ON SINGULAR FIBERINGS

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1. The purpose of this note is to discuss some properties of singular fibrations introduced by Montgomery and Samelson **(4)** and studied by Hu **(2)** and Conner and Dyer **(1)**. This work was motivated by the last paper, where singular fibrations in which the fibre is a sphere were studied. Here more general fibres will be considered.

A fibre space with singularities is a quadruple $[(X, A), (Y, B), \pi, F]$ such that

(1) $\pi : (X, A) \to (Y, B)$ is a proper open onto mapping,

(2) $\pi^{-1}(B) = A$ and π/A is a homeomorphism, and

(3) $\pi/X - A : X - A \to Y - B$ is a fibre mapping with fibre F.

The cohomology used will be the Čech cohomology groups with coefficients in Z_2. Cohomology groups with compact supports will be denoted by a subscript c.

In addition we require that $\pi_1(Y - B)$ acts simply on $H^q(F)$. This will be true in particular if π is a bundle map and the structure group is pathwise connected **(5)**. We also assume that if $\dim F = r$, then $H^r(F) = Z_2$.

2. One of the principal tools we shall use is a truncated exact sequence obtained from the spectral sequence of $(X - A, Y - B, F)$. The following description is from Hu **(2**, p. 241**)**. Let $\lambda \leqslant m \leqslant \mu$ be integers. Then a spectral sequence satisfies the two-term condition $\{\lambda, \mu; 2\}$ if E^2 has the following properties. For each integer m such that $\lambda \leqslant m \leqslant \mu$, $E_{p,q}{}^2 = 0$ if $p + q = m$ and (p, q) is different from two given pairs (a_m, b_m) and (c_m, d_m) where

$$a_m + b_m = m = c_m + d_m, \qquad a_m < c_m.$$

Moreover, the following condition also must be fulfilled:

(1) $E_{p,q}{}^2 = 0$ if $p + q = m - 1$, $p \leqslant a_m - 2$, and $\lambda \leqslant m \leqslant \mu$,

(2) $E_{p,p}{}^2 = 0$ if $p + q = m + 1$, $p \geqslant c_m + 2$, and $\lambda \leqslant m \leqslant \mu$.

Then, if the spectral sequence is a regular δ sequence we have the following exact sequence:

$$E^2_{c_\lambda d_\lambda} \to \dots \to E^2_{c_m d_m} \to \mathfrak{H}_m \to E^2_{a_m, b_m} \to \dots \to E^2_{a_\mu b_\mu}.$$

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We recall that for a fibre space $(X - A, Y - B, F)$ with coefficients in Z_2, we have $\mathfrak{H}_m = H_c^m(X - A)$ and $E_{pq}^2 = H_c^p(Y - B; H^q(F))$, **(3**, pp. 270–271**)**.

Our first theorem is similar to Theorem 1.1 of **(1)**.

THEOREM 1. *If* $[(X, A), (Y, B), \pi, F]$ *is a singular fibering of a finite-dimensional space* X *by an r-dimensional fibre* F *such that for some integer* m, $H^i(X) = 0$ *for* $i \geqslant m$, *then* $H^i(A) = 0$ *for* $i \geqslant m$ *and* $H^i(Y) \simeq H^i(B)$ *for* $i \geqslant m - r$.

Proof. Consider the following diagram

$$\begin{array}{ccccc} H^i(A) & \xrightarrow{\delta^*} & H_c^{i+1}(X - A) & \rightarrow & H^{i+1}(X) \\ \simeq\uparrow \pi_1^* & & \uparrow \pi^* & & \\ H^i(B) & \rightarrow & H_c^{i+1}(Y - B) & \rightarrow & H^{i+1}(Y) \end{array}$$

Since $H^i(X) = 0$, $i \geqslant m$, the homomorphism δ^* is onto for $i \geqslant m - 1$, but π_1^* is an isomorphism. Thus π^* is onto for $i \geqslant m - 1$.

Let s be the least integer such that $H_c^s(Y - B) \neq 0$. Since X is finite-dimensional, Y is also finite-dimensional and hence s exists. We wish to apply the truncated exact sequence to this situation with $\mu = s + r + 1$ and $\lambda = s + r$. If $p + q = r + s + 1$ then either $p > r$ or $q > s$. In both cases $H_c^p(Y - B; H^q(F)) = 0$. If $p + q = r + s$, then if $p > r$ or $q > s$ we have $H^p(Y - B; H^q(F)) = 0$. But if $p = r$ and $q = s$, $H^p(Y - B; H^q(F))$ is not necessarily zero. If we take for our two pairs

$$\begin{pmatrix} s & r + 1 \\ s + 1 & r \end{pmatrix}_{s+r+1} \quad \text{and} \quad \begin{pmatrix} s & r \\ s + 1 & r - 1 \end{pmatrix}_{s+r},$$

then conditions 1 and 2 are satisfied, as is easily checked. Our exact sequence then is

$$(1) \quad H_c^{s+1}(Y - B; H^{r-1}(F)) \rightarrow H_c^{r+s}(X - A) \rightarrow H_c^s(Y - B; H^r(F)) \rightarrow H_c^{s+1}(Y - B; H^r(F)).$$

