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LARGE MATCHINGS IN GRAPHS

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1. Introduction. How large a matching must a graph have?

We consider graphs G (finite, undirected, with no loops or multiple edges), with order nG (always $\geqq 1$) and mG the maximum number of edges in a matching of G. The *matchability* μG of G is the fraction $(2m/n)$ of nodes covered by a maximum matching. For any class S of graphs we define the *matchability* μS, the *essential* (or *limit* or *large-graph*) *matchability* $\mu^* S$, and the class wS of *worst-matched graphs*:

μS is the greatest lower bound of the values μG for $G \in S$, with $\mu S = 0$ if S is empty.

$\mu^* S$ is the least upper bound for positive k of the values $\mu(S_k)$, where S_k comprises all graphs in S of order $\geqq k$.

wS is the class of all $G \in S$ with $\mu G = \mu S$.

We study matchability in the attempt to generalize [**4**] (which looked at matching in order to study critical graphs in coloring problems). We shall seek a nontrivial lower bound for μG in terms of the "local structure" of G. Concerning this "local structure" we assume given both a uniform *lower* bound i and a uniform *upper* bound j for the degrees of all nodes of G (else, for all we know, G has very few edges, making μG close to 0, or G has a small set of nodes of high degree which intercept all edges, again making μG close to 0). We shall also use the additional information that G is h-connected (where $h \leqq i$ and "h-connected" is taken in any of several senses, cf. § 2, 8). Given such h, i, j, and letting T be the class of all graphs satisfying these conditions for h, i, j, we shall determine the values μT, $\mu^* T$, wT: in this sense we shall determine just how the local structure of a graph controls the matchability.

§ 2 gives needed preliminaries. § 3 gives some reductions of the problem. In § 4, our main result, 4.8, gives lower bounds for μ which are shown exact by the examples of § 5. In § 6 we calculate μ^* for cases not already resolved. §§ 7, 8 treat variations of the problems relating to the condition "triangle-free" and to connectivity conditions.

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2. Preliminaries.

2.1 *Generalities*. For any set X, $|X|$ is its cardinal number. For any number i,

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i^* is the least greater odd integer ($= i + 1$ for i even and $i + 2$ for i odd).

We sometimes use a 3-vector formalism. Let U, V be any real 3-vectors. Then: $U = (U_1, U_2, U_3)$, where the U_k are the *coordinates* of U; $U \geqq V$ if and only if $U_k \geqq V_k$ for each $k \in \{1, 2, 3\}$; $U * V$ is the inner product

$$U_1V_1 + U_2V_2 + U_3V_3;$$

O is the vector $(0, 0, 0)$.

2.2 *Graphs.* Let G be any graph. If nodes x and y adjoin in G, we take the edge joining them to be $\{x, y\}$. NG is the set of nodes of G, EG is the set of edges of G, $nG = |NG|$, $eG = |EG|$. dxG is the *degree* (number of incident edges) of the node x in G. Extending this notation, suppose X any set of nodes of G. dXG is the number of edges joining nodes of X to nodes of G not in X. G/X (the "restriction" of G to X, or the "section" subgraph "induced" by X) is the largest subgraph of G having X as nodes. $G - X$ is $G/(NG - X)$.

A *matching* in G is a set of pairwise disjoint edges of G. Matchings having the maximum possible number mG of edges are *maximum* matchings.

When the choice of the graph G either is immaterial or is clear from context we often omit final "G" from the notation and write N, E, n, e, dx, dX, m for NG, EG, etc.

The *union* $H \cup K$ of graphs H, K is the graph G with $NG = NH \cup NK$ and $EG = EH \cup EK$.

If X, Y are disjoint sets with union NG, and every edge of G joins a node of X with a node of Y, G is *bipartite* with *bipartition* (X, Y). If G has bipartition (X, Y) and $dx = i$ and $dy = j$ for each $x \in X$ and $y \in Y$, then G is (i, j)-*bipartite* with (i, j)-*bipartition* (X, Y).

2.3 *Types.* In terms of any integer k we formulate several connectivity and degree conditions for a graph G. Each such condition holds *strictly* if it holds for k but fails for at least one of $k - 1$, $k + 1$.

G is *h-node-connected* if $G - X$ is a nontrivial connected graph whenever X is a set of nodes with $|X| < h$. G is *h-edge-connected* if $dXG \geqq h$ whenever X is a nonempty proper subset of G. G is *h-odd-connected* if both: $dXG \geqq h$ whenever X is a proper subset of G with $|X|$ odd, and when $h \geqq 1$ G is connected. Note that h-node-connected implies h-edge-connected, and that h-edge-connected implies h-odd connected.

