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ANALYTIC TAF ALGEBRAS

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ABSTRACT A strongly maximal triangular AF algebra which is defined by a real-valued cocycle is said to be analytic Formulas for generic cocycles are given separately for both the integer-valued case and the real-valued coboundary case, and also for certain nest algebras In the case of an integer-valued cocycle, there is an associated partial homeomorphism of the maximal ideal space of the diagonal If the partial homeomorphism extends to a homeomorphism, then the algebra embeds in a crossed product This occurs for a large class of subalgebras of UHF algebras, but an example shows that this does not always occur An example is given of a triangular AF algebra which is analytic via a coboundary but is not a nest algebra, also, it is shown that a nest algebra need not be analytic

Motivated by the studies in [Ba] and [Pr1], the development of a general theory of triangular subalgebras of AF algebras was first undertaken in [PPW], and further extended in a number of other papers, for example [HP, MS1, MS2, Po2, Po3, Pr2, Pr3, PW, T, V1, V2]. The theory of TAF algebras parallels that of the σ-weakly closed triangular subalgebras of von Neumann algebras, expounded in [MSS1] and [MSS2]. A TAF subalgebra $\mathcal{T}$ of an AF algebra $\mathfrak{A}$ is said to be *analytic* if there is a one-parameter family $\{\alpha_t\}$ of automorphisms of $\mathfrak{A}$, leaving the diagonal pointwise fixed, such that $\mathcal{T} = \{a \in \mathfrak{A} : \mathrm{sp}_\alpha(a) \subseteq [0,\infty)\}$ (where sp is the Arveson spectrum) [R]. Viewing $\mathfrak{A}$ as a groupoid C^*-algebra $C^*(\mathcal{R})$, there is a subset $\mathcal{P}$ contained in $\mathcal{R}$ such that $\mathcal{T}$ consists of those elements of $C^*(\mathcal{R})$ supported on $\mathcal{P}$ [MS1]. S. Power first noticed that $\mathcal{P}$, which he called the fundamental relation, completely determines $\mathcal{T}$ [Pr3]. From this perspective, the analyticity of $\mathcal{T}$ is equivalent to the existence of a real-valued cocycle d such that $\mathcal{P} = d^{-1}[0,\infty)$. In the case of a σ-weakly closed triangular algebra $\mathcal{T}$, $\mathcal{T}$ is a nest algebra if and only if it is analytic and the cocycle is trivial, *i.e.*, a coboundary [MSS2, Corollary 3.4]. By contrast, in our setting it can happen that $\mathcal{T}$ is trivially analytic but not a nest algebra (Examples 3.7 and 3.17). On the other hand, $\mathcal{T}$ can be a nest algebra without being trivially analytic (Example 3.18).

The class of analytic TAF algebras is properly contained in the strongly maximal ones [SVe, PWo]. In this paper, we concentrate on those analytic TAF algebras which are trivially analytic, nest algebras, or analytic by means of an integer-valued cocycle ($\mathbb{Z}$-analytic). For TAF subalgebras of simple (infinite dimensional) AF algebras, the latter class is disjoint from the first two (Theorem 3.1 and Proposition 4.4). Of course, these do not exhaust the class of analytic TAF algebras (Example 2.3).

Section 2 is concerned with $\mathbb{Z}$-analytic TAF algebras. The main results are Theorems 2.2 and 2.8. The first result gives a simple necessary and sufficient condition for

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$\mathbb{Z}$-analyticity, and provides a useful generic form for a $\mathbb{Z}$-valued cocycle. The latter result answers the question, raised implicitly in [PPW, Examples 1.2 and 1.3], as to which TAF algebras can be imbedded in semicrossed products, at least for those TAF algebras considered in [PPW, § 4]. Specifically, if $\mathfrak{A}$ is the closed union of factors $\mathfrak{A}_n$ with $\mathcal{T} \cap \mathfrak{A}_n$ maximal triangular in $\mathfrak{A}_n$, and if $\mathcal{T}$ is $\mathbb{Z}$-analytic, then there is a homeomorphism ϕ of X such that $\mathcal{P} = \{(x, \phi^n(x)) : x \in X, n \in \mathbb{Z}, n \geq 0\}$. There is, however, an example of a $\mathbb{Z}$-analytic subalgebra $\mathcal{T}$ of a UHF algebra for which no such homeomorphism exists (Example 2.10). This shows that the algebras considered in [PPW, §4] are indeed a special type of strongly maximal TAF algebra.

In Sections 3 and 4, we discuss real-valued coboundaries. Theorem 3.16, the main result, gives necessary and sufficient conditions for an analytic TAF algebra to be generated by a real-valued coboundary, and also gives a generic form for the coboundary. A similar result for certain nest algebras follows in Theorem 4.6. We also give a number of examples illustrating various phenomena, and some computational tools related to the conditions in Theorem 3.16.

1. **Preliminaries.** An *AF algebra* is a C^*-algebra $\mathfrak{A}$ which has an increasing sequence of finite dimensional C^*-subalgebras $\{\mathfrak{A}_n : 1 \leq n < \infty\}$ such that $\mathfrak{A} = \overline{\bigcup_{n=1}^\infty \mathfrak{A}_n}$. If the sequence $\{\mathfrak{A}_n\}$ can be chosen so that each $\mathfrak{A}_n$ is a factor (*i.e.*, $\mathfrak{A}_n \cong \mathbf{M}_{k_n}$, the $k_n \times k_n$ matrix algebra), then $\mathfrak{A}$ is said to be a *UHF algebra*. In this paper, whenever we use the notation $\mathfrak{A} = \overline{\bigcup_{n=1}^\infty \mathfrak{A}_n}$, we will always assume that the sequence $\{\mathfrak{A}_n\}$ is increasing, $\mathfrak{A}$ is unital, and $\mathfrak{A}_1$ contains the unit 1 of $\mathfrak{A}$.

An AF algebra $\mathfrak{A}$ can also be defined as an inductive limit $\varinjlim(\mathfrak{A}_n, j_n)$ of finite dimensional C^*-algebras $\mathfrak{A}_n$ with $j_n \colon \mathfrak{A}_n \hookrightarrow \mathfrak{A}_{n+1}$ a unital C^*-embedding [Br]. Then $\mathfrak{A}_n$ is isomorphic to a C^*-subalgebra $\tilde{\mathfrak{A}}_n$ of $\mathfrak{A}$ such that $\mathfrak{A} = \overline{\bigcup_n \tilde{\mathfrak{A}}_n}$, so in this case we will identify $\mathfrak{A}_n$ and $\tilde{\mathfrak{A}}_n$.

Suppose $\mathfrak{A} = \overline{\bigcup_{n=1}^\infty \mathfrak{A}_n}$ is an AF algebra, and suppose $\mathfrak{D}_n$ is a maximal abelian self-adjoint subalgebra (*masa*) of $\mathfrak{A}_n$ such that $\mathfrak{D}_n \subseteq \mathfrak{D}_{n+1}$ for each n. Let $\mathfrak{D} = \overline{\bigcup_{n=1}^\infty \mathfrak{D}_n}$. Then $\mathfrak{D}$ is a masa (also called the *diagonal*) of $\mathfrak{A}$, and $\mathfrak{D}_n = \mathfrak{D} \cap \mathfrak{A}_n$. By [SV], such a masa always exists, and we will always use the term masa to refer to a masa of this form. Let j_n denote the embedding of $\mathfrak{A}_n$ into $\mathfrak{A}_{n+1}$. If $\mathfrak{A}_n = \bigoplus_{m=1}^{\ell(n)} \mathbf{M}_{k(n,m)}$, then for each n and m, a system of matrix units $\{e_{ij}^{(nm)}\}$ can always be chosen for $\mathbf{M}_{k(n,m)}$ so that each $j_n(e_{ij}^{(nm)})$ is a sum of matrix units of $\mathfrak{A}_{n+1}$, and $\mathfrak{D}$ is the closed linear span of $\{e_{ii}^{(nm)} : 1 \leq n, 1 \leq m \leq \ell(n), 1 \leq i \leq k(n,m)\}$ (see [PPW, §1] for details). Whenever we use matrix units in $\mathfrak{A}$, we will always assume that they are chosen in this manner. Also, we will often write $e_i^{(nm)}$ for $e_{ii}^{(nm)}$ and, if $\mathfrak{A}_n$ is a factor, $e_{ij}^{(n)}$ for $e_{ij}^{(n1)}$.

All subalgebras of AF algebras in this paper will be norm-closed. If $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$ is an AF algebra with masa $\mathfrak{D}$, then a subalgebra $\mathcal{T}$ of $\mathfrak{A}$ is said to be *triangular* AF *(with diagonal* $\mathfrak{D}$), or TAF, if $\mathcal{T} \cap \mathcal{T}^* = \mathfrak{D}$. We often write TUHF instead of TAF if $\mathfrak{A}$ is a UHF algebra. A TAF subalgebra $\mathcal{T}$ of $\mathfrak{A}$ is said to be *maximal triangular* if $\mathcal{T}$ is the only TAF subalgebra containing $\mathcal{T}$. In addition, $\mathcal{T}$ is said to be *strongly maximal triangular*

[PPW, page 105] if the sequence $\{\mathfrak{A}_n\}$ can be chosen so that $\mathcal{T}\cap\mathfrak{A}_n$ is maximal triangular in $\mathfrak{A}_n$ for every n.

Suppose $\mathfrak{A}_n = \bigoplus_{m=1}^{\ell(n)} \mathbf{M}_{k(n,m)}$ for each n and $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$. For each k, let $\mathbf{T}_k$ be the set of upper triangular matrices of $\mathbf{M}_k$. Let $\mathcal{T}_n = \bigoplus_{m=1}^{\ell(n)} \mathbf{T}_{k(n,m)}$, and suppose the embedding $j_n\colon \mathfrak{A}_n \longrightarrow \mathfrak{A}_{n+1}$ takes $\mathcal{T}_n$ to $\mathcal{T}_{n+1}$. Then $\mathcal{T} = \overline{\bigcup_n \mathcal{T}_n}$ is a strongly maximal TAF subalgebra of $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$ by [PPW, Theorem 2.6]. Conversely, suppose $\mathcal{T}$ is a strongly maximal subalgebra of $\mathfrak{A}$ such that $\mathcal{T}_n = \mathcal{T}\cap\mathfrak{A}_n$ is maximal triangular in $\mathfrak{A}_n$ for each n. Then there exist permutation matrices $U_n \in \mathfrak{A}_n$ such that $U_n\mathcal{T}_nU_n^* = \bigoplus_{m=1}^{\ell(n)} \mathbf{T}_{k(n,m)}$, and the following diagram commutes:

$$\begin{array}{ccccccc}
\longrightarrow & \mathfrak{A}_n & \xrightarrow{j_n} & \mathfrak{A}_{n+1} & \longrightarrow \\
& {\scriptstyle \mathrm{Ad}\,U_n}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Ad}\,U_{n+1}} & \\
\longrightarrow & \mathfrak{A}_n & \xrightarrow[\mathrm{Ad}\,R_{n+1}\circ j_n]{} & \mathfrak{A}_{n+1} & \longrightarrow
\end{array}$$

where $R_{n+1} = U_{n+1}j_n(U_n^*)$. Thus, $\mathcal{T}$ is isomorphic to $\overline{\bigcup_n \bigoplus_{m=1}^{\ell(n)} \mathbf{T}_{k(n,m)}}$. This proves the following lemma.

LEMMA 1.1. *Let $\mathcal{T}$ be a strongly maximal triangular subalgebra of $\mathfrak{A} = \overline{\bigcup_{n=1}^{\infty} \mathfrak{A}_n}$ such that $\mathcal{T}\cap\mathfrak{A}_n$ is maximal triangular in $\mathfrak{A}_n$ for each n. Then a system of matrix units can be chosen for $\bigcup_{n=1}^{\infty}\mathfrak{A}_n$ such that if $\mathfrak{A}_n = \bigoplus_{m=1}^{\ell(n)} \mathbf{M}_{k(n,m)}$, then $\mathcal{T}\cap\mathfrak{A}_n = \bigoplus_{m=1}^{\ell(n)} \mathbf{T}_{k(n,m)}$.*

DEFINITION 1.2. A strongly maximal triangular subalgebra $\mathcal{T}$ of a UHF algebra $\mathfrak{A}$ is *strongly maximal triangular in factors* if a sequence $\{\mathfrak{A}_n\}$ can be chosen so that $\mathfrak{A}_n \cong \mathbf{M}_{k_n}$ for each n, $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$, and $\mathcal{T}\cap\mathfrak{A}_n$ is maximal triangular in $\mathfrak{A}_n$ for every n.

We note that it is possible to have a UHF algebra $\mathfrak{A}$ written as $\overline{\bigcup_{n=1}^{\infty}\mathfrak{A}_n}$ where each $\mathfrak{A}_n$ is not a factor. Thus, it does not follow from the definition that a strongly maximal TUHF algebra is strongly maximal triangular in factors. Indeed, we will show in Example 2.10 that this is not true in general. Some of the results in this paper are only valid for TUHF algebras which are strongly maximal triangular in factors.

One of the most important facts in the study of TAF algebras is that the isomorphism class of a TAF algebra depends on the embeddings $j_n\colon \mathfrak{A}_n \hookrightarrow \mathfrak{A}_{n+1}$, even though the isomorphism class of $\mathfrak{A}$ is independent of these embeddings [G, Br, PPW]. We will use two particular embeddings for UHF algebras in a number of examples. The *standard embedding* $\sigma_n\colon \mathbf{M}_{p_n} \hookrightarrow \mathbf{M}_{p_{n+1}}$ is defined by

$$\sigma_n(e_{ij}^{(n)}) = \sum_{t=0}^{q_n-1} e_{i+tp_n,j+tp_n}^{(n+1)}$$

where $q_n = p_{n+1}/p_n$, and the *nest embedding* $\nu_n\colon \mathbf{M}_{p_n} \hookrightarrow \mathbf{M}_{p_{n+1}}$ is defined by

$$\nu_n(e_{ij}^{(n)}) = \sum_{t=1}^{q_n} e_{(i-1)q_n+t,(j-1)q_n+t}^{(n+1)}.$$

$\varinjlim(\mathbf{M}_{p_n},\sigma_n)$ and $\varinjlim(\mathbf{M}_{p_n},\nu_n)$ are both UHF algebras of type $(p_1p_2\cdots)$, and if $\mathcal{T}_n$ denotes the set of upper triangular matrices in $\mathbf{M}_{p_n}$, then $\varinjlim(\mathcal{T}_n,\sigma_n)$ and $\varinjlim(\mathcal{T}_n,\nu_n)$ are (nonisomorphic) TAF algebras.

