SCINTILLATIONS AND MICROLENSING

D.B. MELROSE
Research Centre for Theoretical Astrophysics
School of Physics, University of Sydney

Abstract. It is shown how the theory of scintillation may be applied to treat gravitational lensing. The theory is applied to microlensing by a system of N point masses. It is shown that scintillation theory reproduces a known result for the angular broadening due to multiple microlensing. Some unresolved differences between scintillation and microlensing theories for intensity fluctuations are pointed out.

Key words: microlensing, scintillations

1. Introduction

Gravitational microlensing at large optical depth presents a time consuming numerical problem because of the large number of stars that need to be taken into account (e.g., Paczyński 1986). An analytic treatment is desirable and Katz et al.(1986) developed a multiple-scattering theory to describe the associated angular broadening. A similar theory was used by Deguchi & Watson (1988) to treat intensity fluctuations due to a gaussian source. An alternative analytic approach is developed and explored here: the well-established theory of scintillations is used to treat gravitational lensing. In applying the theory to microlensing by a system of stars, the questions addressed are whether scintillation theory reproduces known results for the angular broadening (Katz et al.1986), and whether it provides any new insight into the intensity fluctuations due to microlensing.

2. Review of Scintillation Theory

In the theory of scintillations, the spectrum of fluctuations in the refractive index of the radiation passing through the turbulent medium is assumed

C. S. Kochanek and J. N. Hewitt (eds), Astrophysical Applications of Gravitational Lensing, 269–274.

https://doi.org/10.1017/S0074180900231513 Published online by Cambridge University Press

to be given. The theory determines various statistical properties of the emerging radiation, such as the angular broadening of a narrow beam and the spectrum of intensity fluctuations, in terms of the given spectrum of fluctuations in refractive index. Let $\delta\mu(\mathbf{x})$ denote the fluctuating part of the refractive index, and let $\phi(\mathbf{x})$ denote the phase of the wave. Denoting the statistical average by angular brackets, the following two (related) correlation functions are assumed to be given:

$$B_n(\mathbf{x}) = \langle \delta\mu(\mathbf{x})\delta\mu(\mathbf{x}' + \mathbf{x})\rangle, \quad D(\mathbf{x}) = \langle [\phi(\mathbf{x}) - \phi(\mathbf{x}' + \mathbf{x})]^2\rangle, \tag{1}$$

where $D(\mathbf{x})$ is the phase structure function. The power spectrum of the refractive index fluctuations, $\Phi_n(\mathbf{K})$, is the Fourier transform of $B_n(\mathbf{x})$.

The mean ray direction is assumed to be along the z-axis, and the fluctuations are projected onto a screen, where they are described in terms of two-dimensional vectors, introduced by writing $\mathbf{x} = (\mathbf{r}, z)$ and $\mathbf{K} = (\mathbf{q}, K_z)$. For isotropic turbulence one introduces

$$A(r) = \frac{1}{2\pi}\int_0^\infty \mathrm{d}q q\, \Phi_n(q, K_z = 0)\, J_0(qr), \tag{2}$$

with $r = (x^2 + y^2)^{1/2}$, $q = (q_x^2 + q_y^2)^{1/2}$. The angular broadening and the intensity fluctuations are then given by the theory in terms of $A(r)$ or

$$D(r) = (8\pi^2 L/\lambda^2)[A(0) - A(r)], \tag{3}$$

where λ is the wavelength of the radiation and L is the distance from the image plane to the screen. For a power law $\Phi_n(\mathbf{q}) \propto q^{-\beta}$, one has $D(\mathbf{r}) \propto r^{\beta-2}$.

3. Scintillations due to Gravitational Lensing

Application of the foregoing theory to gravitational lensing proceeds as follows.

