square of the transition matrix element by the available number of states, i.e. the phase space volume of the final-state hadrons. (One should also multiply by a flux factor for the initial state but this is omitted here.)

The interpretation of the area suppression law as a squared matrix element will be provided in the next chapter.

It is evident, however, that the formulas contain two scales. One of these is the quantity b in the exponent. In the derivation of the formulas we noted that b must be the same for all the breakup vertices. Thus b must be flavor independent and so contain basic information about the color force field for which the string is used as a model.

We note, however, that the other parameter in the fragmentation function, a, has vanished from the expressions in Eq. (10.1) but that the normalisation constant N is still present. (If there are several values of a then the differences $a_{j-1} - a_j$ occur directly in the formulas, cf. Eq. (8.43).)

The normalisation constant N, which occurs together with each of the hadronic state factors, can be thought of as a scale factor for the hadronic states. In the simple picture used up to now, in which we have discussed a $(1 + 1)$-dimensional model, then N is, of course, dimensionless. In the actual $(3 + 1)$-dimensional world then N would need to have the same dimensions as b in order to obtain the correct dimensions of the cross

sections, etc. It is by no means obvious that the scale determining the breakup properties of the string color field, i.e. b, is the same as the scale governing the density of the stable hadronic states.

The reason why the a-dependent factors have vanished from Eq. (10.1) is that this formula is an *exclusive expression* for the probability to produce just the particular set of n particles with energy-momenta $\{p_{oj}\}$, $j = 1, \ldots, n$ and nothing else. Our earlier fragmentation formula for $f(z)$ (Eqs. (8.16), (8.17)) is an *inclusive expression*, i.e. it describes the probability of producing one particular meson independently of whatever will come before (or after); there is, however, an implicit assumption that in general there will be something more. We will now prove that it is this expectation that does in fact produce the a-dependent factors in the Lund model.

1   The case of a single a-parameter and a single hadron

We start by considering the probability for producing a first hadron with energy-momentum

$$(p_{o1+}, p_{o1-}) \equiv (u_1 W_+, m^2/u_1 W_+) \tag{10.4}$$

from the cluster with mass $\sqrt{s}$, independently of what comes after. We will use the same notation as in Chapter 8. Now we must pick out those properties of the expression that are u_1-dependent. Then we integrate and sum over everything else, keeping u_1 and s fixed.

Let us first note that if we sum over *everything*, including even the first-rank hadron, then the only thing that the expression can depend upon is the total squared mass s. (This is Lorentz invariance at work in a situation where the only Lorentz invariant is s.) Therefore we can define the function $g(s)$ as follows:

$$g(s) = \sum_n \int \prod_{j=1}^{n} N d^2 p_{oj} \delta^+(p_{oj}^2 - m^2) \delta \left(\sum_{j=1}^{n} p_{oj} - P_{rest} \right) \exp(-bA_{rest}) \tag{10.5}$$

$$s = P_{rest}^2 \equiv W_+ W_-$$

If we introduce the above parameter u_1 (by means of $\delta(p_{o1}^2 - m^2)d^2 p_{o1} = du_1/u_1$), then noting that the area A_{rest} in Fig. (10.1) can be subdivided in an obvious way to give $A_{rest} = \frac{m^2}{u_1} + A(2 - n)$ we obtain the result

$$g(s) = \int \frac{N du_1}{u_1} \exp \left(-\frac{bm^2}{u_1} \right) h \tag{10.6}$$