ν_n is called the nest embedding because $\varinjlim(\mathcal{T}_n, \nu_n)$ is also a nest algebra. In general, if $\mathcal{N}$ is a set of projections in $\mathfrak{D}$, then we define $\operatorname{Alg}\mathcal{N} = \{a \in \mathfrak{A} : e^{\perp}ae = 0 \text{ for all } e \in \mathcal{N}\}$, where $e^{\perp} = 1 - e$. $\operatorname{Alg}\mathcal{N}$ is a norm-closed algebra, and it is called a *nest algebra* if $\mathcal{N}$ is a linearly ordered set (*nest*) of projections. It follows from [PPW, Example 1.1 and Proposition 2.8] that $\varinjlim(\mathcal{T}_n, \nu_n) = \operatorname{Alg}\mathcal{N}$ for the nest $\mathcal{N} = \{0, \sum_{i=1}^{j} e_i^{(n)} : 1 \leq j \leq p_n, 1 \leq n < \infty\}$. We call this nest the *canonical nest* and $\varinjlim(\mathcal{T}_n, \nu_n)$ the *canonical nest algebra*. On the other hand, if $\mathcal{S}$ is a subset of $\mathfrak{A}$ with $\mathfrak{D} \subseteq \mathcal{S}$, then a projection $e \in \mathfrak{A}$ is *invariant* for $\mathcal{S}$ if $e^{\perp}se = 0$ for all $s \in \mathcal{S}$. The set $\operatorname{Lat}\mathcal{S}$ of invariant projections of $\mathcal{S}$ is a commutative lattice in $\mathfrak{D}$ since $\mathfrak{D} \subseteq \mathcal{S}$ and $\mathfrak{D}$ is a masa.

We will use $\mathcal{W}_{\mathfrak{D}}$ to denote the set of partial isometries $w \in \mathfrak{A}$ such that $w^*\mathfrak{D}w \subseteq \mathfrak{D}$ and $w\mathfrak{D}w^* \subseteq \mathfrak{D}$. Note that the initial and final projections of $w \in \mathcal{W}_{\mathfrak{D}}$ lie in $\mathfrak{D}$. Also, every matrix unit of $\mathfrak{A}$ is an element of $\mathcal{W}_{\mathfrak{D}}$ [PPW, Lemma 3.3]. Two partial isometries $v, w \in \mathcal{W}_{\mathfrak{D}}$ are *orthogonal* if their initial projections are orthogonal (*i.e.*, $v^*vw^*w = 0$) and their final projections are also orthogonal. The sum of orthogonal partial isometries in $\mathcal{W}_{\mathfrak{D}}$ is also in $\mathcal{W}_{\mathfrak{D}}$.

If $\mathfrak{A}_n$ is a factor, we will often use $[n]$ to denote the size of the matrix algebra $\mathfrak{A}_n$ (*i.e.*, $[n] = \sqrt{\dim \mathfrak{A}_n}$). If $\mathfrak{A}$ has a faithful normalized trace tr (in particular, if $\mathfrak{A}$ is UHF), we will make use of the probability measure μ induced on $X = \hat{\mathfrak{D}}$, the spectrum of $\mathfrak{D}$, by tr.

Let $\mathfrak{A}$ be an AF algebra with diagonal $\mathfrak{D}$, and let $X = \hat{\mathfrak{D}}$. Then by the spectral theorem of Muhly and Solel [MS1, Theorem 3.10], elements of $\mathfrak{A}$ can be represented as continuous functions on an AF groupoid $\mathcal{R}$ on X. This representation will play a major role in our discussion. To establish the notations and definitions, we will recall the construction from [MS1].

Let X be a second countable locally compact Hausdorff space. An *r-discrete principal groupoid* $\mathcal{G}$ is an equivalence relation on X with a certain topological structure (see [MS1] for details). Two points $(x_1, y_1), (x_2, y_2)$ in $\mathcal{G}$ are said to be *composable* if $y_1 = x_2$, and in this case $(x_1, y_1) \circ (x_2, y_2) = (x_1, y_2)$. For $(x, y) \in \mathcal{G}$, let $(x, y)^{-1} = (y, x)$. Let $C_c(\mathcal{G})$ be the space of complex continuous functions on $\mathcal{G}$ with compact support. Given $f, g \in C_c(\mathcal{G})$, define

$$(f * g)(x, y) = \sum_z f(x, z)g(z, y),$$

where the summation runs over all z with (x, z), $(z, y) \in \mathcal{G}$, and

$$f^*(x, y) = \overline{f(y, x)}.$$

This makes $C_c(\mathcal{G})$ a $*$-algebra, and the groupoid C^*-algebra $C^*(\mathcal{G})$ is the closure of $C_c(\mathcal{G})$ under a suitable C^*-norm. The space X can be identified with the subset $\mathcal{G}^0 = \{(x, x) : x \in X\}$ of $\mathcal{G}$. Then every continuous function f in $C(X)$ can be identified with a unique function on $\mathcal{G}^0$ (also denoted by f) such that $f(x, x) = f(x)$.

Let $\mathcal{P}$ be an open subset of $\mathcal{G}$. Define $\mathcal{A}(\mathcal{P}) = \{a \in C^*(\mathcal{G}) : a(x, y) = 0 \text{ for all} (x, y) \notin \mathcal{P}\}$. Then $\mathcal{A}(\mathcal{P})$ is a norm-closed bimodule of $C^*(\mathcal{G})$ over $C^*(\mathcal{G}^0)\big(\approx C(X)\big)$. Conversely, by [MS1, Theorem 3.10], every closed $C^*(\mathcal{G}^0)$-bimodule $\mathcal{A}$ of $C^*(\mathcal{G})$ can be represented uniquely in the form $\mathcal{A} = \mathcal{A}(\mathcal{P})$ for an open subset $\mathcal{P}$ of $\mathcal{R}$.

Now suppose $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$ is an AF algebra with masa $\mathfrak{D} = \overline{\bigcup_n \mathfrak{D}_n}$. For each projection p in $\mathfrak{D}$, $\hat{p} = \{x \in X : x(p) = 1\}$ is a closed and open (clopen) subset of X. Let v be a matrix unit in some $\mathfrak{A}_n$. Then we can define a partial homeomorphism h_v [Pr2] from $\widehat{vv^*}$ to $\widehat{v^*v}$ by $h_v(x) = x_v$, where $x_v(d) = x(vdv^*)$. Letting $\hat{v} = \{(x, x_v) : x(vv^*) = 1\} \subseteq X \times X$, we can then define a groupoid $\mathcal{R}$, called an *AF-groupoid*, by

$$\mathcal{R} = \bigcup\{\hat{v} : v \text{ is a matrix unit of some } \mathfrak{A}_n\}.$$

$\mathcal{R}$ is given the smallest topology such that each $\hat{v}$ is clopen. Since any nonzero intersection $\hat{v}_1 \cap \cdots \cap \hat{v}_k$ contains some $\hat{v}_0$, it follows that $\{\hat{v} : v \text{ is a matrix unit of some } \mathfrak{A}_n\}$ is a base for the topology. Given a matrix unit v, let $\chi_{\hat{v}}$ be the characteristic function on $\hat{v}$. Direct computation shows that $\chi_{\hat{u}} * \chi_{\hat{v}} = \chi_{\widehat{uv}}$ and $(\chi_{\hat{v}})^* = \chi_{\widehat{v^*}}$. Hence we can identify v with $\chi_{\hat{v}}$ in $C_c(\mathcal{R})$. This extends to an isomorphism between $\mathfrak{A}$ and the groupoid C^*-algebra $C^*(\mathcal{R})$ [MS2, V2].

THEOREM 1.3 [MS1, MS2]. *Let $\mathfrak{A}$ be an* AF *algebra with diagonal $\mathfrak{D}$ and $X = \hat{\mathfrak{D}}$, let $\mathcal{R}$ be an* AF-*groupoid such that $\mathfrak{A} = C^*(\mathcal{R})$, and suppose $\mathcal{T} = \mathcal{A}(\mathcal{P})$ is a $\mathfrak{D}$-bimodule in $\mathfrak{A}$. Then*

(a) $\mathcal{T}^* = \mathcal{A}(\mathcal{P}^{-1})$. Thus, $\mathcal{T}$ is self-adjoint if and only if $\mathcal{P} = \mathcal{P}^{-1}$.

(b) $\mathcal{T}$ is an algebra if and only if $\mathcal{P} \circ \mathcal{P} \subseteq \mathcal{P}$.

(c) $\mathcal{T}$ is triangular if and only if $\mathcal{P} \cap \mathcal{P}^{-1} = \mathcal{R}^0$.

Furthermore, a TAF *algebra $\mathcal{T}$ is strongly maximal triangular if and only if $\mathcal{P} \cup \mathcal{P}^{-1} = \mathcal{R}$. Note that in this case $\mathcal{P}$ is a clopen subset.*

THEOREM 1.4 [MS2, T, V1]. *A* TAF *algebra $\mathcal{T}$ is strongly maximal triangular in $\mathfrak{A}$ if and only if $\mathcal{T} + \mathcal{T}^*$ is dense in $\mathfrak{A}$.*

Suppose $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$ and $\mathfrak{D} = \overline{\bigcup_n \mathfrak{D}_n}$ as above. Then for every point $x \in X$, there exists a unique sequence $\{e_n : 1 \leq n < \infty\}$ of projections, where e_n is a minimal projection in $\mathfrak{D}_n$ for every n, such that $\{x\} = \bigcap_{n=1}^{\infty} \hat{e}_n$ [SV]. Since each e_n is minimal in $\mathfrak{D}_n$ and $\mathfrak{A}_n \subseteq \mathfrak{A}_{n+1}$, we have $e_n \geq e_{n+1}$ (the usual order of projections in $\mathfrak{D}$) for every n. Conversely, if for each n, e_n is a minimal projection of $\mathfrak{D}_n$ such that $e_n \geq e_{n+1}$, then $\bigcap_{n=1}^{\infty} \hat{e}_n = \{x\}$ for some unique x in X. We will use $x = (\hat{e}_1, \hat{e}_2, \hat{e}_3, \ldots)$ to denote such a correspondence.

Now suppose we write $\mathfrak{A}$ as $C^*(\mathcal{R})$, and let $\mathcal{T} = \mathcal{A}(\mathcal{P})$ be a subalgebra of $\mathfrak{A}$ containing $\mathfrak{D}$. Then $\mathcal{T}$ defines an ordering on $X = \hat{\mathfrak{D}}$ by $x \ll y$ if $(x, y) \in \mathcal{P}$. We will call $\ll$ the *spectrum ordering on X induced by $\mathcal{T}$*. This ordering is reflexive and transitive, and it is antisymmetric if and only if $\mathcal{T}$ is triangular. An equivalent formulation of this concept was first introduced by Power [Pr2], and he proved that this ordering, when viewed as a topological subrelation of $\mathcal{R}$, gives a complete isometric isomorphism invariant for TAF algebras [Pr3]. Note that if $x \ll y$, then there is some n and some matrix unit $v \in \mathcal{T} \cap \mathfrak{A}_n$ such that $(x, y) \in \hat{v} \subseteq \mathcal{P}$, since $\mathcal{P}$ is open.

LEMMA 1.5. *Suppose $x = (\hat{e}_1, \hat{e}_2, \hat{e}_3, \ldots)$ and $y = (\hat{f}_1, \hat{f}_2, \hat{f}_3, \ldots)$. Then $x \ll y$ if and only if there exists an N and a matrix unit v in $\mathcal{T} \cap \mathfrak{A}_N$ such that $e_n = vf_nv^*$ for all $n \geq N$.*

PROOF. Suppose $x \ll y$. Then there exists an N and a matrix unit $v \in \mathcal{T} \cap \mathfrak{A}_N$ such that $(x, y) \in \hat{v}$. Hence, for each $n \geq N$, vv^* and v^*v are projections in $\mathfrak{D}_n$ such that $x \in \widehat{vv^*}$ and $y \in \widehat{v^*v}$. Since e_n and f_n are minimal, we have $e_n \leq vv^*$ and $f_n \leq v^*v$. Now vf_nv^* is a minimal projection in $\mathfrak{D}_n$ such that $x \in (vf_nv^*)^\wedge \subseteq \widehat{vv^*}$. Hence, $e_n = vf_nv^*$ because there is only one minimal projection e in $\mathfrak{D}_n$ such that $x \in \hat{e}$. The proof of the converse is trivial. ∎

LEMMA 1.6. *Suppose that for each n, $\mathfrak{A}_n = \mathbf{M}_{k_n}$ and $\mathcal{T}_n$ is the algebra of upper triangular matrices in $\mathfrak{A}_n$. Let $x_0 = (\hat{e}_1^{(1)}, \hat{e}_1^{(2)}, \hat{e}_1^{(3)}, \ldots)$ and $x_1 = (\hat{e}_{k_1}^{(1)}, \hat{e}_{k_2}^{(2)}, \hat{e}_{k_3}^{(3)}, \ldots)$. Then x_0 is the unique minimal point and x_1 is the unique maximal point in the spectrum ordering $\ll$ defined by $\mathcal{T} = \overline{\bigcup_n \mathcal{T}_n}$.*

PROOF. We first show that x_0 is minimal in the spectrum ordering. Let $y = (\hat{e}_{i_1}^{(1)}, \hat{e}_{i_2}^{(2)}, \ldots)$ such that $y \ll x_0$. By Lemma 1.5, there exists an N and a matrix unit v in $\mathcal{T}_N$ such that $e_{i_n}^{(n)} = ve_1^{(n)}v^*$ for all $n \geq N$. Since $\mathcal{T}_n$ is the algebra of upper triangular matrices, we have $v = \sum_{(i,j)\in S} e_{ij}^{(n)}$ for a nonempty subset S of $\{(i,j) : 1 \leq i \leq j \leq n\}$. Hence, $i_n = 1$ for all $n \geq N$, and therefore $y = x_0$.

For uniqueness, suppose $y = (\hat{e}_{i_1}^{(1)}, \hat{e}_{i_2}^{(2)}, \ldots) \neq x_0$. Then $i_n > 1$ for some n, so there is some $x \in X$ such that $(x, y) \in \hat{e}_{1 i_n}^{(n)}$. Consequently, $x \ll y$ and y is not minimal.