G is *i-lower* if $dx \geqq i$ for every node of G. G is *j-upper* if $dx \leqq j$ for every node of G. A pair (i, j) of integers with $0 \leqq i \leqq j$ is a *degree*. G is of *degree* (i, j) (or is an (i, j)-*graph*) if G is i-lower and j-upper. If G is of degree (i, i), G is of *degree* i or is *i-regular*.

A triple (h, i, j) of integers with $0 \leqq h \leqq i \leqq j$ is a *type*. A graph G is of *type* (h, i, j) (or is an (h, i, j)-*graph*) if G is h-odd-connected and of degree (i, j). The class $T(h, i, j)$ comprises all graphs of type (h, i, j). We write $\mu(h, i, j)$ in place of $\mu(T(h, i, j))$, and similarly for μ^*, w.

3. The problem and some reductions. Our main task in this paper is to determine $\mu(h, i, j)$, $\mu^*(h, i, j)$ and $w(h, i, j)$ for all types (h, i, j). In §§ 7, 8 we show how to calculate μ, μ^*, w for classes S defined by other (perhaps to the reader more "natural") "h-connectivity" and "(i, j)-degree" conditions.

Call a type (h, i, j) *trivial* if $h = 0$ or $i \leqq 1$ or $j \leqq 2$; or if $i = j$ and $i - h$ is odd. For trivial types, calculation of μ, μ^*, w is either trivial or reduces easily to calculation for nontrivial types. Thus, it is easily shown that when $i = j$ and $i - h$ is odd we have $T(h, i, j) = T(h + 1, i, i)$. Suppose $h = 0$: $\mu(0, 0, j)$ and $\mu^*(0, 0, j)$ are 0 and $w(0, 0, j)$ comprises all 0-graphs; for $i \geqq 1$, $\mu(0, i, j)$ and $\mu^*(0, i, j)$ are both $\mu(1, i, j)$, and $w(0, i, j)$ comprises all graphs each of whose components is in $w(1, i, j)$. Suppose now that $h = 1$ and $j \leqq 2$. Up to isomorphism $T(1, 1, 1)$ contains just a single graph, a complete graph on 2 nodes. $T(1, 1, 2)$ comprises all paths and circuits, $T(2, 2, 2)$ comprises all circuits; these two types each satisfy $\mu = 2/3$, $\mu^* = 1$.

The following lemmas treat the remaining trivial types, viz. types $(1, 1, j)$, $j \geqq 3$, and aid later constructions.

3.1 LEMMA. *Every connected j-upper graph G satisfies $jm + 1 \geqq n$.*

Proof. A connected j-upper graph G with fewest edges such that $jm + 1 \geqq n$ fails must be a tree with a non-end node x which adjoins exactly one non-end node. Delete x and all adjoining end nodes and all edges incident to x to obtain a j-upper tree H; since H satisfies $jm + 1 \geqq n$ so does G.

3.2 LEMMA. *For each $j \geqq 2$ there are arbitrarily large j-upper trees satisfying $jm + 1 = n$. In fact, whenever $2 \leqq i \leqq j$ and $p \geqq 0$ we may construct a j-upper tree G with bipartition (X, Y) in which: $dy = j$ for all $y \in Y$, $dx = 1$ or i for all $x \in X$, $m = 1 + (i - 1)p$, $n = jm + 1$.*

Proof. We construct G by induction on p. For $p = 0$ take G a star with center of degree j. Having constructed H for p we construct G for $p + 1$ as follows. Let x be any end node of H and K a graph with no node in common with H and having $i - 1$ components, each a star with center of degree $j - 1$. Obtain G from $H \cup K$ by adding an edge from x to each stellar center of K.

3.3 *Remark.* Can "elementary" methods be used to determine $\mu(h, i, j)$ for non-trivial (h, i, j)? In particular, we know that for appropriate k the edges of a j-upper graph G can be colored with k "colors" so that distinct intersecting edges are never colored alike. Then $m \geqq e/k$, $\mu \geqq 2e/kn$. If G is i-lower, $2e \geqq in$, so $\mu \geqq i/k$. However, even if we use Vizing's theorem and take $k = j + 1$ (see, e.g., [**3**, p. 248]) we find only that $\mu \geqq i/(j + 1)$, in general not a sharp result (except when $j = i$, i even).