The quantity h is given by the same expression for the particles indexed 2 to n as that for the function $g(s)$ for all the particles. There is, however,
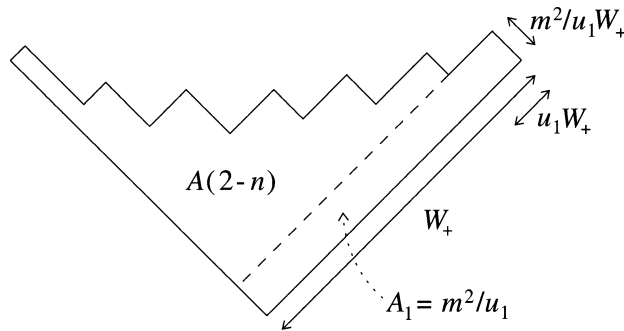
Fig. 10.1. The subdivision of the full cluster area into the area characteristic of the first-rank hadron, A_1, and the area of the remaining ones, $A(2-n)$.

a changed (u_1-dependent) value for the squared mass:

$$s_1 = (P_{rest} - p_{o1})^2 = (W_{+1} - u_1 W_{+1})\left(W_{-1} - \frac{m^2}{u_1 W_{+1}}\right)$$

$$= (1 - u_1)\left(s - \frac{m^2}{u_1}\right) \tag{10.7}$$

This is the squared mass for the hadrons 2 to n; we have taken away the first-rank particle, with a fixed value of u_1. Thus $h \equiv g(s_1)$. Combining these results we obtain an integral equation for g:

$$g(s) = \int \frac{N du_1}{u_1} \exp\left(-\frac{bm^2}{u_1}\right) g\left((1 - u_1)\left(s - \frac{m^2}{u_1}\right)\right) \tag{10.8}$$

To be precise, this integral equation is only valid if s is larger than the square of the single-particle mass. Further, the integration region does not extend all the way down to $u_1 = 0$ because the remainder mass must also be reasonably large.

Nevertheless we note that for large values of s there are solutions of a power character for g (here $\mathscr{C}$ is a constant):

$$g(s) \simeq \mathscr{C} s^a \text{ if } 1 = \int \frac{N du_1}{u_1}(1 - u_1)^a \exp\left(-\frac{bm^2}{u_1}\right) \tag{10.9}$$

Note that this is a requirement on a, i.e. there is a relation between the normalisation constant N, bm^2 and a. This requirement is, of course, nothing other than the original normalisation conditions for f (remember the discussion in Chapter 8).

We could in fact call this property (a form of) *unitarity*: there is a total

probability equal to unity that something will happen in connection with the production process!

We find in this way that the fragmentation function for a finite-mass cluster is formally *s*-dependent, from the normalisation factor h/g:

$$f_s(u_1) = \frac{N du_1}{u_1} \exp\left(-\frac{bm^2}{u_1}\right) \frac{g((1-u_1)(s-\frac{m^2}{u_1}))}{g(s)}$$
$$\simeq f(u_1)\left(1 - \frac{m^2}{su_1}\right)^a \qquad (10.10)$$

In practice the quantity m^2/u_1 is much smaller than s. Therefore we recover our starting expression in Eq. (8.16), i.e. the function f_s tends to f rapidly when $s \gg m^2$.

2 The case of several values of the a-parameter and several kinds of hadron

If there are several *a*-values, a_k, and several flavors and hadrons, we will for simplicity sum over all hadron and flavor indices in the formulas using the convention that the normalisation constants of the fragmentation functions are only nonzero when the hadron index and the flavor indices are compatible. We treat only the case of a single hadron for each flavor combination. The more general case can be inferred from that.

The major difference from the case of a single flavor and hadron is that the hadronic phase space volume is changed (the area-law suppression is the same but with the relevant mass values inserted):

$$d\tau_{0,\alpha} = \delta\left(1 - \sum_1^n u_j\right) \delta\left(s - \sum_1^n \frac{m_{j-1,j}^2}{u_j}\right)$$
$$\times \prod_1^n \frac{N_{f_{j-1},f_j} du_j}{u_j}(u_j)^{a_{f_{j-1}}-a_{f_j}} \qquad (10.11)$$

with the convention that the first-rank hadron has the flavor $f_1 = 0$ and the last one the (anti)flavor corresponding to $f_n = \alpha$.