The proof of the maximality and uniqueness of x_1 is similar. ∎

DEFINITION 1.7. Let $\mathfrak{A} = C^*(\mathcal{R})$ for an AF-groupoid $\mathcal{R}$ on X. A (real-valued) continuous function d on $\mathcal{R}$ is said to be a *cocycle* if $d(x, z) = d(x, y) + d(y, z)$ for all $(x, y), (y, z) \in \mathcal{R}$. If $d(\mathcal{R}) \subseteq \mathbb{Z}$, the integers, then d is said to be an *integer-valued cocycle*. A cocycle d is said to be a *coboundary* if there exists a continuous function b on X such that $d(x, y) = b(y) - b(x)$. Thus, a coboundary is bounded on $\mathcal{R}$, since X is compact. Conversely, if $\mathfrak{A}$ is simple, then every bounded cocycle is a coboundary [R, page 112].

DEFINITION 1.8. A subalgebra $\mathcal{T} = \mathcal{A}(\mathcal{P})$ of $\mathfrak{A} = C^*(\mathcal{R})$ is said to be analytic ($\mathbb{Z}$-analytic) if there exists a (integer-valued) cocycle d such that $\mathcal{P} = d^{-1}[0, \infty)$. In this case, we write $\mathcal{T}_d$ for $\mathcal{T}$. We also say $\mathcal{T}$ is *trivially analytic* if $\mathcal{T} = \mathcal{T}_d$ for some coboundary d. Note that $\mathcal{T}_d$ is triangular if and only if $d^{-1}(\{0\}) = \mathcal{R}^0 (\cong X)$.

COROLLARY 1.9 [V2]. *An analytic* TAF *algebra is strongly maximal triangular.*

PROOF. This follows directly from Theorem 1.3 since $d^{-1}(-\infty, 0] \cup d^{-1}[0, \infty) = \mathcal{R}$. ∎

The converse of Corollary 1.9 is false. Counterexamples are given in [SVe] and [PWo].

REMARK 1.10. Suppose $\mathcal{T} = \mathcal{A}(\mathcal{P})$ is a strongly maximal triangular subalgebra of $\mathfrak{A}$ with diagonal $\mathfrak{D}$. For each n, let $\mathfrak{B}_n = C^*(\mathfrak{A}_n, \mathfrak{D})$. Then there exists a clopen subset $\mathcal{R}_n$ of $\mathcal{R}$ such that $\mathcal{A}(\mathcal{R}_n) = \mathfrak{B}_n$. Note that $\mathcal{R} = \bigcup_n \mathcal{R}_n$. Let $\mathcal{P}_n = \mathcal{P} \cap \mathcal{R}_n$ and $\mathcal{P}_n^+ = \mathcal{P}_n \setminus X$.

Since $\mathcal{T}$ is strongly maximal, we have $\mathcal{R}_n = \mathcal{P}_n^+ \cup X \cup (\mathcal{P}_n^+)^{-1}$. Now suppose that for each n, we can define a cocycle c_n on $\mathcal{R}_n$ such that $c_n(x,y) > 0$ if $(x,y) \in \mathcal{P}_n^+$. Suppose also that for each $(x,y) \in \mathcal{R}$, there is some m such that $(x,y) \in \mathcal{R}_m$ and $c_n(x,y) = c_{n+1}(x,y)$ for all $n \geq m$. Then $d(x,y) = \lim_{n\to\infty} c_n(x,y)$ exists (as a finite number) for every $(x,y) \in \mathcal{R}$, and d is a cocycle on $\mathcal{R}$ such that $\mathcal{T} = \mathcal{T}_d$ since $\mathcal{P} = \bigcup \mathcal{P}_n$. Conversely, if $\mathcal{T} = \mathcal{T}_d$ for some cocycle d, then $c_n = d|_{\mathcal{R}_n}$ is a cocycle on $\mathcal{R}_n$ such that $c_n(x,y) > 0$ for $(x,y) \in \mathcal{P}_n^+$, and $d(x,y) = c_n(x,y)$ for all $n \geq$ some m since $\mathcal{R} = \bigcup \mathcal{R}_n$.

2. **Analytic** TAF **algebras with integer-valued cocycles.** Recall that a $\mathbb{Z}$-analytic TAF algebra is always strongly maximal, by Corollary 1.9. Now suppose $\mathcal{T} = \mathcal{A}(\mathcal{P})$ is a strongly maximal triangular subalgebra of $\mathfrak{A}$ with diagonal $\mathfrak{D}$, and let $\mathcal{R}_n$, $\mathcal{P}_n$, and $\mathcal{P}_n^+$ be defined as in Remark 1.10. For $(x,y) \in \mathcal{P}_n^+$, define

$$\tilde{d}_n(x,y) = \max\{k \geq 1 : \text{there exist } (x_i, x_{i+1}) \in \mathcal{P}_n^+, 1 \leq i \leq k, \\ \text{such that } x_1 = x \text{ and } x_{k+1} = y\}$$

If $(x,y) \in \mathcal{P}_n^+$, then there exists a matrix unit $v \in \mathcal{T}_n \setminus \mathfrak{D}$ such that $(x,y) \in \hat{v}$. Hence $\tilde{d}_n(x,y) > 0$. Since $\mathfrak{A}_n$ is finite dimensional, $\tilde{d}_n(x,y)$ is finite and $\tilde{d}_n(x,y) = \tilde{d}_n(x,z) + \tilde{d}_n(z,y)$ if (x,z) and (z,y) are both in $\mathcal{P}_n^+$. Define d_n on $\mathcal{R}_n$ by

$$d_n(x,y) = \begin{cases} \tilde{d}_n(x,y) & \text{if } (x,y) \in \mathcal{P}_n^+ \\ 0 & \text{if } x = y \\ -\tilde{d}_n(y,x) & \text{if } (y,x) \in \mathcal{P}_n^+. \end{cases}$$

Direct computation shows that d_n is a cocycle on $\mathcal{R}_n$ and $\mathcal{T} \cap \mathfrak{B}_n = \mathcal{T}_{d_n}$. This implies the following result.

PROPOSITION 2.1. *Suppose $\mathcal{T}$ is a strongly maximal triangular subalgebra of $\mathfrak{A}$. Then $\mathcal{T} \cap \mathfrak{B}_n$ is $\mathbb{Z}$-analytic in $\mathfrak{B}_n$ for each n.*

Since $\mathcal{R} = \bigcup_n \mathcal{R}_n$, for each $(x,y) \in \mathcal{R}$ there exists some m such that $d_n(x,y)$ is defined for all $n \geq m$. Furthermore, if $(x,y) \in \mathcal{P}$, then $\{d_n(x,y) : n \geq m\}$ is an increasing sequence.

THEOREM 2.2. *Let $\mathcal{T}$ be a strongly maximal triangular subalgebra of $\mathfrak{A}$. Then $\mathcal{T}$ is $\mathbb{Z}$-analytic if and only if for each $(x,y) \in \mathcal{R}$, $\lim_{n\to\infty} d_n(x,y) = \hat{d}(x,y)$ exists (as a finite number). In this case, $\hat{d}$ is a cocycle on $\mathcal{R}$ and $\mathcal{T} = \mathcal{T}_{\hat{d}}$.*

PROOF. For sufficiency, let $(x,y) \in \mathcal{R}$. Since $d_n(x,y)$ is always an integer, $\lim_{n\to\infty} d_n(x,y)$ exists if and only if the sequence $\{d_n(x,y)\}$ is eventually constant. Thus, $\mathcal{T}$ is $\mathbb{Z}$-analytic by Remark 1.10.

Conversely, suppose $\mathcal{T} = \mathcal{T}_d$ for an integer-valued cocycle d. If $(x,y) \in \mathcal{P}_m$, then $\{d_n(x,y) : m \leq n < \infty\}$ is an increasing sequence bounded above by $d(x,y)$, so $\hat{d}(x,y)$ exists, and it is a cocycle on $\mathcal{R}$ by Remark 1.10. Also, if $(x,y) \in \mathcal{R}$, then

$$\begin{aligned} d(x,y) \geq 0 &\Leftrightarrow (x,y) \in \hat{v} \text{ for some matrix unit } v \in \mathcal{T} \cap \mathfrak{A}_n \\ &\Leftrightarrow d_n(x,y) \geq 0 \text{ for some } n \\ &\Leftrightarrow \hat{d}(x,y) \geq 0. \end{aligned}$$

Therefore, $d^{-1}[0,\infty) = \hat{d}^{-1}[0,\infty)$ and $\mathcal{T}_d = \mathcal{T}_{\hat{d}}$. ∎

Suppose $\mathcal{T} = \mathcal{T}_d$ is $\mathbb{Z}$-analytic. We will call the integer-valued cocycle $\hat{d}$ obtained in Theorem 2.2 the *generic form* of d. For the rest of this section, we will assume that the $\mathbb{Z}$-cocycles are given in generic form. Finally, we note that d is determined by the clopen subset $d^{-1}(\{1\})$.

EXAMPLE 2.3. We will show that the TAF algebra in Example 3.27 of [PPW] is analytic via an unbounded, real-valued cocycle, but it is not $\mathbb{Z}$-analytic.

Let $\mathfrak{A}_n = \mathbf{M}_{2^n}$ with matrix units $\{e_{ij}^{(n)}\}$, and let σ_n and ν_n denote the standard and nest embeddings, respectively. For n even, let $j_n \colon \mathfrak{A}_n \hookrightarrow \mathfrak{A}_{n+2}$ by $j_n = \nu_{n+1} \circ \sigma_n$, so

$$\begin{aligned} j_n(e_{ij}^{(n)}) &= \nu_{n+1} \circ \sigma_n(e_{ij}^{(n)}) \\ &= e_{2i-1,2j-1}^{(n+2)} + e_{2i,2j}^{(n+2)} + e_{2^{n+1}+2i-1,2^{n+1}+2j-1}^{(n+2)} + e_{2^{n+1}+2i,2^{n+1}+2j}^{(n+2)}. \end{aligned}$$

Let $\mathcal{T}_n$ be the upper triangular subalgebra of $\mathfrak{A}_n$ and let $\mathcal{T} = \varinjlim\{(\mathcal{T}_n, j_n) : n \text{ even}\}$. $\mathcal{T}$, viewed as a subalgebra $\mathcal{A}(\mathcal{P})$ of the groupoid C^*-algebra, is supported on $\mathcal{P} = \bigcup\{\hat{e}_{ij}^{(n)} : 1 \le i \le j \le 2^n, n = 2, 4, \ldots\}$.

If $(x,y) \in \hat{e}_{ij}^{(n)}$, define $d(x,y) = (j-i)/(2^{\frac{n}{2}})$. To see that d is well-defined, note that $\hat{e}_{ij}^{(n)} = \hat{e}_{2i-1,2j-1}^{(n+2)} \cup \hat{e}_{2i,2j}^{(n+2)} \cup \hat{e}_{2^{n+1}+2i-1,2^{n+1}+2j-1}^{(n+2)} \cup \hat{e}_{2^{n+1}+2i,2^{n+1}+2j}^{(n+2)}$. Thus, (x,y) also belongs to one of these four sets, say $(x,y) \in \hat{e}_{2i-1,2j-1}^{(n+2)}$. Observe that

$$\frac{(2j-1)-(2i-1)}{2^{\frac{n+2}{2}}} = \frac{2(j-i)}{2 \cdot 2^{\frac{n}{2}}} = \frac{(j-i)}{2^{\frac{n}{2}}}.$$

The same result holds if (x,y) is in any of the other sets. Thus d is well-defined. d is continuous since it is constant on each clopen set $\hat{e}_{ij}^{(n)}$. Also, d is clearly unbounded. Finally, the facts (i) $d(x,y) = 0$ iff $y = x$, and (ii) $d(x,y)+d(y,z) = d(x,z)$ if $(x,y),(y,z) \in \mathcal{R}$, are clear from the definition of d.

Now suppose $(x,y) \in \hat{e}_{ij}^{(n)}$, $i \ne j$. Then $d_n(x,y) = j-i$, but $d_{n+2}(x,y) = 2(j-i)$. Thus, $\lim_{n\to\infty} d_n(x,y)$ does not exist. It follows from Theorem 2.2 that $\mathcal{T}$ is not $\mathbb{Z}$-analytic.

Let X be a compact zero-dimensional space and ϕ a minimal homeomorphism of X (*i.e.*, the ϕ-orbit of each $x \in X$ is dense in X). Then the crossed product $\mathbb{Z} \times_\phi C(X)$ is the C^*-algebra generated by $C(X)$ and a unitary U such that $UfU^* = f \circ \phi$ for $f \in C(X)$. Let $x_0 \in X$. Then the C^*-subalgebra $\mathfrak{A}(\phi, x_0)$ of $\mathbb{Z} \times_\phi C(X)$ generated by $C(X)$ and $UC_0(X) = \{Uf : f \in C(X), f(x_0) = 0\}$ is an AF algebra [Pu] with diagonal $\mathfrak{D} = C(X)$, and the subalgebra $\mathcal{T}(\phi, x_0)$ generated by $C(X)$ and $UC_0(X)$ is a strongly maximal triangular subalgebra of $\mathfrak{A}(\phi, x_0)$ [PPW, Example 1.3].

Let X and ϕ be as given above. Then $\mathcal{G} = \{(x, \phi^n(x)) : x \in X, n \in \mathbb{Z}\}$ is an r-discrete principal groupoid such that $C^*(\mathcal{G}) \cong \mathbb{Z} \times_\phi C(X)$ [R,MS1]. Since both $\mathcal{T}(\phi, x_0)$ and $\mathfrak{A}(\phi, x_0)$ contain $C(X)$, there are open subsets $\mathcal{P}$ and $\mathcal{R}$ of $\mathcal{G}$ such that $\mathcal{T}(\phi, x_0) = \mathcal{A}(\mathcal{P})$ and $\mathfrak{A}(\phi, x_0) = \mathcal{A}(\mathcal{R})$. From Corollary 2.4 of [Po1], we have $\mathcal{P} = \{(x, \phi^n(x)) : \phi^i(x) \ne x_0 \text{ for } 1 \le i \le n, n \ge 0\}$ and $\mathcal{R} = \mathcal{P} \cup \mathcal{P}^{-1}$. Define $d \colon \mathcal{G} \to \mathbb{Z}$ by $d(x, \phi^n(x)) = n$. Then $\mathcal{T} = \mathcal{T}_d$ in $\mathfrak{A} = C^*(\mathcal{R})$. Thus, PROPOSITION 2.4. *Every $\mathcal{T}(\phi, x_0)$ is $\mathbb{Z}$-analytic.*

This leads to the following question: what kind of $\mathbb{Z}$-analytic $\mathcal{T}$ can be represented as $\mathcal{T}(\phi, x_0)$? In Theorem 2.8, we will prove the converse of Proposition 2.4 for a certain class of TUHF algebras.

