CORRESPONDENCE.

ON THE CONVERSION OF ORDINARY ASSURANCES FOR THE WHOLE TERM OF LIFE INTO ENDOWMENT ONES BY APPLICATION OF BONUS.

To the Editor.

SIR,—As some Offices are in the practice of converting policies, originally effected for the whole of life, into endowment assurances payable on the life assured attaining a certain age or death, by application of the bonus, you may possibly deem the following not unworthy of insertion in the *Journal.*

Let us assume that a person, aged x, effects a policy of £1 for the whole of life, and that at the end of every y years a reversionary bonus, B, is declared, the cash values of which are respectively ${}^1b$, ${}^2b$, ${}^3b$, &c., where ${}^1b$, ${}^2b$, &c., represent the values of the bonus at first, second, &c., investigations. It is required to ascertain at what age the policy would become payable, by applying the values ${}^1b$, ${}^2b$, &c., to convert it into an endowment assurance.

A little consideration will at once show, that, at the end of the first y years, and assuming that the policy can be converted into an endowment assurance payable at the age $(x+y+n)$, or at the end of n years from the present, n must be such that the whole-life premium he at present pays, together with the temporary annuity payable *in advance* for n years, which the surrender value of the whole-life policy at end of y years and the value of the bonus will purchase at the age $(x+y)$, will equal the premium which a person aged $(x+y)$ would require to pay for an endowment assurance payable at end of n years or death.

Now $\frac{1}{1+a_x}-d=$ premium for a whole-life policy of £1 payable at death of x.

$1-\frac{1+a_{x+y}}{1+a_x}=$ surrender value of such a policy at end of y years.

$1+a_{x+y\overline{n-1}|}=$ value of temporary annuity of £1, payable in advance, for n years to $(x+y)$.

$\frac{1-\frac{1+a_{x+y}}{1+a_x}+{}^1b}{1+a_{x+y\overline{n-1}|}}=$ value of temporary annuity, payable in advance, for n years, which the surrender value and value of bonus will purchase.

$\frac{1}{1+a_{x+y\overline{n-1}|}}-d=$ premium for an endowment assurance of £1, for n years, to $(x+y)$.

We shall therefore have

$$\frac{1}{1+a_x}-d+\frac{1-\frac{1+a_{x+y}}{1+a_x}+{}^1b}{1+a_{x+y\overline{n-1}|}}=\frac{1}{1+a_{x+y\overline{n-1}|}}-d.$$

$$\therefore\ 1+a_{x+y_{\overline{n-1}|}}+1+a_x-(1+a_{x+y})+{}^1b(1+a_x)=1+a_x,$$

$$\text{and }\ 1+a_{x+y_{\overline{n-1}|}}=1+a_{x+y}-{}^1b(1+a_x).$$

Since, from the tables published, we can readily ascertain the value of $1+a_{x+y}-{}^1b(1+a_x)$, we will get the value of $1+a_{x+y_{\overline{n-1}|}}$; and, by referring to tables of temporary annuities, we will get the age corresponding to this value, and, in consequence, the age at which the policy would become payable. When we have no tables of temporary annuities, we may ascertain the age by the commutation tables thus: by giving to the above formula the D and N values, according to the notation employed by Mr. Chisholm and other writers, we get

$$\frac{N_{x+y}-N_{x+y+n}}{D_{x+y}}=\frac{N_{x+y}}{D_{x+y}}-{}^1b\times\frac{N_x}{D_x},$$

from which we obtain the value of

$$N_{x+y+n}=D_{x+y}\times{}^1b\times\frac{N_x}{D_x};$$

and, as the values of the right-hand side of this equation are known, we get N_{x+y+n}; and, by running the eye down the N column, the nearest value to this will represent *nearly* the age at which the policy would become payable by application of the bonus for the first y years.

The value of N_{x+y+n} may be ascertained in a different manner, thus, B being the *reversionary* bonus:—

$$\text{Since}\qquad {}^1b=B\left\{1-\frac{dN_{x+y}}{D_{x+y}}\right\}=B\left(\frac{D_{x+y}-dN_{x+y}}{D_{x+y}}\right),$$

substituting this value of ${}^1b$ in the equation

$$N_{x+y+n}=D_{x+y}\times{}^1b\times\frac{N_x}{D_x},$$

we get

$$N_{x+y+n}=D_{x+y}\times B\times\left(\frac{D_{x+y}-dN_{x+y}}{D_{x+y}}\right)\times\frac{N_x}{D_x}=B\times(1+a_x)(D_{x+y}-dN_{x+y}).$$

Now, $D_{x+y}-dN_{x+y}=M_{x+y}$ (see *Assurance Magazine*, vol. vi. p. 347),

$$\therefore\ N_{x+y+n}=B(1+a_x)(M_{x+y}).$$

This formula will be found more convenient, for practical purposes, than that of Mr. Sprague in the *Magazine* referred to, when the values allowed for surrender of bonus and the premiums are calculated at the same rate of interest.