We will now also prescribe that the first-rank hadron should have the antiflavor corresponding to β and energy-momentum fraction u_1. We can use a division trick (this time with $1 - u_1$) in the δ-distributions to rearrange them as follows:

$$\delta\left(1 - \sum_1^n u_j\right) \delta\left(s - \sum_1^n \frac{m_j^2}{u_j}\right)$$
$$= \delta\left(1 - \sum_2^n \frac{u_j}{1-u_1}\right) \delta\left(s_1 - \sum_2^n \frac{m_j^2(1-u_1)}{u_j}\right) \quad (10.12)$$

We note firstly the occurrence of s_1 from Eq. (10.7) in the δ-distribution in the second line and secondly that in this way the u_j, $j = 2, \ldots, n$, all occur in the rescaled version $u_j/(1 - u_1) \equiv \zeta_j$. Introducing these rescaled ζ_j we obtain from the phase space factor

$$d\tau_{0,\alpha} = \frac{N_{0,\beta} du_1}{u_1} u_1^{a_0 - a_\alpha} \left(\frac{1 - u_1}{u_1} \right)^{a_\beta - a_\alpha} d\tau_{\beta,\alpha} \qquad (10.13)$$

The interpretation is that the phase space volume of a flavor-ranked string of hadrons, starting at flavor 0 and ending at antiflavor $\bar{\alpha}$, can be rearranged into a product with one factor for a single first-rank hadron with flavors $0, \bar{\beta}$ and the other for the phase space volume of the string $\beta, \bar{\alpha}$.

The area-law suppression factor can be rearranged just as in Eq. (10.6) and we conclude that there is a corresponding integral equation, as for the simpler case in Eq. (10.8):

$$g_{0,\alpha}(s) = \sum_{\beta \in \{f\}} \int du_1 f_{0,\beta}(u_1)(1 - u_1)^{-a_\alpha} g_{\beta,\alpha}(s_1) \qquad (10.14)$$

We have here introduced the fragmentation function for the first-rank hadron and also the quantities g_{f_j, f_k} for the total sum over all possible production contributions starting at flavor f_j and ending at (anti)flavor f_k. Note that the argument of $g_{\beta,\alpha}$ is the reduced squared mass s_1. We must sum over all possible antiflavors of the first-rank hadron. The interesting thing is that this equation has solutions g which, for large values of s, depend solely on the final flavor, a_α, i.e.

$$g_{0,\alpha} \simeq g_{\beta,\alpha} \simeq \mathscr{C}_\alpha s^{a_\alpha} \qquad (10.15)$$

as is easily seen using the result for s_1 in Eq. (10.7) (the factor $(1 - u_1)^{-a_\alpha}$ is compensated by the corresponding factor in $s_1^{a_\alpha}$). There is a requirement again corresponding to unitarity that

$$\sum_{\beta \in \{f\}} \int f_{0,\beta}(u_1) du_1 = 1 \qquad (10.16)$$

i.e. that the total probability is 1 that some flavor β is produced at the first vertex and thereby that there is always a first-rank hadron. The statement that the total sum over all possible productions should depend only upon the final flavor was deduced also in the discussion of the Γ-distribution of the final vertex, in the context of the external-part formulas in the last chapter. We note that Eq. (10.16) must be valid not only for the index 0 but for all the flavor indices in $\{f\}$.

If we analyse our results it is evident that we have repeatedly made use of the fact that *the Lund model fragmentation formulas have simple factorisation properties*. In the next section we will consider the unitarity
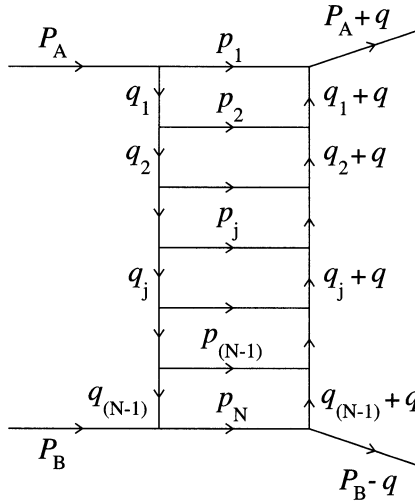
Fig. 10.2. A Feynman diagram describing the scattering $(p_A, p_B) \to (p_A+q, p_B-q)$ with intermediate states and momentum transfers exhibited.

equations for the S-matrix in a quantum field theory. We will show that factorisation properties also in that case lead to very similar results for the correspondence to our state sum.