REMARK 2.5. The strongly maximal triangular algebra $\mathcal{T} = \varinjlim(\mathcal{T}_n, \sigma_n)$, generated by the standard embeddings σ_n and the set of upper triangular matrices $\mathcal{T}_n$ in $\mathbf{M}_{p_n}$, can be represented as $\mathcal{T}(\phi, x_0)$ [PPW, Example 1.2]. Ventura [V2, Example 5.1] has given an explicit formula for an integer-valued cocycle d such that $\mathcal{T} = \mathcal{T}_d$.

DEFINITION 2.6. Given an integer-valued cocycle d on $\mathcal{R}$, let ϕ_d be the partial homeomorphism defined by the clopen subset $d^{-1}(\{1\})$ of $\mathcal{R}$, *i.e.*, $\phi_d(x) = y$, where y is the unique element such that $d(x,y) = 1$. ϕ_d is defined on the open subset $\pi_\ell\big(d^{-1}(\{1\})\big)$, where $\pi_\ell \colon \mathcal{R} \to X$ by $\pi_\ell(x,y) = x$.

LEMMA 2.7. *Let $\mathcal{T}$ be a* TAF *subalgebra of $\mathfrak{A} = C^*(\mathcal{R})$, and let d be an integer-valued cocycle on $\mathcal{T}$ such that $\mathcal{T} = \mathcal{T}_d$. Let $X_{\max}$ (and $X_{\min}$) be the maximal (respectively minimal) points in X with respect to the ordering $\ll$. Then ϕ_d is a homeomorphism from $X \setminus X_{\max}$ to $X \setminus X_{\min}$.*

PROOF. Let $\mathcal{P} = d^{-1}[0,\infty)$, so $\mathcal{T} = \mathcal{A}(\mathcal{P})$. Let $x \in X$. Then $\phi(x)$ is defined iff $d(x,y) = 1$ for some y iff $(x,y) \in \mathcal{P}$ for some $y \neq x$ iff $x \notin X_{\max}$. Similarly, if $y \in X$, then $y = \phi(x)$ for some x iff $y \notin X_{\min}$. ■

THEOREM 2.8. *Let $\mathfrak{A}_n = \mathbf{M}_{[n]}$ for each n and let $\mathcal{T}_n$ ($\mathfrak{D}_n$) be the set of upper triangular (diagonal) matrices in $\mathbf{M}_{[n]}$. Suppose the embedding $\mathfrak{A}_n \hookrightarrow \mathfrak{A}_{n+1}$ takes $\mathcal{T}_n$ into $\mathcal{T}_{n+1}$. Let $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$, $\mathcal{T} = \overline{\bigcup_n \mathcal{T}_n}$, $\mathfrak{D} = \overline{\bigcup_n \mathfrak{D}_n}$, and $X = \hat{\mathfrak{D}}$. Suppose $\mathcal{T} = \mathcal{T}_d$ is $\mathbb{Z}$-analytic, and let $x_0 = (\hat{e}^{(1)}_1, \hat{e}^{(2)}_1, \ldots)$ and $x_1 = (\hat{e}^{(1)}_{[1]}, \hat{e}^{(2)}_{[2]}, \ldots)$. Then*

(a) ϕ_d is a partial homeomorphism from $X \setminus \{x_1\}$ onto $X \setminus \{x_0\}$.

(b) ϕ_d can be extended to a minimal homeomorphism ϕ on X by defining $\phi(x_1) = x_0$.

(c) $\mathfrak{A} \cong \mathfrak{A}(\phi, x_0)$ and $\mathcal{T} \cong \mathcal{T}(\phi, x_0)$.

PROOF. (a) follows from Lemma 2.7 and Lemma 1.6.

For (b), we will first show that the extension $\phi(x_1) = x_0$ is continuous. Let W be an open subset of X containing x_0. We can choose n such that $\hat{e}^{(n)}_1 \subseteq W$. By Theorem 2.2, for each r and s, $1 \leq r, s \leq [n]$, and each $(x,y) \in \hat{e}^{[n]}_{rs}$, we can find some $m \geq n$ such that

$$(*) \qquad (x,y) \in \hat{e}^{(m)}_{ij} \text{ for some } 1 \leq i \leq j \leq [m] \text{ with } j - i = d(x,y).$$

By the compactness of $\bigcup\{\hat{e}^{(n)}_{rs} : 1 \leq r,\, s \leq [n]\}$, we can find a finite number m such that for every r, s, with $1 \leq r,\, s \leq [n]$, and every $(x,y) \in \hat{e}^{(n)}_{rs}$, condition $(*)$ is satisfied.

Now $\hat{e}^{(m)}_{[m]}$ is an open set containing x_1. We are going to show that

$$\phi(\hat{e}^{(m)}_{[m]}) \subseteq \hat{e}^{(n)}_1 \subseteq W.$$

Let $x \in \hat{e}^{(m)}_{[m]}$ with $x \neq x_1$. Then $\phi(x) = \phi_d(x)$. Suppose $\phi(x) \notin \hat{e}^{(n)}_1$. Then there exists some j, $1 < j \leq [n]$, such that $\phi(x) \in \hat{e}^{(n)}_j$. Hence, for some $y \neq \phi(x)$, $\big(y, \phi(x)\big) \in \hat{e}^{(n)}_{1j}$.

Let $d\big(y,\phi(x)\big) = r > 0$. Then by $(*)$, there is some k and r with $1 \le k < k+r \le [m]$ such that $\big(y,\phi(x)\big) \in \hat{e}^{(m)}_{k,k+r}$. Since

$$e^{(m)}_{k,k+r} = e^{(m)}_{k,k+1} \cdot e^{(m)}_{k+1,k+2} \cdots e^{(m)}_{k+r-1,k+r},$$

we have $x = \phi^{-1}\big(\phi(x)\big) = \phi^{r-1}(y) \in \hat{e}^{(m)}_{k+r-1} \neq \hat{e}^{(m)}_{[m]}$, a contradiction. Hence, $\phi(x) \in \hat{e}^{(n)}_{1}$ and ϕ is continuous. Since X is compact Hausdorff and ϕ is bijective, it follows that ϕ is a homeomorphism.

Finally, suppose $z \in X$ and Z is a nonempty open subset of X. Choose n and i, $1 \le i \le [n]$, such that $\hat{e}^{(n)}_{i} \subseteq Z$. $z \in \hat{e}^{(n)}_{j}$ for some j, $1 \le j \le [n]$, so $(y,z) \in \hat{e}^{(n)}_{ij}$ for some $y \in \hat{e}^{(n)}_{i}$. Since either $e^{(n)}_{ij}$ or $e^{(n)}_{ji}$ is in $\mathcal{T}$, we have $y = \phi^{k}(z) \in \hat{e}^{(n)}_{i} \subseteq Z$ for some integer k. Hence, ϕ is minimal.

For (c), let $\mathcal{T} = \mathcal{A}(\mathcal{P})$. We note that $\mathcal{T} \cap \mathfrak{A}_n$ is equal to the set of upper triangular matrices in $\mathfrak{A}_n$ [PPW, Proposition 2.5]. Hence, for $n \ge 1$, $\big(x,\phi^{n}(x)\big) \in \mathcal{P}$ if and only if $\phi^{i}(x) \neq x_0$ for $1 \le i \le n$. Thus, by the discussion preceding Proposition 2.4, $\mathcal{T}(\phi,x_0) \cong \mathcal{T}$ and $\mathfrak{A}(\phi,x_0) \cong \mathfrak{A}$. ∎

COROLLARY 2.9. *Suppose $\mathcal{T}$ is strongly maximal triangular in factors and $\mathcal{T} = \mathcal{T}_d$ is $\mathbb{Z}$-analytic. Then ϕ_d can be extended to a minimal homeomorphism ϕ on X such that $\mathfrak{A} \cong \mathfrak{A}(\phi,x_0)$ and $\mathcal{T} \cong \mathcal{T}(\phi,x_0)$.*

PROOF. Apply Lemma 1.1 and Theorem 2.8. ∎

The previous theorem and corollary depend on the facts that $\mathfrak{A}$ is UHF and $\mathcal{T}$ is strongly maximal triangular in factors. We will next show that not every strongly maximal TUHF algebra is strongly maximal triangular in factors. This same example will then be used to show that the assumption in Corollary 2.9 that $\mathcal{T}$ is strongly maximal triangular in factors is necessary.

EXAMPLE 2.10. (a) A strongly maximal triangular subalgebra of a UHF algebra need not be strongly maximal in factors. Let $\mathfrak{G}_n = \mathbf{M}_{4^n} \oplus \mathbf{M}_{4^n}$ for $n = 0,1,\ldots$, and let I_m denote the identity operator in $\mathbf{M}_m$. Define embeddings $j_n \colon \mathfrak{G}_n \hookrightarrow \mathfrak{G}_{n+1}$ by

$$j_n(A \oplus B) = \begin{pmatrix} A & & & \\ & B & & \\ & & B & \\ & & & A \end{pmatrix} \oplus \begin{pmatrix} B & & & \\ & A & & \\ & & A & \\ & & & B \end{pmatrix}.$$

Define $\mathfrak{A} = \varinjlim(\mathfrak{G}_n, j_n)$. Now let $\mathfrak{A}_n = \mathbf{M}_{2\,4^n}$ and define $J_n \colon \mathfrak{A}_n \hookrightarrow \mathfrak{G}_{n+1} \subseteq \mathfrak{A}_{n+1}$ by $J_n(z) = (z \oplus u_n z u_n^*) \oplus (u_n z u_n^* \oplus z)$, where u_n is the unitary $\begin{pmatrix} 0 & I_{4^n} \\ I_{4^n} & 0 \end{pmatrix}$. Since $J_n|_{\mathfrak{G}_n} = j_n$, it follows that $\mathfrak{A} \cong \varinjlim(\mathfrak{A}_n, J_n)$, *i.e.*, $\mathfrak{A}$ is UHF.

Note that $\mathcal{T}_n$, the set of upper triangular matrices in $\mathfrak{G}_n$, is maximal triangular in $\mathfrak{G}_n$ and $j_n(\mathcal{T}_n) \subseteq \mathcal{T}_{n+1}$. Thus, since each $\mathfrak{G}_n$ is a finite dimensional C^*-algebra, $\mathcal{T} = \overline{\bigcup_n \mathcal{T}_n}$ is strongly maximal triangular in $\mathfrak{A}$. However, $\mathcal{T}$ is not strongly maximal triangular in factors since there are two maximal and two minimal points with respect to the spectrum ordering induced by $\mathcal{T}$. To see this, let $\{e_{ij}^{(nk)}\}$, $k = 1, 2$, be the usual set of matrix units for $\mathfrak{G}_n$. Then the two minimal points are

$$x_0 = (\hat{e}_1^{(01)}, \hat{e}_1^{(11)}, \hat{e}_1^{(21)}, \ldots) \text{ and } x_1 = (\hat{e}_1^{(02)}, \hat{e}_1^{(12)}, \hat{e}_1^{(22)}, \ldots)$$

and the two maximal points are

$$y_0 = (\hat{e}_1^{(01)}, \hat{e}_4^{(11)}, \hat{e}_{16}^{(21)}, \ldots, \hat{e}_{4^n}^{(n1)}, \ldots) \text{ and } y_1 = (\hat{e}_1^{(02)}, \hat{e}_4^{(12)}, \hat{e}_{16}^{(22)}, \ldots, \hat{e}_{4^n}^{(n2)}, \ldots).$$

Since the spectrum ordering is an algebra isomorphism invariant, it follows from Lemmas 1.1 and 1.6 that $\mathcal{T}$ is not isomorphic to an algebra which is strongly maximal triangular in factors.

(b) If $(x, y) \in \mathcal{R}_n$ and $d_n(x, y) = r$, then it follows from the definition of j_n that $d_m(x, y) = r$ for all $m \geq n$. Thus, $\mathcal{T}$ is $\mathbb{Z}$-analytic by Theorem 2.2. However, ϕ_d does not extend to a homeomorphism ϕ of X. To see this, note that $\phi(y_0)$ would have to be either x_0 or x_1. We will show that ϕ is not continuous if $\phi(y_0) = x_0$. The proof is similar for the other choice. Thus, the assumption in Corollary 2.9 that $\mathcal{T}$ is strongly maximal in factors is necessary.

If $\phi(y_0) = x_0$ and ϕ is continuous, then for each $\hat{e}_1^{(n1)}$ there must be some $\hat{e}_{4^m}^{(m1)}$ such that $\phi(\hat{e}_{4^m}^{(m1)}) \subseteq \hat{e}_1^{(n1)}$. But $j_m(e_{4^m}^{(m1)}) = e_{4^m}^{(m+1,1)} + e_{4^{m+1}}^{(m+1,1)} + e_{2\cdot 4^m}^{(m+1,2)} + e_{3\cdot 4^m}^{(m+1,2)}$. Therefore, if $x \in \hat{e}_{4^m}^{(m+1,1)}$, then $\phi(x) \in \hat{e}_{4^m+1}^{(m+1,1)}$. But $\hat{e}_{4^m+1}^{(m+1,1)} \cap \big((j_m \circ \cdots \circ j_n)(e_1^{(n1)})\big)^\wedge = \emptyset$. Thus, ϕ is not continuous.

(c) Let $\mathcal{S}_n$ be the set of upper triangular matrices in $\mathbf{M}_{2\cdot 4^n}$. Note that $\varinjlim(\mathbf{M}_{2\cdot 4^n}, \sigma_n)$ is isomorphic to $\mathfrak{A}$. Let $\mathcal{S}$ be the strongly maximal TAF algebra $\varinjlim(\mathcal{S}_n, \sigma_n)$. Then by Remark 2.5, $\mathcal{S}$ is $\mathbb{Z}$-analytic, so (a) and (b) show that $\mathfrak{A}$ contains two nonisomorphic $\mathbb{Z}$-analytic TAF algebras.