4. Bounds for matchability.

4.1 *Concepts.* In §§ 4, 5, 6 (h, i, j) is a fixed non-trivial type. G a fixed (h, h, j)-graph, and Y a fixed set of nodes of G. G will usually be a graph of type (h, i, j), but we do not assume this now. We now introduce concepts which depend on some or all of h, i, j, G, Y; this dependence will not appear in the notation.

Let K be any subgraph of G. K is *small* if $nK \leqq i$, *large* if $nK > i$, *odd* if nK is odd, *even* if nK is even. A *county* is a component of $G - Y$. cK is the number of odd counties included as subgraphs of K. $rK = cK - |Y \cap NK|$. The *quasi-join* qK of K is the sum for $x \in NK$ of $\max(dxG, i) - dxK$. Note that $qK \geqq d(NK)G$, with equality if and only if all nodes of K have degree $\geqq i$ in G; and $qG = 0$ if and only if G is i-lower, i.e., if and only if G is of type (h, i, j). The *match vector* MK of K is $(nK, qK, -rK)$. We sometimes write M, r for MG, rG, etc.

If, among subsets of NG, the choice of Y makes rG largest possible, Y is *r-maximum*. Y is *singular* if Y is empty and G consists of one large odd county.

4.2 *Procedure.* Observe that a matching fails to cover at least rG nodes, in fact at least rG odd counties. The key result on matching to be used in this paper is the generalized result of Tutte (see [**1**, pp. 179–181]): *if Y is r-maximum, a maximum matching fails to cover exactly rG nodes.* Thus, to obtain a lower bound for μ it suffices to obtain an upper bound for r/n. For Y singular, $r/n = 1/n$; for G nonsingular we shall establish a linear inequality $A * M \geqq 0$, where A is a vector $\geqq O$ and dependent only on (h, i, j); when G is of type (h, i, j) (so that $qG = 0$) we then have $r/n \leqq A_3/A_1$.

The following heuristics motivate the procedures and results.

4.3 *Heuristics.* Which data influence r/n? Consider the match vector of a county K. The larger each coordinate, the more K "helps" the cause of making r/n small. Indeed this is true of the coordinates nK and $-rK$. A large qK helps indirectly: when G is i-lower, $d(NK)G, = qK$, is "evidence" for additional nodes of Y, hence for smaller r and larger n.

We shall define a vector U^0 which will be the "least helpful" match vector of an even county; U^1, U^2, U^3 will be defined similarly for a small odd county, a large odd county (for nonsingular Y) and a node of Y, respectively.

Suppose that G is worst-matched, of type (h, i, j). We find that each county K will have a very special structure, with no even counties, and with $MK = U^1$ (for K small) or $= U^2$ (for K large). Further, when a small county is less help than a large county (or vice versa), G has only small counties (or only large counties). Roughly speaking, when the connectivity h is "big" ($h > h_0$, h_0 defined in 4.4), all counties are rather well-joined to Y and only their size has influence, so that only small counties occur; G is then a bipartite graph (with (i, j)-bipartition $(N - Y, Y)$). When the connectivity h is "small" ($h < h_0$), the fact that G is i-lower forces small counties to be well-joined to Y but not the large counties, and thus only large counties occur (and they have smallest possible size, i^*). When $h = h_0$, both large and small counties can occur.

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4.4 *Definition of* A. In order to define A so that $A * M \geqq 0$, we first observe that $M(G - Y) = \sum (MK : K \text{ a county})$. Hence it suffices to define A so that: $A * MK \geqq 0$ for each county K, and $A * (M - M(G - Y)) \geqq 0$.

We next observe the following facts. Define $U^0 = (2, 0, 0)$, $U^1 = (1, i, -1)$, $U^2 = (i^*, h, -1)$, $U^3 = (1, -j, 1)$. Then:

(0) If K is an even county, $MK \geqq U^0$.

(1) If K is a small odd county, $MK \geqq U^1$.

(2) If K is a large odd county and Y is nonsingular, $MK \geqq U^2$.

(3) $M - M(G - Y) \geqq |Y|U^3$.

Indeed, in the first and third vector coordinate these assertions are trivial, and in the second coordinate (0) is trivial. For (1), qK is the sum of nK terms, each at least $i - (nK - 1)$, where $1 \leqq nK \leqq i$; hence $qK \geqq i$. For (2), $NK \neq NG$, so $qK \geqq d(NK)G \geqq h$. For (3), we must verify that $q(G - Y) - qG \leqq j|Y|$. Now $q(G - Y) - qG \leqq$ sum for $x \in N - Y$ of $dxG - dx(G - Y)$; this sum is dYG, $\leqq j|Y|$.