10.3 The relationship to the unitarity equations for the S-matrix in a quantum field theory

1 The AFS model

It has been known for a long time that it is possible to prove Regge asymptotic behaviour from the unitarity equations for the S-matrix in a quantum field theory. We will, in this and the following subsections, make use of a description similar to the one given in [28].

Amati, Fubini and Stanghellini (AFS), [1], formulated a set of integral equations based upon the so-called *ladder Feynman diagrams* or *multi-peripheral ladder diagrams* (see Fig. 10.2). The starting point is that (the imaginary part of) the elastic scattering amplitude

$$T_{AB} \equiv T(p_A, p_B ; \; p_A + q, p_B - q) \tag{10.17}$$

describing the elastic scattering of the particles A, B with momentum

transfer q must, owing to unitarity, fulfil the following equation:

$$\text{Im}(T_{AB}) = (1/2) \sum_N \int T_N(p_A, p_B; \{p_j\}) T_N^\star(p_A', p_B'; \{p_j\}) d\tau_N \quad (10.18)$$

Here the quantity $d\tau_N$ is the N-particle phase space volume and

$$p_A' = p_A + q \quad \text{such that} \quad p_A'^2 = p_A^2 \quad (10.19)$$

(similarly for p_B'). The point is that all possible hadronic states (denoted $\{p_j\}$ in Eq. (10.18)), fulfilling energy-momentum conservation and capable of being produced from the incoming and outgoing states A, B, should be included in the sum and integral (the p_j being on the mass shell). We note the similarity to the description of a propagator in the Källén-Lehmann representation, where also all possible *intermediate states* occur.

The T (*transition*)-operator is related to the S-operator (which is defined in Chapter 3) in the usual way:

$$S = 1 + iT \quad (10.20)$$

In the AFS model the amplitudes T_N are taken from the ladder diagrams, see Fig. 10.2:

$$T_N^{AFS} = \prod_{j=1}^N \lambda(q_{j-1}, q_j) D(q_j) \quad (10.21)$$

Here $D(q_j)$ is the propagator for the momentum transfer $q_j = q_{j-1} - p_j$ ($q_0 = p_A$, $q_N = -p_B$, for $j = N$ the propagator is equal to 1) at the (Feynman) vertex $j - 1 \rightarrow j$ and $\lambda(q_{j-1}, q_j)$ is the corresponding vertex factor for producing the particle p_j.

We note for future reference that the intermediate N-particle state is in this way built up iteratively, with one particle being produced at a time along a ladder containing the propagators from vertex to vertex.

The expression on the right-hand side of Eq. (10.18), which from now on will be called *rhs*, will have a simple behaviour for a great many production models provided that:

U1 the amplitudes T_N fall off rapidly except when the energy-momentum transfers are small, i.e. $q_j^2 \leq m^2$ with m a typical mass size;

U2 there is no long-range order in the momentum transfers. Thus the amplitude T_N is independent of q_j and q_k if $|j - k| \gg 1$, i.e. the vertices j and k are far from each other in the production process;

U3 the amplitude is not large when the sub-energies $s_{j,j+1} = (p_j + p_{j+1})^2$ of neighboring pairs are large. This means in practice that there are no large rapidity gaps between the produced particles anywhere in the included chains.