REMARK 2.11. Suppose $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$ and $\mathcal{T} \subseteq \mathfrak{A}$ is a TAF algebra with diagonal $\mathfrak{D}$. By Corollary 2.3 of [PPW], we have $\mathcal{T} = \overline{\bigcup_n (\mathcal{T} \cap \mathfrak{A}_n)}$. The last example showed that even when $\mathcal{T}$ is strongly maximal triangular, $\mathcal{T} \cap \mathfrak{A}_n$ need not be maximal triangular in $\mathfrak{A}_n$. By Proposition 2.1 and Corollary 1.9, this complication can be overcome by replacing $\mathfrak{A}_n$ with the infinite dimensional subalgebra $\mathfrak{B}_n = C^*(\mathfrak{A}_n, \mathfrak{D})$. Furthermore, analogous to Lemma 1.1, $\mathcal{T} \cap \mathfrak{B}_n$ can also be put into "upper triangular form". More specifically, suppose $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$, $\mathfrak{D}$ is a masa in $\mathfrak{A}$, and $\mathcal{T}$ is a strongly maximal triangular algebra in $\mathfrak{A}$ with respect to $\mathfrak{D}$. Let $\mathfrak{B}_n = C^*(\mathfrak{A}_n, \mathfrak{D})$ for each n. Then $\mathcal{T} \cap \mathfrak{B}_n$ is maximal triangular in $\mathfrak{B}_n$ and $\mathcal{T} = \overline{\bigcup_n (\mathcal{T} \cap \mathfrak{B}_n)}$. Furthermore, if $\mathfrak{A}_n \cong \bigoplus_{m=1}^{\ell(n)} \mathbf{M}_{k(n,m)}$ for each n, then there exist compact zero dimensional spaces $X_{n,m}$ such that

$$\mathfrak{B}_n \cong \bigoplus_{m=1}^{\ell(n)} \big(C(X_{n,m}) \otimes \mathbf{M}_{k(n,m)}\big)$$

and

$$\mathcal{T} \cap \mathfrak{B}_n = \bigoplus_{m=1}^{\ell(n)} \big(C(X_{n,m}) \otimes \mathbf{T}_{k(n,m)}\big)$$

where $\mathbf{T}_k$ = upper triangular matrices in $\mathbf{M}_k$.

We omit the proof since this result is not needed in the sequel.

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3. **Analytic** TAF **algebras with trivial real-valued cocycles.** This section is motivated by the following question: under what conditions is a TAF algebra $\mathcal{T}$ generated by a real-valued cocycle? In Corollary 1.9, it was shown that a necessary condition is that $\mathcal{T}$ must be strongly maximal triangular. However, this condition is not sufficient. In Theorem 3.16, the main result of this section, we give necessary and sufficient conditions for a strongly maximal TAF algebra to be generated by a real-valued coboundary. In addition, just as in the $\mathbb{Z}$-valued case (Theorem 2.2), this result yields a generic coboundary.

We will first show that trivially analytic TAF algebras cannot be $\mathbb{Z}$-analytic (Theorem 3.1), and we will make some observations on the connection between coboundaries and decreasing sets (defined below). We will also give a number of examples which motivate and illustrate the main result, and we will finish with a computational tool related to Theorem 3.16 (Proposition 3.20).

If $\mathcal{A}$ is a nest subalgebra of a von Neumann algebra $\mathcal{M}$ of the type studied in [MSS1] and [MSS2], then $\mathcal{A}$ is analytic via a coboundary [MSS2, Corollary 3.4]. In [V2, Example 5.2], it was shown that the "canonical" nest algebra Alg $\mathcal{N}$ (defined in §1) in the $\mathbf{M}_{2^\infty}$ UHF algebra is also generated by a coboundary, specifically $d(x,y) = b(y) - b(x)$ with $b(x) = \sup\{\mathrm{tr}(P) : P \in \mathcal{N}, x \notin \hat{P}\}$. However, this coboundary will not work for general nest algebras. In fact, a nest subalgebra of a UHF algebra need not even be analytic, since there are nest algebras which are not strongly maximal [PW, Example 2.26]. Conversely, there are analytic algebras with trivial cocycle which are not nest algebras; in fact, they may have no nontrivial invariant projections (Example 3.7 below). In Section 4, we will investigate the question of when a nest algebra is trivially analytic.

For the remainder of this section, unless otherwise stated $\mathcal{T} = \mathcal{A}(\mathcal{P})$ will be a strongly maximal triangular subalgebra of a simple (infinite dimensional) AF algebra $\mathfrak{A} = C^*(\mathcal{R})$. X will denote the spectrum of $\mathfrak{D} = \mathcal{T} \cap \mathcal{T}^*$. The simplicity of $\mathfrak{A}$ implies that $\mathcal{R}$ is minimal, *i.e.*, $\{y \in X : (x,y) \in \mathcal{R}\}$, the equivalence class of x, is dense in X for each x [R, page 112]. All cocycles and coboundaries will be real-valued. Note that if $d(x,y) = b(y) - b(x)$ is a coboundary, then the range of b is bounded since b is continuous and X is compact.

THEOREM 3.1. *Let $\mathfrak{A}$ be a simple* AF *algebra, and let $\mathcal{T} \subseteq \mathfrak{A}$ be a trivially analytic* TAF *algebra. Then $\mathcal{T}$ is not $\mathbb{Z}$-analytic.*

PROOF. Let $\mathfrak{A} = C^*(\mathcal{R})$ and suppose $\mathcal{T} = \mathcal{A}(\mathcal{P})$, where $d(x,y) = b(y) - b(x)$ for $(x,y) \in \mathcal{R}$ and $\mathcal{P} = d^{-1}[0,\infty)$. If there are distinct points z_i, $i = 0,\ldots,m$, such that $x = z_0 \ll z_1 \ll \cdots \ll z_m = y$, then $\hat{d}(x,y) = \sum_{i=1}^m \hat{d}(z_{i-1}, z_i) \geq m$. Thus, we may assume that for every $(x,y) \in \mathcal{P}$, the set $[x,y] = \{z : x \ll z \ll y\}$ is finite.

Suppose $b(u_0)$ is the minimum of b on X and $b(u_1)$ is the maximum of b on X. Let C be the equivalence class of u_0. If u_1 is in C, then $C = [u_0, u_1]$ is finite, a contradiction. Hence, $u_1 \notin C$. As noted above, $\mathcal{R}$ is minimal since $\mathfrak{A}$ is simple, so there is a sequence $\{x_i\} \subseteq C$ which converges to u_1, and we may further assume that all the x_i's are distinct and satisfy $b(x_1) < b(x_2) < \cdots < b(u_1)$. Choose $\epsilon_i > 0$ such that $b(x_i) + \epsilon_i < b(x_{i+1}) - \epsilon_{i+1}$, and let $W_i = (b(x_i) - \epsilon_i, b(x_i) + \epsilon_i)$. Then we can find a sequence $\{z_i\}$ in the equivalence class of u_1 such that $z_i \in b^{-1}(W_i)$ for all $i \geq 1$. But this implies that $[z_1, u_1]$ is infinite, a contradiction. ∎

In particular, the result is true for infinite dimensional UHF algebras. Note also that the hypotheses are stronger than necessary: only the density of the equivalence classes of u_0 and u_1 is needed in the proof.

DEFINITION 3.2. A set $D \subseteq X$ is *decreasing (relative to $\mathcal{T}$)* if $y \in D$ and $x \ll y$ implies $x \in D$. Similarly, $I \subseteq X$ is *increasing* if $x \in I$ and $x \ll y$ implies $y \in I$.

PROPOSITION 3.3. *If D is a decreasing set, then so is $\overline{D}$. If I is an increasing set, then so is $\overline{I}$.*

PROOF. We only prove the first assertion. Let $z \in \overline{D}$ and suppose $y \ll z$. There is a sequence $\{x_n : 1 \leq n < \infty\} \subseteq D$ such that $x_n \longrightarrow z$. The relation $y \ll z$ is implemented by a partial homeomorphism h_v, $h_v(y) = z$, such that $\hat{v} \subseteq \mathcal{P}$. As $\mathrm{Range}(h_v) = \widehat{v^*v}$ is a clopen subset of X, there is an N such that $x_n \in \mathrm{Range}(h_v)$ for $n \geq N$. Set $y_n = h_v^{-1}(x_n)$. Then $y_n \ll x_n$, so $y_n \in D$, and $\lim y_n = \lim h_v^{-1}(x_n) = h_v^{-1}(z) = y$. Therefore, $y \in \overline{D}$ and $\overline{D}$ is decreasing. ∎

PROPOSITION 3.4. *If $\mathcal{T}$ is generated by a coboundary $d(x,y) = b(y) - b(x)$, then for each r, $b^{-1}(-\infty, r)$ is an open decreasing set and $b^{-1}(-\infty, r]$ is a closed decreasing set. Similarly, each $b^{-1}(r,\infty)$ is an open increasing set and each $b^{-1}[r,\infty)$ is a closed increasing set. In addition, the open decreasing and increasing sets separate the points of X in the sense that if $(x,y) \in \mathcal{P}$, $x \neq y$, then there is an open decreasing set D and an open increasing set I such that $x \in D$, $y \in I$, and $D \cap I = \emptyset$.*

PROOF. Straightforward. ∎

It follows that an analytic algebra generated by a coboundary induces lots of decreasing and increasing sets in X.

EXAMPLE 3.5. In [V2, Example 5.1], it was shown that the maximal triangular subalgebra of a UHF algebra generated by the standard embeddings and the upper triangular matrices is analytic, but that the given cocycle cannot be represented by a coboundary. The previous proposition, however, shows that there is *no* coboundary which generates the algebra, because every open decreasing or increasing set has measure 1. To see this, represent $\mathcal{T}$ as $\overline{\bigcup_n \mathcal{T}_n}$ with $\mathcal{T}_n = \mathrm{span}\{e_{ij}^{(n)} : i \leq j\}$. If W is open and decreasing, then W contains some $\hat{e}_k^{(n)}$, and therefore $\bigcup\{\hat{e}_1^{(n)}, \ldots, \hat{e}_k^{(n)}\} \subseteq W$, so $\mu(W) \geq k/[n]$. But $(\sigma_{m-1} \circ \cdots \circ \sigma_n)(e_k^{(n)}) = \sum_{i=0}^{q-1} e_{i[n]+k}^{(m)}$, where $q = [m]/[n]$, so $\bigcup\{\hat{e}_1^{(m)}, \ldots, \hat{e}_{(q-1)[n]+k}^{(m)}\} \subseteq W$ and therefore $\mu(W) \geq \big((q-1)[n] + k\big)/[m] \geq 1 - [n]/[m] \longrightarrow 1$ as $m \longrightarrow \infty$. A similar proof shows that every open increasing set also has measure 1. Now if $\mathcal{T}$ is generated by a coboundary $d(x,y) = b(y) - b(x)$, choose sets D and I as in Proposition 3.4. Then $\mu(D) = \mu(I) = \mu(X) = 1$ and $D \cap I = \emptyset$, a contradiction.

PROPOSITION 3.6. *If $\mathcal{T}$ is a closed subalgebra of $\mathfrak{A}$ containing $\mathfrak{D}$, then P is an invariant projection for $\mathcal{T}$ if and only if $\hat{P}$ is a clopen decreasing set.*

PROOF. Suppose P is an invariant projection, $y \in \hat{P}$, and $x \ll y$. Then $(x,y) \in \hat{v}$ for some matrix unit $v \in \mathcal{T}$, and therefore $(x,y) \in \widehat{vP}$. But $vP = PvP$ since P is invariant, so $x \in \hat{P}$. Hence, $\hat{P}$ is decreasing, and it is clopen since P is a projection.

On the other hand, if $\hat{P}$ is decreasing but P is not invariant, then $P^{\perp}vP \neq 0$ for some matrix unit $v \in \mathcal{T}$. But then it follows that there is some $(x, y) \in (P^{\perp}vP)^{\wedge}$. Since $P^{\perp}vP \in \mathcal{T}$, we have $x \ll y$ and $y \in \hat{P}$, but $x \notin \hat{P}$, a contradiction. ∎

Thus, in the case of the canonical nest $\mathcal{N}$, with $\mathcal{T} = \operatorname{Alg}\mathcal{N}$, X has lots of clopen decreasing sets. However, as the next example shows, the existence of clopen decreasing sets is not necessary.

EXAMPLE 3.7. There is a TUHF algebra which is analytic via a coboundary but which has no nontrivial invariant projections (and therefore in particular it is not a nest algebra). Let $\mathfrak{A}_n = \mathbf{M}_{4^n}$, $\mathcal{T}_n = \{$upper triangular matrices in $\mathfrak{A}_n\}$, and let U_n be the permutation matrix which satisfies

$$\begin{aligned} &U_n \operatorname{diag}(a_1, \ldots, a_{4^n})U_n^* \\ &\quad = \operatorname{diag}(a_1, a_2, a_3, a_5, a_4, a_6, a_7, a_9, a_8, \ldots, a_{4k-1}, a_{4k+1}, a_{4k}, a_{4k+2}, \ldots, a_{4^n}), \end{aligned}$$

where $\operatorname{diag}(\cdots)$ denotes the diagonal matrix with the given entries. Let $j_n = \operatorname{Ad} U_{n+1} \circ \nu_n$, $j_{1,m} = j_m \circ \cdots \circ j_1$, and note that $j_n \colon \mathcal{T}_n \hookrightarrow \mathcal{T}_{n+1}$, so we can define $\mathcal{T} = \varinjlim(\mathcal{T}_n, j_n)$.

Now $\mathcal{T}$ has no nontrivial invariant projections. For if $P \in \mathfrak{D}_n$ is invariant for $\mathcal{T}_n$, $P \neq 0, 1$, then $P = e_1^{(n)} + \cdots + e_k^{(n)}$ for some $k < 4^n$. But then $j_n(P) = e_1^{(n+1)} + \cdots + e_{4k-1}^{(n+1)} + e_{4k+1}^{(n+1)}$, so $j_n(P)$ is not invariant for $\mathcal{T}_{n+1}$, and thus not for $\mathcal{T}$ as well.

Let $k(n) = (4^n - 1)/3$ and $f_n = \sum_{i=1}^{k(n)} e_i^{(n)}$. Then for every n, f_n is an invariant projection for $\mathcal{T}_n$ and $f_n \leq f_{n+1}$. The set $W = \bigcup_{n=1}^{\infty} \hat{f}_n$ is therefore open and decreasing, and $\mu(W) = \lim_{n\to\infty} \mu(\hat{f}_n) = \lim_{n\to\infty} \operatorname{tr}(f_n) = \lim_{n\to\infty}(4^n - 1)/(3 \cdot 4^n) = \frac{1}{3}$. Thus, X has a nontrivial (neither null nor conull) open decreasing set. Similar arguments show that X has lots of such sets, and in fact it will follow from Theorem 3.16 that $\mathcal{T}$ is induced by a real-valued coboundary (see Example 3.17).