But the first and last terms are zero; therefore

$$H_c^{r+s}(X - A) \simeq H_c^s(Y - B; H^r(F)) = H_c^s(Y - B).$$

By the induction hypothesis $H_c^{r+s}(Y - B) = 0$. But π^* maps this onto $H_c^{r+s}(X - A)$ if $r + s \geqslant m$; therefore, $H_c^{r+s}(X - A) = 0 = H_c^s(Y - B)$. This proves that $s < m - r$ and $H_c^i(X - A) = 0$ if $i \geqslant m$. This, together with $H^i(X) = 0$ if $i \geqslant m$, implies that $H^i(A) = 0$ if $i \geqslant m$. If $i \geqslant m - r$ we have $H^i(Y) \simeq H^i(B)$ since $H_c^i(Y - B) = 0$.

3. In this section some additional results are obtained under the hypothesis that X is a (mod 2) n-sphere. We also assume that $H^r(F) = Z_2$.

THEOREM 2. *Suppose* X *is a cohomology* (mod 2) *n-sphere. Then either* A *is a cohomology n-sphere and* $Y - B$ *is acyclic or the cohomology* dim $A < n - r$ *and* $H_c^{n-r}(Y - B) = Z_2$.

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Proof. As in Theorem 1 we have that $\pi^*: H^i(Y - B) \to H^i(X - A)$ is onto for all i except $i = n$. Theorem 1 implies that $H_c{}^i(X - A) = 0$ for all $i > n - r$ except $i = n$. The exact sequence (1) applies here and we conclude that $H_c{}^m(X - A) \simeq H_c{}^{n-r}(Y - B)$. Consider the sequence

$$H_c^{n+1}(X - A) \leftarrow H^n(A) \leftarrow H^n(X) \xleftarrow{j^*} H_c^n(X - A) \xleftarrow{\delta^*} H^{n-1}(A).$$

Since $\delta^* = \pi^*\delta^*\pi_1{}^{*-1}$ and $H_c{}^n(Y - B) = 0$, we see that δ^* is trivial. Since $H^{n-1}(X) = 0$, therefore $H^{n-1}(A) = 0$. Then $\ker j^* = 0$. But $H^n(X) = Z_2$; therefore j^* is either an isomorphism or is trivial. Hence either $H^n(A) = 0$ and $H_c{}^n(X - A) = Z_2$ or conversely. In the first case $H^i(A) = 0$ for all $i \geqslant n - r$ from the sequence of the pair (X, A). In the second case we have $H_c^n(X - A) \simeq H_c^{n-r}(Y - B) = 0$. Then the induction argument of Theorem 1 applies, proving that $H_c{}^i(Y - B) = 0$ for all $i > 0$.

In the rest of the paper we investigate what happens if A is not a cohomology n-sphere, that is, we require in the remainder of the paper that $H_c{}^s(Y - B) = Z_2$ where $s = n - r$.

THEOREM 3. *If* $r > 1$ *then* $H_c{}^{s-1}(Y - B) = 0$, $H_c{}^{s-2}(Y - B) = H^{r-1}(F)$, $H^{s-2}(B) = H^s(B) = 0$ *and either* $H^s(Y) = Z_2$ *or* $H^{s-1}(B) = Z_2$ (*the other group in each case being zero*).

Proof. We wish to apply the two-term exact sequence to this situation. We take $\mu = r + s$ and $\lambda = r + s - 2$ where $a = n - r$. Then, as before

$$\begin{pmatrix} s & r \\ s+1 & r-1 \end{pmatrix}$$

is an acceptable p, q pair for $m = \mu$. For $m = r + s - 1$ the only interesting pair will be

$$\begin{pmatrix} s-1 & r \\ s & r-1 \end{pmatrix}.$$

For $m = r + s - 2$ the interesting pair will be

$$\begin{pmatrix} s-2 & r \\ s-1 & r-1 \\ s & r-2 \end{pmatrix}.$$

We shall use the first two to show $H^{s-1}(Y - B) = 0$ and then the two-term condition will be satisfied by

$$\begin{pmatrix} s-2 & r \\ s & r-2 \end{pmatrix}.$$

$$H_c{}^s(Y - B; H^{r-1}(F)) \to H_c^{n-1}(X - A) \to H_c{}^{s-1}(Y - B; H^r(F)) \to H_c{}^{s+1}(Y - B; H^{r-1}(F)).$$