3.1. REFRACTIVE INDEX FLUCTUATIONS

The refractive index variation due to a weak gravitational field is given by $\delta\mu(\mathbf{x}) = -2\Phi(\mathbf{x})/c^2$, where $\Phi(\mathbf{x})$ is the Newtonian gravitational potential. The potential is related to the mass density, $\eta(\mathbf{x})$, by $\nabla^2\Phi(\mathbf{x}) = -4\pi G\eta(\mathbf{x})$, whose Fourier transform gives $\tilde{\Phi}(\mathbf{q}) = 4\pi G\tilde{\eta}(\mathbf{q})/|\mathbf{q}|^2$. Thus $A(r)$ in (2) is given by

$$A(r) = \frac{(8\pi)^2 G^2}{c^4}\int \frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2}\, \exp(i\mathbf{q}\cdot\mathbf{r})\, \frac{C_\eta(\mathbf{q})}{|\mathbf{q}|^4}, \tag{4}$$

where $C_\eta(\mathbf{q})$ is the Fourier transform of the correlation function for the density fluctuations.

3.2. A SYSTEM OF POINT MASSES

For a system of N point masses, with the ith mass, M_i, at position $\mathbf{x} = \mathbf{x}_i$, one finds

$$A(r) = \frac{(8\pi)^2 G^2}{\pi R^2 L c^4} \int \frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2} \frac{1}{q^4} \left\langle \sum_{i,j=1}^{N} M_i M_j \, \exp[i\mathbf{q}\cdot(\mathbf{r}+\mathbf{r}_i-\mathbf{r}_j)] \right\rangle , \quad (5)$$

where $\pi R^2 L$ is the volume of the system and R is its radius.

3.3. CUTOFF

The integral (5) diverges, and needs to be cut off to obtain a finite result. Here the integral is cut off at $q < q_0$, with $q_0 = 1/R$. With $r_{ij} = |\mathbf{r}+\mathbf{r}_i-\mathbf{r}_j|$ the integral gives (Katz et al.1986)

$$\int_{q_0}^{\infty} \frac{\mathrm{d}q}{q^3} J_0(qr_{ij}) = \frac{1}{2q_0^2}\left[1 - \frac{q_0^2 {r_{ij}}^2}{2} \ln\left(\frac{2e^{1-\gamma}}{q_0 r_{ij}}\right) + \cdots\right] . \quad (6)$$

Then (5) gives

$$A(r) = \frac{8G^2}{Lc^4}\left[\left(\sum_i M_i\right)^2 - \sum_{i,j} \frac{M_i M_j r_{ij}}{R^2} \ln\left(\frac{3.05\,R}{r_{ij}}\right)\right] . \quad (7)$$

The first term in the square brackets arises from the $i = j$ term in the sum in (5); this term does not contribute to the scintillations. The remaining term is associated with the $N(N-1)/2$ pairwise combinations of point masses. Thus the scintillations may be attributed to the net effect of lensing by $\sim N^2$ two-point-mass systems.

3.4. STATISTICAL AVERAGING

The average of (5) or (7) over a collection of N identical stars, each of mass M, may be performed by retaining one term, say the ij term, writing $\Delta\mathbf{r} = \mathbf{r}_i - \mathbf{r}_j$, and then averaging over $\Delta\mathbf{r}$. A conventional phase-averaging procedure in scintillation theory is over a random phase ϕ that satisfies $\left\langle e^{i\phi}\right\rangle = \exp(-\frac{1}{2}\langle\phi^2\rangle)$. The average over the ij term in (5), with $\Delta\mathbf{r} = \mathbf{r}_i - \mathbf{r}_j$, is then achieved by identifying the phase as $\mathbf{q}\cdot\Delta\mathbf{r}$. One has $\langle(\mathbf{q}\cdot\Delta\mathbf{r})^2\rangle = \frac{1}{2}q^2\langle(\Delta\mathbf{r})^2\rangle$. Thus (5) is replaced by $2N$ times the statistical average of of the ij term, giving

$$A(r) = \frac{64G^2M^2N}{R^2Lc^4}\int_{R^{-1}}^{\infty} \frac{\mathrm{d}q}{q^3} J_0(qr) e^{-q^2R_0^2} \approx \frac{64G^2M^2N}{R^2Lc^4}\int_{R^{-1}}^{R_0^{-1}} \frac{\mathrm{d}q}{q^3} J_0(qr), \quad (8)$$

with $R_0^2 = \frac{1}{4}\langle(\Delta\mathbf{r})^2\rangle \sim R^2/N$ of order the mean square separation between the stars. Then using (6), (8) gives

$$A(r) \simeq -\frac{16G^2M^2}{R_0^2Lc^4}\, r^2 \ln(3.05\, N^{1/2}), \tag{9}$$

where the numerical factor in the argument of the logarithm is chosen to facilitate comparison with Katz et al.(1986).