DEFINITION 3.8. For $x = (\hat{e}_1, \hat{e}_2, \ldots)$, set $E_{n,x} = \bigcup_{v \in \mathcal{T} \cap \mathcal{W}_{\mathfrak{T}}} \{\widehat{ve_n v^*}\}$ and $E_x = \bigcap_{n=1}^{\infty} E_{n,x}$. Then $E_{n,x}$ is the smallest open decreasing set containing $\hat{e}_n$, so E_x is a decreasing G_δ subset of X containing $\{x\}$. Similarly, letting $J_{n,x} = \bigcup_{v \in \mathcal{T} \cap \mathcal{W}_{\mathfrak{T}}} \{\widehat{v^* e_n v}\}$ and $J_x = \bigcap_{n=1}^{\infty} J_{n,x}$, it follows that J_x is an increasing G_δ set containing $\{x\}$. Equivalently, E_x is the intersection of all open decreasing sets containing x, since each open decreasing set contains some $E_{n,x}$, and likewise J_x is the intersection of all open increasing sets containing x. Thus, it is clear that E_x and J_x are independent of the sequence $\{\mathfrak{A}_n : 1 \leq n < \infty\}$; that is, if $\{\mathfrak{G}_n : 1 \leq n < \infty\}$ is another such sequence with $\mathfrak{G}_n \cap \mathfrak{D}$ a masa in $\mathfrak{G}_n$, then $E_x(\{\mathfrak{G}_n\}) = E_x(\{\mathfrak{A}_n\})$.

PROPOSITION 3.9. *If $\mathcal{T}$ is strongly maximal triangular in $\mathfrak{A}$, then $E_x \cup J_x = X$ for each $x \in X$.*

PROOF. Suppose $x = (\hat{e}_1, \hat{e}_2, \ldots)$, and let y be any point in X. For each n, there is some $z_n \in \hat{e}_n$ such that $(y, z_n) \in \mathcal{R}$, since $\mathcal{R}$ is minimal. Now if $(y, z_n) \in \mathcal{P}$, then $y \in E_{n,x}$. Likewise, if $(z_n, y) \in \mathcal{P}$, then $y \in J_{n,x}$. Thus, $y \in E_{n,x} \cup J_{n,x}$ for all n, and therefore

$$y \in \bigcap_{n=1}^{\infty} (E_{n,x} \cup J_{n,x}) = \Big(\bigcap_{n=1}^{\infty} E_{n,x}\Big) \cup \Big(\bigcap_{n=1}^{\infty} J_{n,x}\Big) = E_x \cup J_x$$

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since the sets $\{E_{n,x} : 1 \leq n < \infty\}$ and $\{J_{n,x} : 1 \leq n < \infty\}$ are nested. It follows that $X = E_x \cup J_x$. ∎

From now on, we will assume that $\mathfrak{A}$ has a faithful trace, and μ will denote the measure on X induced by the normalized trace tr. Let $\mathcal{T}$ be a strongly maximal TAF algebra in $\mathfrak{A}$, and define $b_0 \colon X \longrightarrow [0, 1]$ by $b_0(x) = \mu(E_x)$.

LEMMA 3.10. *If $\mu(E_x \cap J_x) = 0$ for all $x \in X$, then $b_0 \in C(X)$.*

PROOF. Suppose $x = (\hat{e}_1, \hat{e}_2, \ldots)$. Since $\mu(E_x) = \mu(\bigcap_{n=1}^{\infty} E_{n,x}) = \lim_{n\to\infty} \mu(E_{n,x})$, it follows that given $\epsilon > 0$, there is an N such that if $n \geq N$, then $|\mu(E_{n,x}) - b_0(x)| < \epsilon$. Thus if $y \in \hat{e}_n$, then $b_0(y) \leq \mu(E_{n,x})$, and hence $b_0(y) \leq b_0(x) + \epsilon$. Therefore b_0 is upper semicontinuous. Now the same argument shows that the function $x \longrightarrow \mu(J_x)$ is upper semicontinuous (replace $E_{n,x}$ and E_x by $J_{n,x}$ and J_x). Since $\mu(E_x \cap J_x) = 0$ by hypothesis, and Proposition 3.9 implies that $\mu(E_x \cup J_x) = \mu(X) = 1$, it follows that $b_0(x) = 1 - \mu(J_x)$ is also lower semicontinuous, and hence continuous. ∎

REMARK 3.11. The converse of the lemma is false. If $\mathcal{T}$ is the algebra in Example 3.5, then every nonempty open decreasing or increasing set in X has measure one. Therefore, $\mu(E_x) = \mu(J_x) = 1$ and necessarily $\mu(E_x \cap J_x) = 1$. Thus, in this case, $b_0 \in C(X)$ is constant.

PROPOSITION 3.12. *Suppose c is a continuous function such that $c(x) \leq c(y)$ if $(x, y) \in \mathcal{P}$. Then for each $y \in X$ we have*

(a) $c^{-1}\big(-\infty, c(y)\big) \subseteq E_y \subseteq c^{-1}\big(-\infty, c(y)\big]$

(b) $c^{-1}\big(c(y), \infty\big) \subseteq J_y \subseteq c^{-1}\big[c(y), \infty\big)$

PROOF. First, we claim $c(x) \leq c(y)$ for all $x \in E_y$. Given $\epsilon > 0$ there is, by continuity of c, a projection $e \in \mathfrak{D}$ with $y \in \hat{e}$ such that $|c(x) - c(y)| < \epsilon$ for all $x \in \hat{e}$. Now $E_y \subseteq F = \bigcup_{v \in \mathcal{W}_{\mathfrak{D}} \cap \mathcal{T}} \{\widehat{vev^*}\}$. Thus, if $x \in F$, then $x \ll x'$ for some $x' \in \hat{e}$. Hence $c(x) \leq c(x') < c(y) + \epsilon$. As $\epsilon > 0$ was arbitrary, $c(x) \leq c(y)$. Similarly, it can be shown that $c(x) \geq c(y)$ for all $x \in J_y$. Since $X = E_y \cup J_y$, (a) and (b) follow. ∎

In particular, the preceding proposition is true for $c = b_0$ if $\mu(E_x \cap J_x) = 0$ for all x.

LEMMA 3.13. *There is a countable collection $\{\tau_n : 1 \leq n < \infty\}$ of measure-preserving homeomorphisms of X such that for each $x \in X$,*

(i) $\big(x, \tau_n(x)\big) \in \mathcal{R}$, and

(ii) $\tau_n(x) \neq \tau_m(x)$ if $n \neq m$.

PROOF. We first prove the result in the UHF case, so assume that $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$ where the $\mathfrak{A}_n$'s are factors. If $e_\iota^{(n)} = e_{\iota_1}^{(n+1)} + \cdots + e_{\iota_k}^{(n+1)}$, set $v_{n,\iota} = \sum_{j=1}^{k-1} e_{\iota_j \iota_{j+1}}^{(n+1)} + e_{\iota_k \iota_1}^{(n+1)}$ and $v_n = \sum_{\iota=1}^{[n]} v_{n,\iota}$. Observe that $v_n \in \mathcal{W}_{\mathfrak{D}}$ is unitary, so that if τ_n is the partial homeomorphism corresponding to v_n, defined by $\tau_n(x)(d) = x(v_n d v_n^*)$, then τ_n is a homeomorphism of X. If $e \in \mathcal{P}(\mathfrak{D})$, then $\mu\big(\tau_n(\hat{e})\big) = \operatorname{tr}(v_n^* e v_n) = \operatorname{tr}(e v_n v_n^*) = \operatorname{tr}(e) = \mu(\hat{e})$. As any open set in X is a union of a countable number of clopen subsets, τ_n is measure-preserving on open sets and hence measure-preserving.

Let $x \in X, x \in \hat{e}_i^{(n+1)}$. Then $\tau_n(x) \notin \hat{e}_i^{(n+1)}$, but if $m > n$ it follows from the construction that $\tau_m(x) \in \hat{e}_i^{(n+1)}$. Thus $\tau_n(x) \neq \tau_m(x)$. It is clear that $\big(x, \tau_n(x)\big) \in \mathcal{R}$.

Now suppose that $\mathfrak{A}$ is a simple AF algebra which is not UHF. Then there exists a sequence $\{\mathfrak{A}_n\}$ such that $\mathfrak{A}_n = \oplus_{m=1}^{\ell(n)} \mathbf{M}_{k(n,m)}$ with $\ell(n) > 1$. Fix n. By [Br, Corollary 3.5], there is some $r > n$ such that the multiplicity of the embedding of each $\mathbf{M}_{k(n,m)}$ in each $\mathbf{M}_{k(r,p)}$ (in the sense of the Bratteli diagram of $\mathfrak{A}$) is at least 1. Consequently, there is some $s > r$ such that the multiplicity of the embedding of each $\mathbf{M}_{k(n,m)}$ in each $\mathbf{M}_{k(s,q)}$ is at least 2. It follows that $e_i^{(nm)} = \sum_{q=1}^{\ell(s)} (e_{q_1}^{(sq)} + \cdots + e_{q_k}^{(sq)})$ with $q_k \geq 2$ for each q. Now v_n can be defined in a similar manner as before, and the result follows. ∎

COROLLARY 3.14. *Let $E \subseteq X$ be measurable with $\mu(E) > 0$. Then there are distinct points $x, y \in E$ such that $(x, y) \in \mathcal{R}$.*

PROOF. Suppose E contains no two points x, y such that $(x, y) \in \mathcal{R}$. Then, with notation as in Lemma 3.13, $\tau_n(E) \cap \tau_m(E) = \emptyset$ for $n \neq m$. For if there exist $x, y \in E$ such that $\tau_n(x) = \tau_m(y)$, then $(x, y) \in \mathcal{R}$ since $\big(x, \tau_n(x)\big) \in \mathcal{R}$ and $\big(\tau_m(y), y\big) \in \mathcal{R}$. Thus, $1 = \mu(X) \geq \mu\big(\bigcup_{n=1}^{\infty} \tau_n(E)\big) = \sum_{n=1}^{\infty} \mu\big(\tau_n(E)\big) = \sum_{n=1}^{\infty} \mu(E) = \infty$. This contradiction completes the proof. ∎

LEMMA 3.15. *Suppose $\mathcal{T}$ is analytic with cocycle $d(x, y) = c(y) - c(x)$, $c \in C(X)$. Then $\mu(E_x \cap J_x) = 0$ for all $x \in X$.*

PROOF. Let $\mathcal{P} = d^{-1}[0, \infty)$, so $\mathcal{P} \cup \mathcal{P}^{-1} = \mathcal{R}$ and $\mathcal{P} \cap \mathcal{P}^{-1} = X = d^{-1}(\{0\})$. Suppose that $\mu(E_{x_0} \cap J_{x_0}) > 0$ for some $x_0 \in X$. By the last corollary, there is a pair $(x, y) \in \mathcal{P}$, $x \neq y$, such that $x, y \in E_{x_0} \cap J_{x_0}$. Hence $c(x) < c(y)$. Now exactly one of the following holds: (i) $c(y) \leq c(x_0)$; (ii) $c(x) \leq c(x_0) < c(y)$; or (iii) $c(x_0) < c(x)$. In case (i), let $\alpha \in \mathbb{R}$ satisfy $c(x) < \alpha < c(y)$. Then $c^{-1}(\alpha, \infty)$ is open, increasing, and contains x_0. Thus $J_{x_0} \subseteq c^{-1}(\alpha, \infty)$. But $x \notin c^{-1}(\alpha, \infty)$, whereas $x \in J_{x_0}$. The other cases are treated similarly. ∎

We can now prove the main result.

THEOREM 3.16. *Let $\mathfrak{A}$ be a simple* AF *algebra with faithful trace, and let $\mathcal{T} = \mathcal{A}(\mathcal{P}) \subseteq \mathfrak{A}$ be a strongly maximal* TAF *algebra. With $\mathfrak{D} = \mathcal{T} \cap \mathcal{T}^*$ and $X = \hat{\mathfrak{D}}$, let $\{J_n\}_{n=1}^{N}$ $(N \leq \infty)$ be an enumeration of the increasing clopen subsets of X, and set*

$$b(x) = \mu(E_x) + \sum_{n=1}^{N} 2^{-n} \chi_{J_n}(x).$$

Then $\mathcal{T}$ is analytic via a real-valued coboundary iff

(i) $\mu(E_x \cap J_x) = 0$ for all $x \in X$, and

(ii) if $(x, y) \in \mathcal{P}$, $x \neq y$, then $b(x) < b(y)$.

If (i) and (ii) are satisfied, then $d(x, y) = b(y) - b(x)$ is a coboundary for $\mathcal{T}$; that is, b is continuous and $\mathcal{P} = d^{-1}[0, \infty)$.

PROOF. Suppose first that $\mathcal{T}$ is analytic with cocycle $d'(x, y) = c(y) - c(x)$, where $c \in C(X)$. By Lemma 3.15, condition (i) is satisfied. Thus by Lemma 3.10, the function

$b_0(x) = \mu(E_x)$ is continuous. Now $\sum_{n=1}^{N} 2^{-n}\chi_{J_n}$ is continuous, if N is finite, since J_n is clopen; but if $N = \infty$, the sum is the uniform limit of partial sums and thus is again continuous. It follows that $b \in C(X)$.

Let $(x,y) \in \mathcal{P}, x \neq y$. If there is an n such that y belongs to the increasing set J_n and $x \notin J_n$, then $b(y)-b(x) \geq 2^{-n}$. So we may suppose that every increasing clopen set containing y also contains x. It follows that $\mathrm{Range}(c) \cap \big(c(x), c(y)\big) \neq \emptyset$, for otherwise $c^{-1}[c(y), \infty) = c^{-1}\big(c(x), \infty\big)$ is a clopen increasing set containing y but not x. So let $\alpha \in \mathrm{Range}(c)$, $c(x) < \alpha < c(y)$. By Proposition 3.12, $c^{-1}\big(-\infty, c(y)\big) \subseteq E_y$ and $c^{-1}\big(-\infty, c(x)\big] \supseteq E_x$. Thus

$$E_y \setminus E_x \supseteq c^{-1}\big(-\infty, c(y)\big) \setminus c^{-1}\big(-\infty, c(x)\big] \supseteq c^{-1}\big(c(x), c(y)\big)$$

which is a nonempty open subset of X and consequently has positive measure. As $E_x \subseteq E_y$, $\mu(E_y) - \mu(E_x) = \mu(E_y \setminus E_x) \geq \mu\Big(c^{-1}\big(c(x), c(y)\big)\Big) > 0$. This implies $b(y) - b(x) > 0$.