Hence it suffices to define A so that $A * U^k \geqq 0$ for each $k \in \{0, 1, 2, 3\}$. We first define vectors $A^1 \geqq O$, $A^2 \geqq O$ such that $A^1 * U^1 = A^1 * U^3 = 0 = A^2 * U^2 = A^2 * U^3$. For this it suffices to take $A^1 = (j - i, 2, j + i)$, and $A^2 = (j - h, i^* + 1, i^*j + h)$. Finally, if we put $h_0 = \frac{1}{2}(j + i - i^*(j - i))$, we have $A^1 * U^2 = 2(h - h_0) = -A^2 * U^1$, so it suffices to take $A = A^1$ for $h \geqq h_0$ and $A = A^2$ for $h < h_0$.

4.5 THEOREM. *Suppose* Y *nonsingular. Then* $A * M \geqq 0$.

4.6 COROLLARY (Tutte). *Suppose* $h = i = j$. *Then* $r \leqq 1$. *Further, if* i *is odd,* $\mu G = 1$, *and if* i *is even* $\mu G \geqq i/(i + 1)$.

Proof. When Y is singular, $rG = 1$. When Y is nonsingular, 4.5 yields $-2ir \geqq 0$, whence $r \leqq 0$. When i is odd, n (the number of nodes of odd degree) must be even, whence r must be even, hence $\leqq 0$, so $\mu = 1$. When i is even, the complete graph on $i + 1$ nodes is the unique example of a worst-matched singular G.

Henceforth in §§ 4–6 we assume that $h \neq j$. *Hence each* $A_k > 0$.

4.7 *Remarks.* When Y is singular, we have n odd and $\geqq i^*$, $r = 1$, and $A * M \geqq A_1 n - A_3$. Thus, by 4.5 the inequality $A * M \geqq 0$ fails, if at all, only for Y singular and G one of a few graphs satisfying $i^* \leqq n(\text{odd}) < A_3/A_1$. These conditions are impossible for $j \geqq i + 2$ or for $j = i(\text{odd})$; but possible (in fact with $n = i^*$) when $j = i + 1$ or when $j = i$ (even).

When G is i-lower, we have: $qG = 0$, $A * M = A_1 n - A_3 r$; further, when Y is r-maximum, $A * M \geqq 0$ if and only if $\mu \geqq 1 - (A_1/A_3)$.

These remarks yield the following main result.

4.8 THEOREM. *Suppose* G *of type* (h, i, j). *Then:*

(1) *For* $j \geqq i + 2$, $\mu \geqq 2i/(i + j)$.

(2) *For* $j = i$ (*odd*), $\mu \geqq 1 - (A_1/A_3)$.
(3) *For* $j = i + 1$, *or for* $j = i$ (*even*), $\mu \geqq 1 - (1/i^*)$.

4.9 *Remarks.* (1) The bound in 4.8 (3) is clearly exact. In § 5 we verify that the bounds of 4.8 (1) and (2) are exact. (2) Taking $i = 2$, 4.8 (1) specialises to the main result 3.5 of [**4**]. (3) When $j \geqq i + 2$ we see that a singular graph G of type (h, i, j) satisfies $\mu \leqq 2i/(i + j)$ just when $j = i + 2$ (i even) and G is complete on $i + 1$ nodes. (4) For $j = i$ (even), A_3/A_1 is at most its value for $h = i - 2$, viz. $\frac{1}{2}(ii^* + i - 2)$. For $j = i + 1$, A_3/A_1 is at most $2i + 1$. (5) Suppose an (h, h, j)-graph satisfies $A * M < 0$. We must have Y singular, and $qG < h$ (else $M \geqq U^2$ and then $A * M \geqq 0$). Suppose $h = 1$. Then $qG = 0$ and G is i-lower. From 4.7 and the triviality of $(1, i, i)$ (i even), we must have $j = i + 1$. We must have $n = i^*$: else $M - U^2 \geqq (2, -1, 0)$, and since both $A * U^2 \geqq 0$ and $A * (2, -1, 0) \geqq 0$ we would have $A * M \geqq 0$. When i is even, G must be complete on $i + 1$ nodes. Specializing to type $(1, 2, 3)$: *every connected graph of degree* $(2, 3)$ *other than a triangle satisfies* $\mu \geqq 4/5$. [**4**, 3.6], conjectured that every such graph satisfies $\mu > 2/3$.