4. Angular Broadening

In scintillation theory angular broadening is described by the mean square fluctuations in the angular deviation of a ray, $\langle(\delta\theta)^2\rangle = -L\nabla^2 A(r)$, which with (9) for $A(r)$ gives

$$\langle(\delta\theta)^2\rangle(\mathbf{r}) = 2\theta_\mathrm{R}^2 \ln(3.05\, N^{1/2}), \quad \theta_\mathrm{R} = \frac{4GM}{R_0c^2} = \frac{4GM}{Rc^2}\, N^{1/2}, \tag{10}$$

where θ_R corresponds to a ray passing a mass M with an impact parameter R_0. The result (10) was derived by Katz et al.(1986) using a model for multiple scattering by point masses. This confirms that scintillation theory reproduces a known result in microlensing. Note that (9) includes only the effect of impact parameters between R_0 and R. Scintillation theory does not apply for impact parameters $\ll R_0$, which corresponds to large-angle scattering by single stars.

5. Fluctuations in intensity

The intensity fluctuations are described by (e.g., Prokhorov et al.1975)

$$B_I(\mathbf{r}) = \frac{\langle I(\mathbf{r}')I(\mathbf{r}'+\mathbf{r})\rangle - \langle I(\mathbf{r})\rangle^2}{\langle I(\mathbf{r})\rangle^2} = \int \frac{\mathrm{d}^2\mathbf{q}}{(2\pi)^2}\, e^{-i\mathbf{q}\cdot\mathbf{r}}\, W(\mathbf{q}), \tag{11}$$

$$W(\mathbf{q}) = \int \mathrm{d}^2\mathbf{r}\, e^{i\mathbf{q}\cdot\mathbf{r}} \left\{ e^{[-D(r)-D(r_\mathrm{F}^2 q)+\frac{1}{2}D(|\mathbf{r}+r_\mathrm{F}^2\mathbf{q}|)+\frac{1}{2}D(|\mathbf{r}-r_\mathrm{F}^2\mathbf{q}|)]} - 1 \right\}, \tag{12}$$

where $W(\mathbf{q})$ is the power spectrum of the intensity fluctuations and $r_\mathrm{F} = (L/k)^{1/2}$ is the Fresnel scale. The phase structure function (3) corresponding to (7) or (8) contains terms $\propto r^2$ and $\propto r^2 \ln r$. The former cancels and only the latter contributes to (12). This corresponds to scintillations due to a power-law spectrum with $\beta = 4$ (Goodman & Narayan 1985). This case is not amenable to a simple analytic treatment, but several relevant results can be inferred from the existing literature.

https://doi.org/10.1017/S0074180900231513 Published online by Cambridge University Press

5.1. SMOOTH POWER SPECTRUM

For $\beta \neq 4$ a natural scale, $r = r_{\rm diff}$, is defined by $D(r_{\rm diff}) = 1$, and then (12) implies peaks in the the power spectrum $W(\mathbf{q})$ at the refractive, $q_{\rm ref} \sim r_{\rm diff}/r_{\rm F}^2$, and diffractive, $q_{\rm diff} \sim 1/r_{\rm diff}$, scales. As $\beta = 4$ is approached the refractive and diffractive peaks recede to $q = 0$ and $q = \infty$, leaving a smooth spectrum of intensity fluctuations with no natural scale. Only the end points, $R^{-1} < q \lesssim R_0^{-1}$, can lead to significant features in the power spectrum.

5.2. THE MARGINAL DIFFRACTAL OF BERRY (1979)

The level of the intensity fluctuations for a point source has been estimated by a careful consideration of how the limit $\beta \to 4$ is approached. Berry (1979), who referred to the case $\beta = 4$ as the "marginal diffractal", showed that the asymptotic form (for $L \to \infty$) of the intensity fluctuations gives (cf. also Jakeman & Jefferson 1984)

$$I_2 = B_I(0) + 1 = 2, \tag{13}$$

where I_2 is the second moment of the intensity, with mean intensity $I_1 = 1$.