At the end of another y years, an investigation takes place, at which a cash bonus of ${}^2b$ is declared; it is required to find the age at which the policy would *now* become payable by application thereof.

Let us assume that the policy will become payable at the age of $(x+2y+m)$, or at the end of m years from the present. It will be evident that m must be such that the whole-life premium he at present pays, together with the temporary annuity payable in advance for m years to

$(x+2y)$, which can be purchased by the value of the bonus, the surrender value of the endowment assurance, in the previous case, at end of y years, and the value of the unexpired portion of the temporary annuity (the difference between the endowment assurance and whole-life premiums) in the previous case, will equal the premium for an endowment assurance to $(x+2y)$ for m years. This, when converted into an algebraic form, becomes

$$\frac{1}{1+a_x}-d+\frac{1-\dfrac{1+a_{x+2y\overline{n-y-1}|}}{1+a_{x+y\overline{n-1}|}}+{}^2b+(1+a_{x+2y\overline{n-y-1}|})\left(\dfrac{1}{1+a_{x+y\overline{n-1}|}}-d-\dfrac{1}{1+a_x}+d\right)}{1+a_{x+2y\overline{m-1}|}}$$

$$=\frac{1}{1+a_{x+2y\overline{m-1}|}}-d.$$

$$\therefore\ 1+a_{x+2y\overline{m-1}|}+1+a_x-\frac{(1+a_x)(1+a_{x+2y\overline{n-y-1}|})}{1+a_{x+y\overline{n-1}|}}+{}^2b(1+a_x)$$

$$+\frac{(1+a_x)(1+a_{x+2y\overline{n-y-1}|})}{1+a_{x+y\overline{n-1}|}}-1+a_{x+2y\overline{n-y-1}|}=1+a_x,$$

$$\text{and }\ 1+a_{x+2y\overline{m-1}|}=1+a_{x+2y\overline{n-y-1}|}-{}^2b(1+a_x).$$

Giving these values, as before, their commutation equivalents, we get

$$\frac{N_{x+2y}-N_{x+2y+m}}{D_{x+2y}}=\frac{N_{x+2y}-N_{x+y+n}}{D_{x+2y}}-{}^2b\frac{N_x}{D_x};$$

$$\therefore\ N_{x+2y+m}=N_{x+y+n}+D_{x+2y}\times{}^2b\times\frac{N_x}{D_x}.$$

It has already been shown that $N_{x+y+n}=B(1+a_x)M_{x+y}$.

Now,
$${}^2b=B\left(1-\frac{dN_{x+2y}}{D_{x+2y}}\right),$$

$$\therefore\ D_{x+2y}\times{}^2b\times(1+a_x)=(D_{x+2y}-dN_{x+2y})B\times(1+a_x)=B(1+a_x)M_{x+2y};$$

$$\therefore\ N_{x+2y+m}=B(1+a_x)M_{x+y}+B(1+a_x)M_{x+2y}=B(1+a_x)(M_{x+y}+M_{x+2y}).$$

The age corresponding to the value of N_{x+2y+m} will be the age at which the policy would now become payable.

In a precisely similar manner we may ascertain the age at which another y years' bonus will make policy payable. Assume that it will be at the end of q years from the present, we shall therefore get

$$N_{x+3y+q}=B(1+a_x)(M_{x+y}+M_{x+2y}+M_{x+3y});$$

and so on, until the difference between the ages corresponding to the values of N_{x+ty+s} and M_{x+ty} is less than y, t representing the number of investigations that have taken place.

If the bonus is declared annually, instead of at the end of every y years,

$$N_{x+t+s}=B(1+a_x)\{R_{x+1}-R_{x+t+1}\}.$$

It will be evident from the foregoing, that when we know the age at which the whole-life policy would become payable by application of the bonus, and the age at entry, we can ascertain the rate of reversionary bonus necessary for this purpose.

Since $N_{x+ty+s}=B(1+a_x)\{M_{x+y}+M_{x+2y}+M_{x+3y}+\ldots M_{x+ty}\}$, dividing both sides of the equation by $(1+a_x)(M_{x+y}+M_{x+2y}+\ldots M_{x+2y}$, we get

$$B=\frac{N_{x+ty+s}}{(1+a_x)(M_{x+y}+M_{x+2y}+M_{x+3y}+\ldots M_{x+ty})}.$$

As B represents the reversionary bonus for y years, $\frac{B}{y}$ will represent the rate for each year.

The above formulæ will only apply when the reversionary bonus declared at each investigation is the same, when it is different we get

$$N_{x+ty+s}=(1+a_x)\{{}^1BM_{x+y}+{}^2BM_{x+2y}+{}^3BM_{x+3y}+\ldots{}^tBM_{x+ty}\},$$

where ${}^1B$, ${}^2B$, &c., represent the reversionary bonuses declared at first, second, &c., investigations. This formula will also enable us to find the age at which the policy would become payable when the premiums and cash values of bonus are calculated at *different* rates of interest, as we can readily find what reversionary bonus can be given, at the same rate of interest as that on which the premiums are based, for the cash values of the bonus.