5. Examples. We have what seem to be good bounds for $\mu(h, i, j)$ but for most types we do not know yet that the bound is in fact exact, nor do we know $\mu^*(h, i, j)$. We know that among (h, i, j)-graphs with singular r-maximum only finitely many satisfy $\mu \leqq 1 - (A_1/A_3)$. If we seek many examples of equality, we must consider nonsingular Y. *In* § 5 *we assume* G *of type* (h, i, j).

5.1 *Definition.* Y is *exact* if and only if $A * M = 0$. G is *exact* if some subset of NG is exact.

5.2 *Remark.* If Y is exact and $n > A_3/A_1$, then Y is r-maximum and $\mu G = 1 - (A_1/A_3)$. This assertion has an easy proof but a notable effect: to construct examples of graphs with $\mu = 1 - (A_1/A_3)$ we need only construct (large enough) G with a set of nodes Y satisfying certain readily verified structural criteria. Namely, consulting the argument in 4.4 we see that nonempty Y is exact if and only if all the following conditions hold: if $h > h_0$ there are no large counties; if $h < h_0$ there are no small counties; every small county K satisfies $nK = 1$, $d(NK)G = i$; every large county K satisfies $nK = i^*$, $d(NK)G = h$; every node of Y has degree j and adjoins no other node of Y.

5.3 *Definition.* A nontrivial type (h, i, j) is *special* if $h = 1$ and $j \leqq i + 1$, provided that $(i, j) \neq (2, 3)$ and $\neq (3, 4)$.

5.4 *Preview.* We shall show that, for every nontrivial type (h, i, j) with $h \neq j$; (1) *An exact graph exists.* (2) *Infinitely many nonisomorphic exact graphs exist if* (h, i, j) *is not special.* (3) *If* (h, i, j) *is special, all exact graphs are of order* $A_3 = i^*j + 1$ *and satisfy* $|Y| = 1$. Indeed, for $j = i$ (odd), all exact graphs are isomorphic.

By 5.4 (1) we see that in cases (1) and (2) of 4.8 we have $\mu = 1 - (A_1/A_3)$, and the worst-matched (h, i, j)-graphs are the exact graphs. By 5.4 (2), for nonspecial (h, i, j), $\mu^*(h, i, j) = 1 - (A_1/A_3)$. 5.4 (3) follows from 5.2. In § 6 we compute $\mu^*(h, i, j)$ for (h, i, j) special.

We now verify 5.4 (1) and (2) by construction of (h, i, j) graphs G with exact set Y.

5.5 *Construction for* $h \geqq h_0$. We take G an i-node-connected bipartite graph with (i, j)-bipartition $(N - Y, Y)$. For $i = 1$ there is only one possible such graph: a star with central node y of degree j, with $Y = \{y\}$. For $i \geqq 2$ and any positive integer p we may take NG the set of numbers (nonnegative integers) $< (i + j)p$ and Y the even numbers $< 2ip$; to describe the edges, write $+'$ for addition (mod $2ip$) (so that $s +' t$ is the unique z with $0 \leqq z < 2ip$ such that $2ip$ divides $(s + t) - z$). G has edges of the following two kinds:

(1) $\{y, y +' 1 +' 2k\}$, $y \in Y$ and $0 \leqq k < i$;

(2) $\{x, y\}$ where $x \geqq 2ip$, $y \in Y$ and $y \equiv x \pmod p$.

5.6 *Construction for* $h < h_0$. Let H be a bipartite graph with (h, j)-bipartition $(NH - Y, Y)$, as constructed in 5.5 for type (h, h, j). We obtain G by "replacing" each node x of $NH - Y$ by a copy K_x of a certain graph K of order i^*.

K has nodes $\{1, 2, \ldots, i^*\}$. Let T be the set $\{1, 2, \ldots, h\}$. K will be a graph with $i - 1 \leqq dxK \leqq j - 1$ for $z \in T$ and $i \leqq dzK \leqq j$ for $z \in NK - T$. We construct the *complement* L of K. When $h = 1$ and i is odd, L has (all but one) components of order 2 and a star component of order 3 with center node 1. In all other cases, $L = (L/T) \cup (L - T)$, where:

(1) For h even: L/T is a 1-graph, $L - T$ is a 0-graph.

(2) For h odd, i even: L is a 0-graph.

(3) For h odd and $\geqq 3$, i odd: L/T is a circuit, $L - T$ is a 1-graph.