These assumptions are at the basis of Gribov's Reggeon calculus, and all of them seem very natural. It should be noted, however, that in QCD the momentum transfers are in general larger than allowed in Gribov's basic assumptions so there is need for some caution in applying the rules to QCD.

In particular one can show from the assumptions U1–U3 that the equations will lead to Regge behaviour, i.e. that

$$\mathrm{Im}\,T_{AB} \sim s^{\alpha} \tag{10.22}$$

The parameter(s) α will in general depend upon the quantum numbers of the particles A, B and also upon the squared momentum transfer $t = q^2$ in the process. The way to obtain the result in Eq. (10.22) is, in the AFS model, to make use of the following simple factorisation property of the amplitude in Eq. (10.21):

$$T_N^{AFS} = \lambda(p_A^2, q_1^2) D(q_1^2) T_{N-1}^{AFS} \tag{10.23}$$

Introducing this factorisation property one obtains immediately an integral equation for the right-hand side, $(rhs)^{AFS}$, of Eq. (10.18) ($q = 0$, $\mathcal{N}$ a numerical constant and $s_1 = (P_{tot} - p_1)^2$ as in Eq. (10.7)):

$$(rhs)^{AFS}(s) = \int \mathcal{N} dp_1 \delta(p_1^2 - m^2)$$
$$\times |\lambda(p_A^2, (p_A - p_1)^2)|^2 |D((p_A - p_1)^2|^2 (rhs)^{AFS}(s_1) \tag{10.24}$$

The similarity between this expression and the integral equation(s) for g and $g_{\alpha,\beta}$ in Eqs. (10.8) (10.15) are obvious. With a few manipulations one obtains the desired power behaviour in $s = P_{tot}^2$, the power-law parameter α_0 being determined from the eigenvalues of the equation, [1].

2   A detour into transverse dimensions

In this section we consider the extension of the results in Eq. (10.24) to nonzero values of the momentum transfer q. This is a preliminary to the extension of the Lund model formulas to a $(3 + 1)$-dimensional world. A second reason is that in this way there emerges a simple and intuitively appealing picture of the behaviour of multiparticle production models. We start with a brief discussion of the influence of the requirements U1–U3 on the results.

In order to make the integrals in the formulas for the function rhs convergent it is necessary to have a fast falloff in q_1^2, i.e. to make use of the requirement U1 above. It is possible to have more complex factorisation properties than in the simple AFS model, i.e. one may introduce short-range correlations between the vertices. The requirement U2 does,

however, ensure the possibility of still writing integral equations (albeit, in this case, systems of integral equations).

The requirement U3 is needed for more subtle reasons and is only necessary when considering transverse dimensions also. (It is instructive to show that in a $(1 + 1)$-dimensional scenario the requirements U1 and U3 are equivalent and the reader is urged to do that.) The (general) requirement U3 is intended to solve a type of *transmission problem* by means of the *law of large numbers*.

Energy-momentum conservation at every (Feynman) vertex means, according to Fig. 10.2, that the individual momentum transfers at the jth cell of the ladder must fulfil

$$q'_j = q_j + q \qquad (10.25)$$

Therefore in order to transmit the momentum transfer q across the ladder it is necessary that neither q_j^2 nor $(q + q_j)^2$ should imply strong suppression of the vertex and propagator functions for the different steps. Therefore q^2 must be limited in size in accordance with the requirement U1 above. Actually, as we will see, the restrictions on q are essentially stronger because in a chain with n vertices there will be n requirements to accommodate.

Further, the momentum transfer q must be spacelike to keep the particles A and B on the mass shell. At high energies it is even necessary that q *should be almost transversely directed*, i.e. q should be almost orthogonal to the direction $\mathbf{p}_A + \mathbf{p}_B$ (from now on the *beam direction*). To prove this let us go to the cms of the two particles A and B; they have energy-momenta in a lightcone frame $p_A = (W, m^2/W, \mathbf{0}_t)$, $p_B = (m^2/W, W, \mathbf{0}_t)$. Then if q has large components along any of the lightcones, the mass-shell conditions for A and B, Eq. (10.19), cannot be fulfilled for large values of W and small values of q^2.