In this latter case, however, possibly the formula, a modification of that of Mr. Sprague,

$$N_{y+ty+s}=(1+a_x)\{{}^1b.D_{y+y}+{}^2b.D_{x+2y}+\ldots{}^tb.D_{x+ty}\}$$

would be better, ${}^1b$, ${}^2b$, &c., being the cash values.

As an example of the preceding, let it be required to find the age at which a whole-life policy for £1 on the life of a person aged 20, would be converted into an endowment assurance, by application of the reversionary bonus, declared quinquennially, at the rate of £2 per cent. per annum, Carlisle 4 per cent.

At the end of the first five years we shall have

$$N_{25+n}=B(1+a_{20})(M_{25});$$

$$B=\cdot 1,\quad 1+a_{20}=19\cdot 3617,\quad \text{and } M_{25}=623\cdot 86064.$$

$$\therefore\quad N_{25+n}=\cdot 1\times 19\cdot 3617\times 623\cdot 86064=1207\cdot 9.$$

On referring to the N column, we find that 69·9 is the nearest age corresponding to this value, which is the age at which the whole-life policy would become payable by application of the first five years' bonus.

At the second investigation we shall have

$$N_{30+m}=B(1+a_{20})\{M_{25}+M_{30}\}=1207\cdot 9+1\cdot 93617\times 545\cdot 1327$$
$$=2263\cdot 3705,$$

from which we find that policy would now become payable at (say) 64·8.

At the third investigation we shall have

$N_{35+q} = B(1+a_{20})(M_{25}+M_{30}+M_{35}) = 2263{\cdot}3705 + 1{\cdot}93617 \times 468{\cdot}2037$
$= 3169{\cdot}8931,$

from which we find that policy would now become payable at (say) 61·6.

At the fourth investigation we shall have

$$N_{40+e} = B(1+a_{20})(M_{25}+M_{30}+M_{35}+M_{40})$$
$$= 3169{\cdot}893 + 1{\cdot}93617 \times 403{\cdot}57 = 3951{\cdot}2746,$$

from which we find that policy would now become payable at (say) 59·3.

At the fifth investigation we shall have

$$N_{45+e} = B(1+a_{20})(M_{25}+M_{30}+M_{35}+M_{40}+M_{45})$$
$$= 3951{\cdot}27 + 1{\cdot}93617 \times 339{\cdot}12 = 4607{\cdot}8712,$$

from which we find that policy would now become payable at (say) 57·7.

At the sixth investigation we shall have

$$N_{50+} = B(1+a_{20})(M_{25}+M_{30}+M_{35}+M_{40}+M_{45}+M_{50})$$
$$= 4607{\cdot}871 + 1{\cdot}93617 \times 288{\cdot}67 = 5166{\cdot}7987,$$

from which we find that policy would now become payable at (say) 56·3.

At the seventh and last investigation, as the difference between the ages corresponding to values N_{55+s} and M_{55} is now less than 5, we get

$$N_{55+s} = B(1+a_{20})(M_{25}+M_{30}+M_{35}+M_{40}+M_{45}+M_{50}+M_{55})$$
$$= 5166{\cdot}7987 + 1{\cdot}93617 \times 248{\cdot}22 = 5647{\cdot}3987,$$

from which we find that the whole-life policy would, on the assumptions stated, become payable at *nearly* age 55·3.

This result may obviously be verified by ascertaining what endowments, payable at this age, could be purchased by the cash values of the bonus at each investigation, and adding thereto the surrender value of the whole-life policy on the life of 20 at end of 35·3 years.

This rule is derived from the formula

$$N_{x+ty+s} = (1+a_x)\{{}^1bD_{x+y} + {}^2bD_{x+2y} + \ldots {}^tbD_{x+ty}\}.$$

Dividing both sides of this equation by D_{x+ty+s} and $1+a_x$, changing the signs and adding 1 to each, we get

$$1 - \frac{D_x}{N_x} \cdot \frac{N_{x+ty+s}}{D_{x+ty+s}} = 1 - \left\{\frac{{}^1bD_{x+y} + {}^2bD_{x+2y} + \ldots {}^tbD_{x+ty}}{D_{x+ty+s}}\right\},$$

$$\therefore\ 1 - \frac{D_x}{N_x} \cdot \frac{N_{x+ty+s}}{D_{x+ty+s}} + \frac{{}^1bD_{x+y} + {}^2bD_{x+2y} + \ldots {}^tbD_{x+ty}}{D_{x+ty+s}} = 1 \text{ (the sum assured).}$$

Now, $1 - \frac{D_x}{N_x} \cdot \frac{N_{x+ty+s}}{D_{x+ty+s}}$ = surrender value of policy on life of x at the age $(x+ty+s)$, and $\frac{{}^1bD_{x+y} + {}^2bD_{x+2y} + \ldots {}^tbD_{x+ty}}{D_{x+ty+s}}$ = endowments payable at the age $(x+ty+s)$, which the values of the bonus at each investigation will purchase.

I remain, your obedient servant,

Glasgow, October 24th, 1866. T. M.

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