For each $x \in NH - Y$, let K_x be an isomorph of K such that K_x has no node in common with H or with any $K_{x'}$, $x' \neq x$. Obtain G from the union H/Y and of the K_x $(x \in NH - Y)$ by adding for each x a set E_x of h edges obtained as follows. Let $y_{1x}, \ldots, y_{hx}$ be the distinct nodes which adjoin x in H and let E_x comprise all edges $\{k_x, y_{kx}\}$ $(1 \leqq k \leqq h)$, where k_x is the node k in the copy K_x of K.

The construction makes G h-node-connected, because H is h-node-connected and so is the graph K^* obtained by adding to K a new node z and joining z to each node of T.

6. Special types. We have now determined μ, μ^*, w for all nontrivial types except μ^* for special types (cf. 5.3, 5.4). Now assume (h, i, j) special. Only finitely many nonisomorphic exact graphs exist. To deal with this difficulty, we modify the procedure of § 4 for the vector A. We shall define a vector B and a constant B_0 such that $B_k > 0$ for $k \in \{0, 1, 2, 3\}$ and such that the inequality $B * M \geqq -B_0$ holds, with equality for infinitely many $(1, i, j)$-graphs. We then will have $\mu^*(h, i, j) = 1 - (B_1/B_3)$.

6.1 *Definitions*. The B_k are defined so that:

$$B * (U^2 - U^1) = 0 = B * (U^3 + (j-1)U^1);\ B_0 = -B * U^1.$$

These conditions imply that $B * U^2 = -B_0$, $B * U^3 = (j-1)B_0$.

It suffices to define:

$$B_0 = 2(h_0 - 1);\ B_1 = (j-2)(i-1);\ B_2 = (j-2)(i^* - 1);$$
$$B_3 = i + i^*(ij - i - j).$$

6.2 THEOREM. *Suppose Y nonsingular. Then $B * M \geqq -B_0$.*

Proof. By the statements (0)–(3) of 4.4 and the conditions of 6.1, $B * M \geqq B_0((j-1)|Y| - c)$. Obtain H from G by contracting counties to single nodes H is connected, so $nH \leqq eH + 1$ (with equality if and only if H is a tree). Now $nH \geqq cG + |Y|$, $eH \leqq \sum(dyG : y \in Y) \leqq j|Y|$. Hence

$$(j-1)|Y| - c \geqq -1,$$

so $B * M \geqq -B_0$.

6.3 *Remarks*. (1) In particular, taking $i = j = 3$: all connected 3-upper graphs satisfy $9m + 1 \geqq 2e + n$; all connected 3-graphs satisfy $9m + 1 \geqq 4n$, and $\mu^*(1, 3, 3) = 8/9$.

(2) As noted in 4.9 (5) (for A in place of B) every connected j-upper graph G with $B * M < -B_0$ is i-lower of order i^* (so that for i even, G is complete on $i + 1$ nodes).

6.4 *Definition*. Suppose G of type $(1, i, j)$. Y is *B-exact* if $B * M = -B_0$. G is *B-exact* if some subset of NG is B-exact.

6.5 *Remark*. Analogously with 5.2 *et seq.*, if Y is B-exact, and $nG > (B_3 - B_0)/B_1$ then Y is nonsingular and r-maximum and then $\mu G = 1 - ((B_1 + (B_0/nG))/B_3)$. Further, nonempty Y is B-exact if and only if all the following conditions hold: every small county K satisfies $nK = 1$, $d(NK)G = i$; every large county K satisfies $nK = i^*$, $d(NK)G = 1$; every node of Y has degree j and adjoins no other node of Y; and each node $y \in Y$ is a cut node of G (or, equivalently, the result H of contracting counties to single nodes is a tree).

We now wish to construct infinitely many B-exact graphs.

6.6 *Construction*. Let H be a tree (with $Y \subseteq NH$) as constructed in 3.2 for (i, j). Obtain G by "replacing" each end node x with a copy of the graph K, with K and "replacement" defined as in 5.6.

6.7 *Alternative construction*. Call a connected j-upper graph G *closed* if G is i-lower, and *open* otherwise. Define $M'G = (nG, qG, 2mG - nG)$ (so that $MG = M'G$ if Y is r-maximum). Two graphs with the same value for $A * M'$

(or $B * M'$) are *A-equivalent* (or B-equivalent). We know that for nonspecial type infinitely many graphs of that type can be A-equivalent (with $A * M' = 0$), and that for a special type infinitely many can be B-equivalent (with $B * M' = -B_0$).