The last term is zero and $H_c^{n-1}(X-A)=0$; therefore $H_c^{s-1}(Y-B; H^r(F))=0$. But this is just $H_c^{s-1}(Y-B)$. Hence we can add the next term and we have

$$\mathrm{H}_c^s((Y-B);H^{r-2}(F))\to H_c^{n-2}(X-A)\to H_c^{s-2}(Y-B;H^r(F))\to H_c^s(Y-B;H^{r-1}(F))\to 0.$$

Therefore $H_c^{s-2}(Y-B;H^r(F))\simeq H_c^s(Y-B;H^{r-1}(F))$. But this is just $H_c^{s-2}(Y-B)\simeq H^{r-1}(F)$. The final statement comes from considering

$$\begin{array}{ccccccc} & & H_c^{s-1}(X-A) & \to & H^{s-1}(X) & \to & H^{s-1}(A) \\ & & \uparrow & & \pi_1^*\uparrow & & \uparrow\pi^* \\ H^{s-2}(B) & \to & H_c^{s-1}(Y-B) & \to & H^{s-1}(Y) & \xrightarrow{i^*} & H^{s-1}(B) \end{array}$$

$$\begin{array}{ccccccccc} \to & H_c^s(X-A) & \to & H^s(X) & \to & H^s(A) & \to & H_c^{s+1}(X-A) \\ & \uparrow & & \uparrow & & \uparrow & & \\ \to & H_c^s(Y-B) & \to & H^s(Y) & \to & H^s(B) & & \end{array}$$

We see that $H^s(B)\simeq H^s(A)=0$. That $H^{s-2}(B)=0$ follows by the same argument used to show that $H^{n-1}(A)=0$. Now i^* is an isomorphism into, but since π_1^* is trivial and π^* is also an isomorphism we see that i^* is trivial. Hence $H^{s-1}(Y)=0$. Now we are left with

$$0\to H^{s-1}(B)\to Z_2\to H^s(Y)\to 0.$$

Hence either $H^{s-1}(B)$ or $H^s(Y)$ vanishes and the other equals Z_2.

If we further restrict F we can get more specific information. The following theorems are examples of such results.

THEOREM 4. *Suppose $H^s(Y)=0$ and $H^{r-3}(F)=H^{r-4}(F)=0$ (for example, if $r=2$), then we have the exact sequence*

$$0\to H_c^{s-5}(Y-B)\to H_c^{s-2}(Y-B)\xrightarrow{\pi^*} H_c^{s-2}(X-A)\to 0.$$

In addition $H_c^{s-3}(Y-B)=0$ and $H_c^{s-4}(Y-B)\simeq H_c^{s-2}(Y-B;H(F))$.

Proof. By Theorem 3 $H_c^{s-1}(Y-B)=0$. Consider

$$\begin{pmatrix} s-2 & r \\ s & r-2\end{pmatrix},\quad \begin{pmatrix} s-3 & r \\ s-2 & r-1\end{pmatrix},\quad \begin{pmatrix} s-4 & r \\ s-3 & r-1 \\ s-2 & r-2\end{pmatrix},\quad \begin{pmatrix} s-5 & r \\ s-4 & r-1 \\ s-3 & r-2\end{pmatrix}.$$

Using the first two-pair and the truncated exact sequence we shall show that $H_c^{s-3}(Y-B)=0$. The result then follows from the sequence by using the remaining pair.

Now for the proof that $H_c^{s-3}(Y-B)=0$.

Using the first two-pair, and setting $r=2$, the truncated exact sequence becomes

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$$H_c^{s-2}(Y-B;H^{r-1}(F)) \to H_c^{s-1}(X-A) \to H_c^{s-3}(Y-B)$$
$$\to H_c^{s}(Y-B) \xrightarrow{\pi^*} H_c^{s}(X-A) \to 0.$$

We have a zero for the last map since π^* is onto. But $H_c^s(Y-B) = Z_2$ and since $H^s(Y) = 0$, $H^{s-1}(B) = Z_2 = H^{s-1}(A)$. Hence π^* is also an isomorphism. From Theorem 3 we see that $H_c^{s-1}(X-A) = 0$; hence $H_c^{s-3}(Y-B) = 0$. This also implies $H_c^{s-3}(X-A) = 0$.

THEOREM 5. *If $H^{r-1}(F) = 0$ and $r > 4$, then $H_c^{s-4}(Y-B) = H^{r-3}(F)$, $H_c^{s-3}(Y-B) \simeq H^{r-2}(F)$, and $H_c^{s-2}(Y-B) = 0 = H^{s-3}(B) = H^{s-2}(Y) = H^{s-1}(Y)$.*

The proof of this result is similar to the above and since the result is rather special we omit the proof.

4. We now consider some applications. In this section, X is always a (mod 2) n-sphere. We shall need the following lemma.