We will therefore use the approximation that $q \simeq \mathbf{q}_t$. It is useful to divide the hadronic phase space volume, $d\tau_N$, into transverse (t) and longitudinal (l) parts with respect to the beam direction:

$$d\tau_N = d\tau_{Nt} d\tau_{Nl} \quad \text{so that} \quad d\tau_{Nt} = \prod_{j=1}^{N} d^2 p_{tj} \delta \left(\sum_{j=1}^{N} \mathbf{p}_{tj} \right) \qquad (10.26)$$

We also observe from the ladder graph in Fig. 10.2 that the hadronic transverse momenta can be expressed as

$$\mathbf{p}_{tj} = \mathbf{q}_{t(j-1)} - \mathbf{q}_{tj}, \quad j = 1, \dots, N \qquad (10.27)$$

This ensures that the transverse momentum conservation δ-distribution in Eq. (10.26) is fulfilled. We may evidently introduce as integration variables the transverse components of the momentum transfers q_j, $j = 1, \dots, N-1$ instead of the corresponding components for the hadrons, p_j.

The next step is to introduce the (transverse) Fourier transforms of the transition amplitudes in Eq. (10.18):

$$T_N(p_A, p_B; p_j) = \int F_N(\mathbf{b}_j, p_{lj}) \prod_{j=1}^{N-1} \exp(i\mathbf{q}_{tj} \cdot \mathbf{b}_j) d^2 b_j \qquad (10.28)$$

(the index l is for longitudinal). There is a corresponding result for the amplitude $T_N^{\star}(p_A', p_B'; p_j)$ in terms of (the complex conjugate of) the same function:

$$T_N^{\star} = \int F_N^{\star}(\mathbf{b}_j', p_{lj}) \prod_{j=1}^{N-1} \exp[-i(\mathbf{q}_{tj} + \mathbf{q}) \cdot \mathbf{b}_j'] d^2 b_j' \qquad (10.29)$$

An essential point is *the appearance of the q-dependence as a common factor in the above equation*:

$$\exp\left[-i\left(\sum_{j=1}^{N-1} \mathbf{b}_j'\right) \cdot \mathbf{q}\right] \qquad (10.30)$$

because if we now perform the integrals on the right-hand side of Eq. (10.18) we obtain a very pretty description:

$$(\text{rhs})(s, q^2) = \int \exp[-i\mathbf{b} \cdot \mathbf{q}] d^2 b \mathscr{F}(\mathbf{b}, s)$$

$$\mathscr{F} = \sum_N \int d\tau_{Nl} \prod_{j=1}^{N-1} (2\pi)^2 d^2 b_j \delta\left(\sum_{j=1}^{N-1} \mathbf{b}_j - \mathbf{b}\right) |F_N(\mathbf{b}_j, p_{lj})|^2 \quad (10.31)$$

We have here repeatedly made use of the well-known Fourier distribution identity

$$\int d^2 q_t \exp\left[i(\mathbf{b} - \mathbf{b}') \cdot \mathbf{q}_t\right] = (2\pi)^2 \delta(\mathbf{b} - \mathbf{b}') \qquad (10.32)$$

to make the identification $\mathbf{b}_j = \mathbf{b}_j'$.

The whole mathematical game has been to introduce instead of the transverse momentum transfers $\mathbf{q}_{tj}$ their canonically conjugate correspondences, the *impact-space vectors* $\mathbf{b}_j$ of the different *links* in the ladder graphs. We find that *the full q-dependence* (in the transverse approximation) *is described by the Fourier transform with respect to the sum of all the individual impact-space vectors for the different links*.