For the other direction, if (i) and (ii) are satisfied, then we know from the above that b is continuous, and (ii) implies $\mathcal{P} = d^{-1}[0, \infty)$, so $\mathcal{T} = \mathcal{T}_d$ is the analytic TAF algebra with cocycle $d(x,y) = b(y) - b(x)$. ∎

EXAMPLE 3.17. We will use Theorem 3.16 to show that the TAF algebra $\mathcal{T}$ given above in Example 3.7 is analytic via a real-valued coboundary. Note that $b(x) = \mu(E_x)$ since X has no clopen decreasing sets. Now if $(x,y) \in \mathcal{P}, x \neq y$, then $(x,y) \in \hat{e}^{(n)}_{i_n k_n}$ for some i_n, k_n, n, with $i_n < k_n$. It then follows from the definition of the embedding j_n that $(x,y) \in \hat{e}^{(n+1)}_{i_{n+1} k_{n+1}}$ with $k_{n+1} \geq i_{n+1} + 3$. Furthermore,

$$\mu(E_x) \leq \frac{i_{n+1}}{4^{n+1}} + \frac{1}{4^{n+2}} + \frac{1}{4^{n+3}} + \cdots = \frac{i_{n+1}}{4^{n+1}} + \frac{1}{3 \cdot 4^{n+1}}$$

and

$$\mu(E_y) \geq \frac{k_{n+1} - 1}{4^{n+1}} - \frac{1}{4^{n+2}} - \frac{1}{4^{n+3}} - \cdots = \frac{k_{n+1} - 1}{4^{n+1}} - \frac{1}{3 \cdot 4^{n+1}},$$

so $\mu(E_x) < \mu(E_y)$.

Suppose $x = (\hat{e}^{(1)}_{i_1}, \hat{e}^{(2)}_{i_2}, \ldots)$, and let $\mathcal{T}_n = \mathcal{T} \cap \mathfrak{A}_n$. For each $m \geq n$, define the sets $E^{(m)}_{n,x} = \bigcup_{v \in \mathcal{T}_m \cap \mathcal{W}_{\mathfrak{T}}} \{(ve^{(n)}_{i_n} v^*)^\wedge\}$ and $J^{(m)}_{n,x} = \bigcup_{v \in \mathcal{T}_m \cap \mathcal{W}_{\mathfrak{T}}} \{(v^* e^{(n)}_{i_n} v)^\wedge\}$. Note that $E_{n,x} = \bigcup_{m \geq n} E^{(m)}_{n,x}$ and $J_{n,x} = \bigcup_{m \geq n} J^{(m)}_{n,x}$. Then for $i_n \neq 1, [n]$,

$$E^{(n)}_{n,x} \cap J^{(n)}_{n,x} = \{\hat{e}^{(n)}_{i_n}\},$$

$$E^{(n+1)}_{n,x} \cap J^{(n+1)}_{n,x} = \bigcup_{k=4i_n-5}^{4i_n+1} \{\hat{e}^{(n+1)}_k\},$$

$$E^{(n+2)}_{n,x} \cap J^{(n+2)}_{n,x} = \bigcup_{k=16i_n-25}^{16i_n+5} \{\hat{e}^{(n+1)}_k\}, \text{ etc.}$$

Thus, $\mu(E_{n,x} \cap J_{n,x}) = 4^{-n} + 2 \cdot 4^{-(n+1)} + 2 \cdot 4^{-(n+2)} + \cdots = \frac{5}{3}4^{-n}$. In the other two cases, the formulas are similar and $\mu(E_{n,x} \cap J_{n,x}) = \frac{4}{3}4^{-n}$. It follows that $\mu(E_x \cap J_x) = 0$ since $E_x \cap J_x = \bigcap_n (E_{n,x} \cap J_{n,x})$. Thus, conditions (i) and (ii) of Theorem 3.16 are satisfied, so $\mathcal{T}$ is trivially analytic.

EXAMPLE 3.18. Consider the algebra $\mathcal{T}_{(\alpha)}$ in [PW, Theorem 2.24] with $0 < \alpha < 1$. $\mathcal{T}_{(\alpha)}$ is a strongly maximal TAF nest subalgebra of the 2^∞ UHF algebra which has the property $\sup\{\mathrm{tr}(P) : P \in \mathrm{Lat}\,\mathcal{T}_{(\alpha)}, P < 1\} = \alpha$. We will briefly describe the construction. Let $Q(N)$ be a permutation matrix in $\mathbf{M}_{2N}$ such that

$$Q(N)\,\mathrm{diag}(a_1^{(1)}, a_2^{(1)}, a_1^{(2)}, a_2^{(2)}, \ldots, a_1^{(N)}, a_2^{(N)})Q(N)^t = \mathrm{diag}(a_1^{(1)}, a_1^{(2)}, \ldots, a_1^{(N)}, a_2^{(1)}, a_2^{(2)}, \ldots, a_2^{(N)}),$$

where $\mathrm{diag}(b_1, \ldots, b_\ell)$ denotes the diagonal matrix with entries $b_1, \ldots, b_\ell$. Let $R(n,m) = I_{2m} \oplus Q(2^n - m)$ for each n, $1 \leq m < 2^n$. If $\alpha = \sum_{n=1}^\infty \frac{k_n}{2^n}$ is the nonterminating binary expansion of α, let $M_n = \sum_{\iota=1}^n 2^{n-\iota} k_\iota$ and define the embedding $j_n \colon \mathbf{M}_{2^n} \hookrightarrow \mathbf{M}_{2^{n+1}}$ by $j_n = \mathrm{Ad}\,R(n, M_n) \circ \nu_n$. Let $\mathfrak{A} = \varinjlim(\mathbf{M}_{2^n}, j_n)$.

Define $P_j^{(n)} = \sum_{\iota=1}^j e_\iota^{(n)}$, $j = 1, \ldots, M_n$, and $\mathcal{M} = \{0, 1, P_j^{(n)} : j = 1, \ldots, M_n, n = 1, 2, \ldots\}$. Then it was shown in [PW, Theorem 2.24] that $\mathcal{M}$ is a maximal nest such that $\sup\{p \in \mathcal{M} : p < 1\} = \alpha$, and that $(\mathrm{Alg}\,\mathcal{M}) \cap \mathbf{M}_{2^n}$ is the set of upper triangular matrices in $\mathbf{M}_{2^n}$ (so $\mathcal{T}_{(\alpha)} = \mathrm{Alg}\,\mathcal{M}$ is a strongly maximal TAF algebra).

Let $E = \bigcup_n \hat{P}_{M_n}^{(n)}$. $\mu(E) = \alpha < 1$, so there exists some $x_0 \in X \setminus E$. For each n, let i_n be the integer satisfying $x_0 \in \hat{e}_{\iota_n}^{(n)}$. Note that $i_n > M_n$. Now if W is an open decreasing set containing x_0, then $W \supseteq \bigcup\{\hat{e}_1^{(n)}, \ldots, \hat{e}_{\iota_n}^{(n)}\}$ for some n. But $(j_{m-1} \circ \cdots \circ j_n)(e_{\iota_n}^{(n)}) \supseteq \hat{e}_{2^m - 2^n + \iota_n}^{(m)}$, so $W \supseteq \bigcup\{\hat{e}_1^{(m)}, \ldots, \hat{e}_{2^m - 2^n + \iota_n}^{(m)}\}$. Thus, $\mu(W) \geq (2^m - 2^n + i_n)/2^m \longrightarrow 1$ as $m \longrightarrow \infty$. It follows that $\mu(W) = 1$, and therefore $\mu(E_{x_0}) = 1$.

Now suppose V is an open increasing set containing x_0. Every nonempty open set intersects E because $\mathcal{M}$ is a maximal nest, so therefore there is some $x_1 \in V \cap E$. Let $e_k^{(n)}$ be a matrix unit satisfying $x_1 \in \hat{e}_k^{(n)} \subseteq V \cap E$. Then $\bigcup\{\hat{e}_k^{(n)}, \ldots, \hat{e}_{[n]}^{(n)}\} \subseteq V$ and $\bigcup\{\hat{e}_1^{(n)}, \ldots, \hat{e}_{k-1}^{(n)}\} \subseteq E$. It follows that $\mu(V) \geq 1 - \alpha$, and therefore $\mu(J_{x_0}) \geq 1 - \alpha$. Consequently, $\mu(E_{x_0} \cap J_{x_0}) \geq 1 - \alpha > 0$, so Theorem 3.16 implies that $\mathcal{T}_{(\alpha)}$ is not generated by a real-valued coboundary.

Since a trivially analytic algebra need not have any increasing clopen sets (Example 3.7), the $\mu(E_x)$ term in the definition of $b(x)$ in Theorem 3.16 is necessary. The following example shows that the series $\sum_{n=1}^N 2^{-n}\chi_{J_n}(x)$ is also necessary, even for "nice" nest algebras.

EXAMPLE 3.19. Let $\mathfrak{A}_n = \mathbf{M}_{2^n}$ with the usual matrix units $\{e_{ij}^{(n)}\}$, ν_n be the nest embedding, and $\mathfrak{A} = \varinjlim(\mathfrak{A}_n, \nu_n)$. Define

$$\mathcal{M}_n = \Big\{0, 1, \sum_{\iota=1}^k e_\iota^{(n)}, \sum_{\iota=1}^{2^{n-1}} e_\iota^{(n)} + \sum_{j=1}^m e_{2^n - j}^{(n)} : 1 \leq k, m \leq 2^{n-1}\Big\}.$$

Note that $\mathcal{M}_n \subseteq \mathcal{M}_{n+1}$, so $\mathcal{M} = \bigcup_{n=1}^\infty \mathcal{M}_n$ is a nest. Write $\mathrm{Alg}\,\mathcal{M}$ as $\mathcal{A}(\mathcal{P})$.

Let $x = (\hat{e}_1^{(1)}, \hat{e}_2^{(2)}, \hat{e}_4^{(3)}, \ldots, \hat{e}_{2^{n-1}}^{(n)}, \ldots)$ and $y = (\hat{e}_2^{(1)}, \hat{e}_4^{(2)}, \hat{e}_8^{(3)}, \ldots, \hat{e}_{2^n}^{(n)}, \ldots)$. Then $(x,y) \in \mathcal{P}$, $x \neq y$, but $\mu(E_x) = \mu(E_y)$ since $E_y = E_x \cup \{y\}$. However, $\mathrm{Alg}\,\mathcal{M}$ is analytic via a coboundary by Theorem 4.6 below (or 3.16), since $C^*(\mathcal{M}) = \mathfrak{D}$. It is interesting to compare this with the canonical nest algebra, for which $\mu(E_x)$ *does* generate a coboundary.

This illustrates, as seen in [PW], the importance of the embedding of the nest $\mathcal{M}$ in $\mathfrak{D}$ in relation to the groupoid $\mathcal{R}$.

We will conclude this section by developing another method for computing $\mu(E_x)$ (Proposition 3.20). This is useful in the application of Theorem 3.16, as will be seen in Example 3.21. For the remainder of this section, $\mathfrak{A} = C^*(\mathcal{R})$ will be UHF and $\mathcal{T} = \mathcal{A}(\mathcal{P}) \subseteq \mathfrak{A}$ will be a TAF algebra which is strongly maximal triangular in factors, *i.e.*, there is an increasing sequence $\{\mathfrak{A}_n : 1 \leq n < \infty\}$ of finite dimensional factors such that $\mathfrak{A} = \overline{\bigcup_n \mathfrak{A}_n}$ and $\mathcal{T} \cap \mathfrak{A}_n$ is maximal triangular in $\mathfrak{A}_n$. Let $\mathfrak{D} = \mathcal{T} \cap \mathcal{T}^*$ and $X = \hat{\mathfrak{D}}$. By Lemma 1.1, we can choose a set of matrix units $\{e_{ij}^{(n)} : 1 \leq i,j \leq [n], 1 \leq n < \infty\}$ for $\mathfrak{A}$ such that $\mathcal{T} \cap \mathfrak{A}_n$ is spanned by $\{e_{ij}^{(n)} : 1 \leq i \leq j \leq [n]\}$. If $x = (e_{k(1,x)}^{(1)}, e_{k(2,x)}^{(2)}, \ldots) \in X$, define

$$b^+(x) = \limsup_n \frac{k(n,x)}{[n]}, \text{ and } b^-(x) = \liminf_n \frac{k(n,x)}{[n]},$$

and recall that $b_0(x) = \mu(E_x)$.

PROPOSITION 3.20. *The following are equivalent:*

(a) $\mu(E_x \cap J_x) = 0$ *for all* $x \in X$.

(b) b_0 *is continuous and* $b^+(x) = b^-(x) = b_0(x)$ *for all* $x \in X$.

(c) b_0 *is continuous and the sequences* $\{\frac{k(n,x)}{[n]}\}$ *and* $\{\mu(E_{n,x})\}$ *converge uniformly to* $b_0(x)$.

(d) b_0 *is continuous and* $\mathrm{Range}(b_0) = [0,1]$.

(e) b_0 *is continuous and* $\mu\big(b_0^{-1}(\{\alpha\})\big) = 0$ *for each* $\alpha \in [0,1]$.

PROOF (a) $\Rightarrow$ (b). If $E_{n,x}$ (resp., $J_{n,x}$) is the decreasing (resp., increasing) open set generated by $e_{k(n,x)}^{(n)}$, then $E_{n,x} \supseteq \bigcup_{j \leq k(n,x)} \hat{e}_j^{(n)}$ and $J_{n,x} \supseteq \bigcup_{j \geq k(n,x)} \hat{e}_j^{(n)}$ since $\mathcal{T} \cap \mathfrak{A}_n$ is the full upper triangular subalgebra of the factor $\mathfrak{A}_n$. Thus, $\mu(E_{n,x}) \geq \frac{k(n,x)}{[n]}$ and $\mu(J_{n,x}) \geq 1 - \frac{k(n,x)-1}{[n]}$. As $\{E_{n,x} : 1 \leq n < \infty\}$ is nested, $\lim_{n \to \infty} \mu(E_{n,x})$ exists, so $\mu(E_x) = \lim_{n \to \infty} \mu(E_{n,x}) \geq \limsup_{n \to \infty} \frac{k(n,x)}{[n]} = b^+(x)$. Similarly,

$$\begin{aligned}
\mu(J_x) &\geq \limsup_{n \to \infty} \left(1 - \frac{k(n,x) - 1}{[n]} \right) \\
&\geq 1 - \liminf_{n \to \infty} \frac{k(n,x) - 1}{[n]} \\
&\geq 1 - b^-(x).
\end{aligned}$$

Since $E_x \cup J_x = X$ and $\mu(E_x \cap J_x) = 0$, then $1 = \mu(E_x) + \mu(J_x) \geq b^+(x) + 1 - b^-(x)$, *i.e.*, $b^+(x) - b^-(x) \leq 0$. As the opposite inequality is clear, it follows that $b^+(x) = b^-(x)$, and the proof shows that the common value equals $b_0(x)$. Finally, b_0 is continuous by Lemma 3.10.