5.3. INTERPRETATION OF THE MARGINAL CASE

To understand the significance of (13) to gravitational microlensing, one needs to interpret it in terms of a physical model for microlensing. The following remarks describe an unsuccessful attempt to do this.

The intensity fluctuations may be attributed to caustics, and described in terms of the probability distribution $p(A)$ of a magnification A (e.g., Vietri & Ostriker 1983; Rauch et al.1992). The mean intensity (for a source of unity intensity, $I_0 = 1$) is $\langle I \rangle = \langle A \rangle = 1/(1-\tau)^2$ for $\tau = \pi n r_{\rm E}^2 \ll 1$, where n is the number density of stars and $r_{\rm E} = (4GML/c^2)^{1/2}$ is the Einstein radius. A weakness (in the present context) of scintillation theory is that no distinction is made between $\langle I \rangle$ and I_0, and the theory needs to be modified to take account of $\langle A \rangle \neq 0$. The mean square intensity is dominated by individual microlensing events with large amplifications. The probability distribution $p(A) \propto 1/A^3$ for large A applies to both a point-mass lens and (approximately) to the two-point-mass lens systems (Schneider & Weiß 1986) relevant to (5). The integral $\langle I^2 \rangle = \int dA\, A^2 p(A)$ needs to be cut off at some $A_{\rm max}$, and then $\langle I^2 \rangle$ depends logarithmically on the cutoff value $A_{\rm max}$. It is not obvious how this model can reproduce (13), that is, $\langle I^2 \rangle = 2$. Further thought needs to be given to the interpretation of (13) in the context of gravitational microlensing.

5.4. INTENSITY FLUCTUATIONS FOR A GAUSSIAN SOURCE

A model for the intensity fluctuations in a gaussian source was presented by Deguchi & Watson (1988). Their averaging procedure produces an equation, their (16), of the form (12), but with only the $D(r)$-term in the exponent (the terms involving r_F do not appear). Deguchi & Watson set $D(\mathbf{r}) \propto r^2$, as in (9), but $D(r) \propto r^2$ cancels in (12) (e.g., Goodman & Narayan 1985), and only the term $D(r) \propto r^2 \ln r$ from (7) contributes to the intensity fluctuations (Berry 1979). Thus there is an inconsistency between the statistical averaging procedure adopted by Deguchi & Watson (1988), in which the Fresnel scale does not appear, and that used in scintillation theory, which depends explicitly on r_F.

6. Conclusions

1) Scintillation theory may be used to treat gravitational lensing by identifying the refractive index fluctuations in terms of the gravitational potential.
2) Application to multiple microlensing by a system of point masses reproduces a known result (Katz et al.1986) for the angular broadening.
3) Scintillation theory suggests a smooth power spectrum of intensity fluctuations (the refractive and diffractive scales disappear).
4) There are unresolved difficulties in the treatment of intensity fluctuations. (a) Scintillation theory implies an asymptotic variance of unity (Berry 1979), but a simple model for caustic-induced magnifications does not reproduce this result. (b) The averaging procedure of Deguchi & Watson (1988) is not compatible with the result (12) of scintillation theory.

In summary, scintillation theory can be used to treat statistical microlensing. However, there are some specific difficulties and inconsistencies related to intensity fluctuations. The resolution of these difficulties is likely to provide deeper insight into statistical microlensing theory.

References

Berry, M.V., 1979, J. Phys. A: Math. Gen., 12, 781
Deguchi, S., & Watson, W.D., 1988, ApJ, 335, 67
Goodman, J., & Narayan, R., 1985, MNRAS, 214, 519
Jakeman, E., & Jefferson, J.H., 1984, Optica Acta, 31, 853
Katz, N., Balbus, S., & Paczyński, B., 1986, ApJ, 306, 2
Paczyński, B., 1986, ApJ, 301, 503
Prokhorov, A.M., et al., 1975, Proc. IEEE, 63, 790
Rauch, K.P., Mao, S., Wambsgauss, J., & Paczyński, B., 1992, ApJ, 386, 30
Schneider, P., & Weiß, A., 1986, A&A, 164, 237
Vietri, M., & Ostriker, J.P., 1983, ApJ, 267, 488