The question then arises of the distribution of the sum of these individual impact-space vectors. Now we can again make use of the two requirements U1 and U2 above. Equation (10.28) was used to define the distributions F_N and we can of course invert it by means of the Fourier

transform relations into

$$F_N(\mathbf{b}_j) = \int T_N(p_A, p_B; \mathbf{q}_{jt}, p_l) \prod_{j=1}^{N-1} \frac{d^2 q_{jt}}{(2\pi)^2} \exp[-i(\mathbf{q}_{jt} \cdot \mathbf{b}_j)] \quad (10.33)$$

Then using the properties of the Fourier transforms we can deduce that the two requirements U1 and U2 will have the following implications for the impact-space vectors $\mathbf{b}_j$.

$U^i 1$. The function(s) F_N are smooth and well-behaved distributions in $\mathbf{b}_j$ with, typically $\mathbf{b}_j^2 \simeq 1/m^2$, i.e. the inverse of the q_j^2.

$U^i 2$. The correlations $\mathbf{b}_j \, \mathbf{b}_k = 0$ if $|j - k| \gg 1$.

Now, finally, comes the basic use of the requirement U3: it means that the major contributions to the integrals come from situations very similar to the one in the Lund model, i.e. *there will be no major contributions to the integrals from small multiplicities*. If this were not the case then there would be many large rapidity gaps. In the Lund model picture there would then have to be many small values of the proper times of the vertices, i.e. we would be far from the usual hyperbola breakup.

(In fact, owing to U1, nor will there be major contributions from very large multiplicities, for which, in accordance with our findings for the Lund model, it is necessary to have large values of the proper time or equivalently of the momentum transfers Γ.) This *large-multiplicity requirement* is necessary because we are now going to make use of the law of large numbers, which (apart from some mathematical epsilontics) reads as follows.

- Consider the distribution of a quantity Σ, which is the sum of a (sufficiently) large number of *independent* stochastic variables:

$$\Sigma = \sum_{j=1}^{n} \rho_j \quad (10.34)$$

Each of the ρ_j is distributed in one way or another with a mean $\langle \rho_j \rangle$ and a variance σ_j^2. Then Σ is distributed according to a gaussian:

$$\frac{dP}{d\Sigma} = \frac{1}{\sqrt{2\pi n \sigma_0^2}} \exp\left[-\frac{(\Sigma - n\Sigma_0)^2}{2n\sigma_0^2} \right] \quad (10.35)$$

The centre and the width of the Σ-distribution are given by

$$n\Sigma_0 = \sum_{j=1}^{n} \langle \rho_j \rangle, \quad n\sigma_0^2 = \sum_{j=1}^{n} \sigma_j^2 \quad (10.36)$$

In our case, the impact-space vectors evidently will all, due to symmetry, have a vanishing mean value. Depending upon the length of the correlations, according to $U^i 2$ they can be subdivided into groups which are independent (we assume each to have a correlation length $\langle \mathbf{r}^2 \rangle$). If the number of such groups is $n(s) \gg 1$ for a given value of the squared cms energy s we may apply the law of large numbers.

In this way we obtain a very general result for the distribution $\mathscr{F}$ in Eq. (10.31), i.e. a gaussian distribution in the total impact-space vector $\mathbf{b}$.

The method we have used can in somewhat vague language be described as follows.

- We go over to impact-parameter space and consider the building up of the total impact parameter $\mathbf{b}$ as a Brownian motion along the chain with each impact parameter (or for short-range correlations each group of) impact-parameter vector(s) $\mathbf{b}_j$ contributing in a random way.

This is a very general feature of transmissions through many steps: they tend to randomise rather quickly and then only the general mean and variance are noticeable (together with the number of steps). Note that if there are major contributions from the fluctuations down to a small number of steps these statements may not be true.