PROPOSITION. (*Within the type* $(1, i, j)$) *every open graph is B-equivalent with infinitely many open graphs and with infinitely many closed graphs.*

Proof. Consider two "graph extension relations":

(1) H *ope-extends* G if G is a subgraph of H and $NH - NG$ comprises exactly j distinct nodes $x_1, \ldots, x_j$; and for some node x_0 of G with $dx_0G < i$ we have $EH - EG$ comprising all pairs $\{x_1, x_k\}$ with $0 \leqq k \leqq j$, $k \neq 1$.

(2) H *clo-extends* G if H results from G by "replacing" some node x with $dxG = 1$ by a copy of K, as in 5.6.

Note that if H ope-extends G then H is open and H and G are B-equivalent (since $M'H - M'G = (j-1)U^1 + U^3$); and if H clo-extends G then H and G are B-equivalent (since $M'H - M'G = U^2 - U^1$) and H is "more closed" than G: fewer nodes have degree $<i$ in H than do in G. To obtain an open graph H_1 and a closed graph H_2 each B-equivalent with G, take H_1 any iterated ope-extension of G, take H_1' an iterated ope-extension of H_1 in which all nodes of degree $<i$ have degree 1, and take H_2 an appropriate closed iterated clo-extension of H_1. The closed graphs so obtained from the trivial graph are precisely the B-exact graphs.

7. Triangle-free graphs. How are μ, μ^*, w affected if in §2 we add to the definition of $T(h, i, j)$ the requirement that the member graphs be triangle-free? The answer is easily determined when (h, i, j) is trivial or $j \geqq i + 2$ or $h = j$. Indeed (except when $j = 2$) the values of μ, μ^* are not altered at all, and when $j \geqq i + 2$, w is altered only in that, for i even and $j = i + 2$, w now excludes complete graphs on $i + 1$ nodes.

For remaining types, i.e. nontrivial types with $j \leqq i + 1$ and $h \neq j$, one may carry out the procedures of §§4–6 with the definitions of 4.1 and 4.4 slightly modified as follows: "small" now means $n \leqq 2i$, "large" means $n > 2i$,

$$U^2 = (2i + 1, h, -1), \quad h_0 = i(i + 1 - j),$$
$$A^2 = (j - h, 2i + 2, (2i + 1)j + h).$$

The other statements of 4.1 remain in force. Theorem 4.5 remains true. Its proof (in 4.4) has to be modified to show that an odd county (triangle-free) K with $i < nK < 2i$ satisfies $qK \geqq i$. To show this, write s for $(nK - 1)/2$ and note that K would include a triangle if more than s nodes x satisfied $dxK > s$. Hence, at least $s + 1$ nodes x of K satisfy $\max(dxG, i) - dxK \geqq i - s$; whence $qK \geqq (s + 1)(i - s)$. Since $i < nK < 2i$, $qK \geqq i$.

When $j = i + 1$, $A = A^1$ and we have $A * M \geqq 0$ even for Y singular (with strict inequality with Y singular and $n > 2i + 1$). Hence, for $j = i + 1$,

$\mu = 2i/(2i + 1)$; and w consists of graphs which are (i, j)-bipartite or of order $2i + 1$. When $j = i$, $A = A^2 = (i - h, 2i + 2, 2i^2 + i + h)$. As in 5.6 we obtain graphs which make exact the bound $\mu \geqq 1 - (A_1/A_3)$, except that now: H is chosen of even order (so that "replacement" preserves h-odd-connectedness), K is chosen of order $2i$, $NK = \{1, 2, \ldots, 2i\}$ and G is obtained by adding sets E_x to the disjoint union of H and of all the K_x. To describe E_x and EK, write s for $(i - h)/2$: s is a positive integer. E_x comprises all $\{x, k_x\}$, where $1 \leqq k \leqq s$ or $i + 1 \leqq k \leqq i + s$. EK comprises all $\{x_k, x_l\}$, where: $k \leqq i$ and $l > i$ and not both $k \leqq s$ and $l = i + k$. G is in general *not* h-edge-connected but is h-odd-connected: the proof uses the fact that H, K are both h-odd-connected and of even order. For μ^* when $j = i$ and (h, i, j) is special, i.e., is $(1, i, i)$ with i odd and $\geqq 3$—we carry out the procedure of § 6 with