(b) $\Rightarrow$ (c). The sequence $f_n(x) = \mu(E_{n,x})$ is decreasing to $b_0(x) = \mu(E_x)$. By Dini's Theorem, the convergence is uniform. Similarly, the decreasing sequence $\mu(J_{n,x})$ converges uniformly to $\mu(J_x) = 1 - b_0(x)$. Because $1 - \mu(J_{n,x}) \leq \frac{k(n,x)}{[n]} \leq \mu(E_{n,x})$, it follows that $\{\frac{k(n,x)}{[n]} : 1 \leq n < \infty\}$ converges uniformly to $b_0(x)$.

(c) $\Rightarrow$ (d) Since Range(b_0) is closed, it is enough to show Range(b_0) is dense Let $\alpha \in [0,1]$ and $\epsilon > 0$ be given Choose N so that for $n \geq N$, $|\frac{k(n,x)}{[n]} - b_0(x)| < \frac{\epsilon}{2}$ for all $x \in X$ We may assume $[N] > \frac{2}{\epsilon}$ Thus, for $n \geq N$, there is some $k \in \{1, 2, \ldots, [n]\}$ with $|\frac{k}{[n]} - \alpha| < \frac{\epsilon}{2}$ If $x \in \hat{e}_k^{(n)}$, i e, $k(n,x) = k$, then $|b_0(x) - \alpha| \leq |b_0(x) - \frac{k(n,x)}{[n]}| + |\frac{k(n,x)}{[n]} - \alpha| < \epsilon$

(d) $\Rightarrow$ (e) Let $\alpha \in (0,1)$, and choose any $\epsilon \in (0,1)$ and $0 \leq \gamma < \alpha < \beta \leq 1$ with $\beta - \gamma = \epsilon$ Then there are $y, z \in X$ with $b_0(y) = \beta$ and $b_0(z) = \gamma$ Since $b_0^{-1}(-\infty, \beta) \subseteq E_y$ and $E_z \subseteq b_0^{-1}(-\infty, \gamma]$ by Proposition 3 12, we have $b_0^{-1}(\{\alpha\}) \subseteq b_0^{-1}(-\infty, \beta) \setminus b_0^{-1}(-\infty, \gamma] \subseteq E_y \setminus E_z$ Thus, $\mu\big(b_0^{-1}(\{\alpha\})\big) \leq \mu(E_y \setminus E_z) = \mu(E_y) - \mu(E_z) = \beta - \gamma = \epsilon$ Since $\epsilon > 0$ was arbitrary, $\mu\big(b_0^{-1}(\{\alpha\})\big) = 0$ The cases $\alpha = 0$ and $\alpha = 1$ are treated similarly

(e) $\Rightarrow$ (a) This follows from Proposition 3 12 ∎

EXAMPLE 3 21 We will use Theorem 3 16 and Proposition 3 20 to show that the TAF algebra in Example 2 3, generated by alternating the standard and nest embeddings, is not trivially analytic Using the notation from Example 2 3 and letting $m = n/2$, consider the point $x = (\hat{e}_2^{(2)}, \hat{e}_{12}^{(4)}, \hat{e}_{24}^{(6)}, \hat{e}_{176}^{(8)}, \ldots, \hat{e}_{k_m}^{(2m)}, \ldots)$, where $k_m = 2k_{m-1}$ if m is odd and $k_m = 2([2m-2] + k_{m-1}) = 2(4^{m-1} + k_{m-1})$ if m is even Now for m odd,

$$\frac{k_m}{[2m]} = \frac{2k_{m-1}}{4^m} = \frac{2(2(4^{m-2} + k_{m-2}))}{4^m} = \frac{1}{4} + \frac{1}{4}\frac{k_{m-2}}{[2(m-2)]}$$

Since $k_1/[1] = 2/4 = 1/2$, it follows that $k_3/[3] \leq 1/4 + 1/8 = 3/8 < 1/2$, and therefore $k_m/[2m] \leq 3/8$ for all odd m Thus, $c^-(x) \leq 3/8$

On the other hand, if m is even, then

$$\frac{k_m}{[2m]} = \frac{2(4^{m-1} + k_{m-1})}{4^m} = \frac{2(4^{m-1} + 2k_{m-2})}{4^m} = \frac{1}{2} + \frac{1}{4}\frac{k_{m-2}}{[2(m-2]}$$

Since $k_2/[2] = 12/16 = 3/4 > 1/2$, it follows that $k_4/[4] \geq 1/2 + 1/8 = 5/8 > 1/2$, and therefore $k_m/[2m] \geq 5/8$ for all even m Thus, $c^+(x) \geq 5/8$ It now follows from Proposition 3 20 that $\mu(E_x \cap J_x) \neq 0$, so Theorem 3 16 implies that $\mathcal{T}$ cannot be trivially analytic

General alternating algebras were classified in [Po3] and [HP] The definition is the same as in Example 2 3 except that we allow $\mathfrak{A}_n$ to be any $\mathbf{M}_{p_n}$, with $p_n | p_{n+1}$ Then Example 2 3 can be generalized to prove that these algebras are analytic, and an argument similar to the last example shows that they are not trivially analytic

As mentioned in § 1, a bounded cocycle is trivial A recent result of Solel shows that if an analytic subalgebra of a simple AF algebra is generated by an unbounded cocycle, then it cannot be trivially analytic [S] Thus, the conclusion of Example 3 21 also follows from this result

4. **Analytic nest algebras with trivial real-valued cocycles.** As already indicated, some nest subalgebras of UHF algebras are analytic, but not all. This raises the question of determining which nest algebras are analytic. We will give several results concerning this problem, culminating in Theorem 4.6. Again, $\mathfrak{A}$ will always represent a simple AF algebra, but the nest algebras we consider will not necessarily be triangular.

Suppose $\mathcal{M}$ is a nest. Then $\mathcal{M}$ induces a partial order on X by $x < y$ if there is some $P \in \mathcal{M}$ such that $x \in \hat{P}$, $y \notin \hat{P}$. This is different from the partial order $x \ll y$ if $(x, y) \in \mathcal{P}$ (where $\operatorname{Alg} \mathcal{M} = \mathcal{A}(\mathcal{P})$).

PROPOSITION 4.1. *Suppose* $\operatorname{Alg} \mathcal{M} = \mathcal{A}(\mathcal{P})$ *is a nest subalgebra of* $\mathfrak{A} = C^*(\mathcal{R})$. *Then*

$$x < y,\ (x, y) \in \mathcal{R} \Rightarrow x \ll y \text{ with } x \neq y.$$

PROOF. $x \neq y$ since $x \in \hat{P}$ and $y \notin \hat{P}$. $(x, y) \in \mathcal{R}$ implies that $(x, y) \in \hat{e}_{ij}^{(n)}$ for some i, j, n, so $(x, y) \in (Pe_{ij}^{(n)}P^{\perp})^{\wedge}$. Then $x \ll y$ since $Pe_{ij}^{(n)}P^{\perp} \in \operatorname{Alg} \mathcal{M}$. ∎

COROLLARY 4.2. *Suppose* $\operatorname{Alg} \mathcal{M}$ *is induced by a real-valued cocycle* d. *Then*

$$x < y,\ (x, y) \in \mathcal{R} \Rightarrow d(x, y) > 0.$$

PROOF. By the previous proposition, $d(x, y) \geq 0$. If $d(x, y) = 0$, then $(x, y) \in \hat{v}$ for some matrix unit $v \in (\operatorname{Alg} \mathcal{M}) \cap (\operatorname{Alg} \mathcal{M})^*$. It follows that $(x, y) \in (PvP^{\perp})^{\wedge}$ for some $P \in \mathcal{M}$, so $0 \neq PvP^{\perp}$. This implies that $0 \neq P^{\perp}v^*P$, *i.e.*, $v^* \notin \operatorname{Alg} \mathcal{M}$, a contradiction. Therefore, $d(x, y) > 0$. ∎

The converse of Proposition 4.1 is false in general. For example, it is false for the multiplicity 2 nest algebra in Example 2.29 of [PW]. However, if $C^*(\mathcal{M}) = \mathfrak{D}$, the converse is true by Proposition 4.5 below.

PROPOSITION 4.3. *Suppose* $\mathcal{M}$ *is a finite nest:* $0 = M_0 < M_1 < \cdots < M_n = 1$. *Then* $\operatorname{Alg} \mathcal{M}$ *is* $\mathbb{Z}$*-analytic via the coboundary* $d(x, y) = z(y) - z(x)$ *with* $z\colon X \longrightarrow \mathbb{Z}$ *defined by* $z(x) = \inf\{k : x \in \hat{M}_k\}$.

PROOF. This follows easily from Proposition 4.1 and the fact that for each k, $\operatorname{Alg} \mathcal{M}$ contains $(M_k - M_{k-1}) \operatorname{Alg} \mathcal{M}(M_k - M_{k-1})$. ∎

On the other hand,

PROPOSITION 4.4. *If* $\mathcal{M}$ *is an infinite nest, then* $\operatorname{Alg} \mathcal{M}$ *is not* $\mathbb{Z}$*-analytic.*

PROOF. Assume $\operatorname{Alg} \mathcal{M}$ is analytic via an integer-valued cocycle d. Now $\mathcal{M}$ contains an increasing or decreasing sequence, so suppose $\{P_i : 1 \leq i < \infty\}$ is an increasing sequence with $P_1 \neq 0$. Set $P_0 = 0$. Since X is compact, there is some $y \in X \setminus (\bigcup_{i=1}^{\infty} \hat{P}_i)$. By minimality, for each i there is some $x_i \in (P_i - P_{i-1})^{\wedge}$ such that $(x_i, y) \in \mathcal{R}$. Now for each positive integer k, $d(x, y) = \sum_{i=1}^{k-1} d(x_i, x_{i+1}) + d(x_k, y) \geq k$ by Corollary 4.2. Since k is arbitrary, $d(x, y) = \infty$, a contradiction. A similar argument works if $\mathcal{M}$ contains only decreasing sequences. ∎

PROPOSITION 4 5 *Suppose $C^*(\mathcal{M}) = \mathfrak{D}$ Then*

$$x \ll y,\ x \neq y \Rightarrow x < y \text{ and } (x, y) \in \mathcal{R}$$

PROOF Since $C^*(\mathcal{M}) = \mathfrak{D}$, Alg $\mathcal{M}$ is triangular (see [PW, § 2]) If $x \ll y$ and $x \neq y$, then there exist i, j, and n such that $(x, y) \in \hat{e}_{ij}^{(n)}$ with $e_{ij}^{(n)} \in \operatorname{Alg}\mathcal{M}$ and $i \neq j$ $e_i^{(n)} \in C^*(\mathcal{M})$, so [PW, Lemma 3 1] implies that there are projections $P, Q \in \mathcal{M}$ such that $x \in (P - Q)^\wedge \subseteq \hat{e}_i^{(n)}$ (note the assumption in [PW, Lemma 3 1] that $\mathfrak{A}$ is UHF is not necessary, the proof works in the general AF case) Likewise, there are projections $R, S \in \mathcal{M}$ such that $y \in (R-S)^\wedge \subseteq \hat{e}_j^{(n)}$ Since $(P-Q)(R-S) = 0$, either $Q < P \leq S < R$ or $S < R \leq Q < P$ But the latter case implies that $y \in \hat{R}$ and $x \notin \hat{R}$, so $y < x$, and therefore $y \ll x$ by Proposition 4 1, a contradiction Therefore, $x < y$ since $x \in \hat{P}$ and $y \notin \hat{P}$

Alternatively, the result can be proved by applying [PW, Corollary 3 12] Again, the proof of that corollary works in the general AF case ∎

In [PW, Corollary 3 13], it was shown that if $C^*(\mathcal{M}) = \mathfrak{D}$, then Alg $\mathcal{M}$ is strongly maximal triangular We can now obtain the following stronger result

THEOREM 4 6 *Suppose $\mathfrak{A}$ is simple and $\mathcal{M}$ is a nest satisfying $C^*(\mathcal{M}) = \mathfrak{D}$ Let $\{P_i \quad 1 \leq i < \infty\}$ be the set of projections in $\mathcal{M}$ Then* Alg $\mathcal{M}$ *is analytic via the real-valued coboundary $d(x, y) = \tilde{b}(y) - \tilde{b}(x)$, where*

$$\tilde{b}(x) = \sum_{n=1}^{\infty} 2^{-n} \chi_{\widehat{P_n^\perp}}(x)$$

PROOF As in the proof of Theorem 3 16, $\tilde{b} \in C(X)$ By Proposition 4 5, if $x \ll y$ and $x \neq y$, then $\tilde{b}(y) - \tilde{b}(x) > 0$ If $x = y$, then certainly $\tilde{b}(y) - \tilde{b}(x) = 0$ Conversely, if $(x, y) \in \mathcal{R}$ and $\tilde{b}(y) - \tilde{b}(x) > 0$, then $x < y$, so $x \ll y$ with $x \neq y$ by Proposition 4 1 Finally, suppose $(x, y) \in \mathcal{R}$ with $\tilde{b}(y) - \tilde{b}(x) = 0$ Now either $(x, y) \in \mathcal{P}$ or $(y, x) \in \mathcal{P}$ since Alg $\mathcal{M}$ is strongly maximal triangular But then if $x \neq y$, it follows from the above that $\tilde{b}(y) - \tilde{b}(x) > 0$ in the first case and $\tilde{b}(x) - \tilde{b}(y) > 0$ in the second case, a contradiction Therefore, $\tilde{b}(y) - \tilde{b}(x) \geq 0$ with $(x, y) \in \mathcal{R}$ if and only if $(x, y) \in \mathcal{P}$ ∎

COROLLARY 4 7 *If $C^*(\mathcal{M}) = \mathfrak{D}$, then* Alg $\mathcal{M}$ *is analytic via a real valued coboundary $d(x, y) = \tilde{b}(y) - \tilde{b}(x)$ which satisfies*

$$x < y \Leftrightarrow \tilde{b}(y) > \tilde{b}(x)$$

Note that the function $\tilde{b}$ in Theorem 4 6 is the same as the function b in Theorem 3 16 without the $\mu(E_x)$ term If $C^*(\mathcal{M}) = \mathfrak{D}$, there are enough increasing clopen sets so that the $\mu(E_x)$ term is not needed

ADDENDUM In Example 3 18, we showed that the algebra $\mathcal{T}_{(\alpha)}$ is not trivially analytic It has recently been shown in [PWo] that this algebra is not even analytic Therefore, this provides an example of a strongly maximal nest algebra which is not analytic

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