LEMMA 8. *If $H^s(Y) \neq 0$, Y satisfies Poincaré duality, and B is connected, we have $H^1(Y) = 0$, and $H^2(Y) = H^1(F)$.*

Proof. Since B is connected and $H^{s-1}(Y) = 0$ we have $H^1(Y) = 0$. Hence $H_c^1(Y-B) = 0$. Hence we have

$$0 \to H^1(F) \to H_c^2(Y-B) \xrightarrow{\pi^*} H_c^2(X-A) \to 0. \tag{2}$$

The last zero follows from the fact that π^* is onto.

We also have the diagram

$$0 \to H^1(B) \xrightarrow{\delta^*} H_c^2(Y-B) \xrightarrow{j^*} H^2(Y) \to 0$$
$$0 \to H^1(A) \to H_c^2(X-A) \to 0$$

π^* is an isomorphism on the image of δ^*. Hence $\ker \pi^* \cap \operatorname{im} \delta^* = 0$. We also have $\ker \pi^* \oplus \operatorname{im} \delta^* = H_c^2(Y-B)$. Hence j^* is an isomorphism on $\ker \pi^*$. But from (2) we see that $\ker \pi^* = \operatorname{im} \tau$ and τ is an isomorphism into, hence $\operatorname{im} j^* \simeq H^1(F)$.

4.1. *If Y has dimension 3, $r \geqslant 2$, $H^3(Y) \neq 0$, then Y is a* (mod 2) *3-sphere and B is acyclic with $\tilde{H}^0(B) = H^{r-1}(F)$ ($\tilde{H}^0$ is the reduced group).*

This follows directly from Theorem 3.

4.2. *If Y has dimension 3 and $H^3(Y) = 0$, then B is a* (mod 2) *cohomology 2-sphere, Y is acyclic, and $H^1(F) = 0$.*

Proof. We know that $H_c^2(Y-B) = 0$ and $H_c^{s-2}(Y-B) = H^{r-1}(F) = H^1(F)$. Consider the truncated sequence for

$$\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$$

We have

$$H_c^3(Y - B; H^{-1}(F)) \to H_c^2(X - A) \to H_c^1(Y - B; H^1(F))$$
$$\to H_c^3(Y - B; H^0(F)) \xrightarrow{\pi^*} H_c^3(X - A) \to H_c^1(Y - B; H^2 f)).$$

Now π^* is the projection map, which is an isomorphism here if $H^3(Y) = 0$. Therefore $0 \to H_c^1(Y - B; H^1(F)) \to 0$ or $H^1(F) \otimes H^1(F) = 0$; hence $H^1(F) = 0$.

4.3. *If Y is 4-dimensional, satisfies Poincaré duality, $r \geqslant 2$, and B is connected, then B is acyclic and $H^2(Y) = H^1(F)$ and $H^3(Y) = H^1(Y) = 0$.*

4.4. *If Y is 4-dimensional, $H^4(Y) = 0$, F is 2-dimensional, then F is a cohomology 2-sphere, B is a cohomology 3-sphere, and Y is acyclic.*

Proof. Theorem 4 implies $H_c^{s-3}(Y - B) = 0$ and $H_c^{s-2}(Y - B; H^1(F)) \simeq H^0(Y - B) = 0$. But since $H_c^{s-2}(Y - B) \simeq H^1(F)$, we see that $H^1(F) = 0$. Therefore $H_c^{s-2}(Y - B) = 0$. Hence if $H^4(Y) = 0$, Y is acyclic and B is a cohomology 3-sphere.

4.5. *If* dim $Y = 5$, $H^3(Y) = 0$, *then*

$$H^4(Y) = H^3(Y) = H^2(Y) = H^3(B) = H^1(B) = 0,$$

$$H^2(B) = H^1(F), \qquad H^4(B) = H^0(B) = Z_2, \qquad H^1(Y) = H^1(F) \otimes_{Z_2} H^1(F).$$

This follows as above from Theorem 4.

4.6. *If Y is a cohomology 5-manifold and $r \geqslant 2$ and B is connected, then $H^4(Y) = H^1(Y) = 0$ and $H^2(Y) = H^3(Y) = H^1(F)$. B has cohomology dimension $\leqslant 1$. If $H^{r-1}(F) = 0$, then $H^1(B) = H^2(F)$.*

Remark. In **(4)** Montgomery and Samelson suggest that it seems likely that the singular set of a fibering of a sphere must be a cohomology sphere. Results 4.1, 4.2, 4.3, and 4.4 answer the question affirmatively if the dimension of Y is less than or equal to 4. But 4.5 shows that if there is a singular fibration of S^7 with a 2-dimensional fibre which is not a cohomology sphere then the conjecture is false. We have not been able to construct such a map.

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