3 Coming back to the Regge phenomenology

After having obtained a very general distribution for $\mathscr{F}$ we will go back to the unitarity equations, as well as to the representation in Eq. (10.31), to obtain

$$(rhs)(s, q^2) = (rhs)(s, q^2 = 0) \exp \left[\frac{-n(s) \langle \mathbf{r}^2 \rangle \mathbf{q}_t^2}{2} \right] \qquad (10.37)$$

i.e. we perform a straightforward gaussian integral.

For the case when the correlations, according to U2, are reasonably short-range one expects $n(s) = c \langle n \rangle$, i.e. this number should be proportional to the mean multiplicity at that energy.

In any of the models which fulfils not only U1 and U2 but also U3, this mean multiplicity will grow logarithmically in s. We therefore expect that the Regge parameters α in Eq. (10.22) are generally *linear functions of t* (note that $\exp(-C \ln s) = s^{-C}$). Conventionally these are written as

$$\alpha(t) = \alpha_0 + \alpha' t \qquad (10.38)$$

with $t = q^2 \simeq -\mathbf{q}_t^2$. Here α_0 is called the *intercept* and the parameter α' *the Regge slope*). This latter parameter should according to our formula

be proportional to $\left\langle \mathbf{r}^2 \right\rangle /2$ and therefore depend both upon the value of the (average) transverse momentum and upon the size of the short-range correlations.

It is known phenomenologically that for most *Regge trajectories*, as the $\alpha(t)$-line in Eq. (10.38) is called, this parameter $\alpha' \sim 1~(\text{GeV}/c)^{-2}$. However, there is one trajectory, equipped with the vacuum quantum numbers, called the *Pomeron*. The Pomeron has an essentially smaller slope, $\alpha'_P \simeq 0.25~(\text{GeV}/c)^{-2}$.

In this way both the elastic and the total cross sections are dominated by power behaviour in s when they are expressed in terms of the matrix elements of T as

$$
\begin{aligned}
\frac{d\sigma_{el}}{dt} &= \frac{|T_{AB}|^2}{16\pi s^2} \propto s^{2\alpha(t)-2} \\
\sigma_{tot} &= \frac{\operatorname{Im} T_{AB}}{s} \propto s^{\alpha(t)-1}
\end{aligned}
\tag{10.39}
$$

The second line of Eq. (10.39) is of course nothing other than a statement that the sum over all states, like in Eq. (10.18), includes everything that can happen; the factor $1/s$ is the flux factor of the incoming state. For the first line of Eq. (10.39) it is necessary also to know something about the real part of T_{AB}, but fortunately the latter turns out to give only a small correction.

Similar power results for the high-energy behaviour are obtained from potential scattering models by rewriting the scattering amplitude using the Sommerfeld-Watson transform, which was originally developed for light-scattering, [46]. These features are, however, outside the scope of this book. One very important result in this connection is that the Regge trajectories $\alpha(t)$ also contain information on the bound-state spectrum of the potential. It is possible to show that $\alpha(t)$ is an analytical function of t in the potential scattering models. In particular, when the squared momentum transfer t (which in the case discussed up to now must be negative) is continued to positive values then

$$
\alpha(t = m^2) = j
\tag{10.40}
$$

with j the angular momentum of the bound state with mass m.

Regge behaviour has also been proposed and investigated in more complex processes than elastic scattering, e.g. in charge exchange processes such as $\pi^- p \rightarrow \pi^0 n$ (p, n stand for proton and neutron, respectively). In this case the use of the *ρ-trajectory* has provided a very good description of all available experimental data. From the experimental analysis, this trajectory actually does exhibit the straight-line behaviour expected in Eq. (10.38), and with the constraint from Eq. (10.40) (that the ρ-meson spin is 1) the value of $\alpha_{0\rho}$ has been decided as being close to 0.5.

The ρ-trajectory is, in accordance with the flavor composition of ρ as a mixture of $u\bar{u}$ and $d\bar{d}$, related to the most common flavors. It should be more than a coincidence that the phenomenological values for the Lund model parameter a, which are obtained from studies that also include gluonic radiation, tend to demand a value just above 0.5.