$$B = ((i - 1)(i - 2), 2i(i - 2), i(2i^2 - 3i - 1))$$

and

$$B_0 = 2(h_0 - 1) = 2(i - 1).$$

8. Variations.

8.1 *Density*. Let (h, i, j) be a type. A class S of graphs is *dense* (*for* (h, i, j)) if $S \subseteq T(h, i, j)$ and $\mu^*S = \mu^*(h, i, j)$. We contend that almost any non-empty "interesting" class S which one is likely to define by "h-connectivity" and "(i, j)-degree" conditions can in fact be verified to be dense (or sometimes dense for $(h + 1, i, j)$). Now, for any class S known to be dense we not only know μ^*S but also can determine μS (and then wS). The reason for this is that we now know (or can readily deduce) that for certain (known) numbers $b \geqq 0$ and $n_0 > 0$ every (h, i, j)-graph of order $\geqq n_0$ satisfies $\mu \geqq \mu^*(h, i, j) - (b/n)$; and the set of (h, i, j)-graphs of order $\geqq n_0$ for which equality holds is dense and comprises graphs of known structure. Hence, to find μS we need only try to find graphs of S for which $\mu < \mu^*(h, i, j)$. When $b = 0$ (which is the case for all nontrivial nonspecial types with $h \neq j$) we need examine only the finitely many graphs of S of order $< n_0$. Even when $b > 0$ it suffices to find but one graph G_1 in S with $\mu G_1 < \mu^*(h, i, j)$ and thereafter to examine only the finitely many other graphs of S of order $< \max(n_0, n_1)$, where $n_1 = b/(\mu^*(h, i, j) - \mu G_1)$.

To verify density of S it suffices to check that $S \subseteq T(h, i, j)$ and to construct arbitrarily large graphs G in S with μG "close" to $\mu^*(h, i, j)$. We give one example of such construction.

8.2 *Construction. Suppose* (h, i, j) *a type with* $i < j$ *and* f, g *integers with* $2 \leqq f \leqq g \leqq h$. *We may construct an* (h, i, j)*-graph* G *as large as desired and with* μG *as close as desired to* $\mu^*(h, i, j)$, *and with* G *strictly* f*-node-connected, strictly* g*-edge-connected, strictly* h*-odd-connected.* Indeed, G can be constructed "almost biregular": each node but at most four will have degree i or j.

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Further, if type $T(h, i, j)$ is defined to comprise only triangle-free graphs, G can be constructed triangle-free.

From 5.5 or 5.6, there is a large h-node-connected (h, i, j)-graph H_0 of even order with μH_0 "close" to $\mu^*(h, i, j)$. Let T be a "large enough" set of nodes of degree i in H_0, no pair of which adjoin each other or a common third node. Obtain from H_0 a graph H_1 by adding a new node x and joining x to i nodes of T. Define t to be 0 if f is even and 1 if f is odd. For any integer k and set X and graph H write X^k for $Xx\{k\}$ and H^k for the isomorph of H under the map

$$[x \to (x, k) : x \in NH].$$

Let G_1 be the union of the graphs $H_0{}^1$, $H_t{}^2$, $H_1{}^3$, $H_1{}^4$.

We obtain G from G_1 by joining: $H_0{}^1$ to $H_t{}^2$ with an "f-node-cut"; $H_t{}^2$ to $H_1{}^3$ with a "g-edge-cut"; and $H_1{}^3$ to $H_1{}^4$ with an "h-edge-cut". More precisely, let X be a set of f new nodes not in G_1. Obtain G by adding to G_1 the nodes X and new edges as follows: join each $x \in X$ to $[j/2]$ nodes of T^1 and to $j - [j/2]$ nodes of T^2, and join T^2 to T^3 with g edges and T^3 to T^4 with h edges, with at most one new edge incident with any node in any of the T^k.

When T is large enough, G can be taken "almost biregular" by adding edges joining nodes within each T^k. This can be done so that: at most one node in each T^k does not have degree i or degree j, global connectivities are not altered, and triangles are not introduced.

8.3 *Full localization.* One might object to our definition of $T(h, i, j)$ for $h > 1$ on the grounds that h-odd-connectivity is not a "local" property, i.e. a property of connected graphs which depends only on the connected subgraphs of diameter less than some fixed bound D. However, given (h, i, j) with $1 < h < j$ we can find D such that all arguments and proofs of §§ 4–7 (and values of μ, μ^*, w) are valid with $T(h, i, j)$ replaced with the larger class of all connected graphs G having the following local property: G is of degree (i, j) and every connected proper odd-order subgraph K of diameter $<D$ satisfies $d(NK)G \geqq h$. It suffices to take D so that $A * (D, 1, -1) > A * U^2$, i.e., so that $D > U_1{}^2 + (h - 1)A_2/A_1$. If we are willing to take D somewhat larger, we can "localize" all connectivities in the above Construction 8.2. In that construction we can replace each of the H^k for $k \neq 1$ by fixed graphs of not too large order, independent of $H_0{}^1$.

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