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# A PROPERTY OF ANALYTIC FUNCTIONS WITH HADAMARD GAPS

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In this paper we obtain a sufficient and necessary condition for an analytic function $f$ on $D$ with Hadamard gaps, that is, for $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$ satisfying $n_{k+1}/n_k \geqslant \lambda > 1$ for all $k$, to belong to a kind of space consisting of analytic functions on $D$. The special cases of these spaces are $BMOA$ and $VMOA$. In view of our result we can answer the open question given recently by Stroethoff.

## 1. INTRODUCTION

Let $D = \{z\colon |z| < 1\}$ be the open disc in the complex plane. For an analytic function $f$ on $D$ we set

$$\|f\|_{BMOA} = \sup_{\lambda\in D} \left( \frac{1}{2\pi} \int_0^{2\pi} \left| f(\varphi_\lambda(e^{i\theta})) - f(\lambda) \right|^2 d\theta \right)^{1/2},$$

where

$$\varphi_\lambda(z) = \frac{z-\lambda}{1-\overline{\lambda}z}, \qquad z \in D.$$

The space $BMOA$ is the set of all analytic functions $f$ on $D$ for which $\|f\|_{BMOA} < \infty$. Contained in $BMOA$ is the subspace $VMOA$, the set of all analytic functions $f$ on $D$ for which

$$\lim_{|\lambda|\to 1-0} \left( \frac{1}{2\pi} \int_0^{2\pi} \left| f(\varphi_\lambda(e^{i\theta})) - f(\lambda) \right|^2 d\theta \right) = 0.$$

It is well-known that for every analytic function $f$ on $D$ (see [2]),

$$\text{(1)} \qquad \|f\|_{BMOA} \approx \sup_{\lambda\in D} \left( \int_D |f'(z)|^2 \left(1 - |\varphi_\lambda(z)|^2\right) dA(z) \right)^{\frac{1}{2}}$$

and $f \in VMOA$ if and only if

$$\text{(2)} \qquad \lim_{|\lambda|\to 1-0} \int_D |f'(z)|^2 \left(1 - |\varphi_\lambda(z)|^2\right) dA(z) = 0,$$

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where $A$ denotes the Lebesgue area measure and "$\approx$" means equivalently (see [4]). The Bloch space $\mathcal{B}$ is the set of all analytic functions $f$ on $D$ for which $\|f\|_{\mathcal{B}} = \sup_{z \in D} \left(1 - |z|^2\right) |f'(z)| < \infty$. Contained in $\mathcal{B}$ is the little Bloch space $\mathcal{B}_0$, the set of all analytic functions $f$ on $D$ for which $\lim_{|z| \to 1-0} \left(1 - |z|^2\right) |f'(z)| = 0$. We know that for $0 < p < \infty$ (see [4]),

$$\|f\|_{\mathcal{B}} \approx \sup_{\lambda \in D} \left( \int_D |f'(z)|^p \left(1 - |z|^2\right)^{p-2} \left(1 - |\varphi_\lambda(z)|^2\right)^2 dA(z) \right)^{1/p} \tag{3}$$

and $f \in \mathcal{B}_0$ if and only if

$$\lim_{|\lambda| \to 1-0} \int_D |f'(z)|^p \left(1 - |z|^2\right)^{p-2} \left(1 - |\varphi_\lambda(z)|^2\right)^2 dA(z) = 0. \tag{4}$$

It has been known to us that $\mathcal{B}$ and $BMOA$ share many analogous properties, as do $\mathcal{B}_0$ and $VMOA$. Comparing the above equivalence (1) with (3), as well as (2) with (4) when $p = 2$, Stroethoff [4] offered the following open question:

Question: Let $0 < p < \infty$ and let $f$ be an analytic function on $D$. Are the following true?

(i) $$f \in BMOA \iff \sup_{\lambda \in D} \int_D |f'(z)|^p \left(1 - |z|^2\right)^{p-2} \left(1 - |\varphi_\lambda(z)|^2\right) dA(z) < \infty,$$

(ii) $$f \in VMOA \iff \lim_{\lambda \to 1-0} \int_D |f'(z)|^p \left(1 - |z|^2\right)^{p-2} \left(1 - |\varphi_\lambda(z)|^2\right) dA(z) = 0.$$

For $p = 2$ the above question has a positive answer. Making use of the fact that there is a constant $C$ such that

$$\|f\|_{\mathcal{B}} \leqslant C \|f\|_{BMOA}$$

for every analytic $f$ on $D$, we know a partial answer to the question: for an analytic function $f$ on $D$ and $0 < p \leqslant 2$ the conditions in (i) and (ii) are sufficient for containment in $BMOA$ and $VMOA$, respectively; for $2 \leqslant p < \infty$ the conditions in (i) and (ii) are necessary for $f$ to belong to $BMOA$ and $VMOA$, respectively.

Let $0 < p < \infty$. For an analytic function $f$ on $D$ we set

$$\|f\|_{B^p} = \sup_{\lambda \in D} \left( \int_D |f'(z)|^p \left(1 - |z|^2\right)^{p-2} \left(1 - |\varphi_\lambda(z)|^2\right) dA(z) \right)^{1/p}. \tag{5}$$

We define the space $B^p$ to be the set of all analytic functions $f$ on $D$ for which $\|f\|_{B^p} < \infty$ and define $B_0^p$ to be the subspace of $B^p$, the set of all analytic functions $f$ on $D$ for which

(6) $$\lim_{|\lambda|\to 1-0} \int_D |f'(z)|^p \left(1-|z|^2\right)^{p-2} \left(1-|\varphi_\lambda(z)|^2\right) dA(z) = 0.$$

It is clear that for $0 < p < \infty$, $B^p \subset \mathcal{B}$ and $B_0^p \subset \mathcal{B}_0$, especially $B^2 = BMOA$, $B_0^2 = VMOA$. The known partial answer now can be expressed as: for $0 < p < 2$,

(7) $$B^p \subset BMOA, \qquad B_0^p \subset VMOA;$$

for $2 < p < \infty$,

(8) $$BMOA \subset B^q, \qquad VMOA \subset B_0^q.$$

According to our definition, Stroethoff's question becomes: are the above inclusions strict?

In this paper we give a sufficient and necessary condition for an analytic function with Hadamard gaps to belong to $B^p$ or $B_0^p$. In view of the result it is easy to conclude that the above inclusions (7) and (8) are strict. Hence we get a negative answer to the question in general.

## 2. Main result

Our main result is the following theorem.

**Theorem 1.** *Let $0 < p < \infty$. If $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$ is analytic on $D$ and has Hadamard gaps, that is, if*

$$\frac{n_{k+1}}{n_k} \geqslant \lambda > 1, \qquad (k = 1, 2, \ldots),$$

*then the following statements are equivalent:*

$$\text{(I)}\quad f \in B^p; \qquad \text{(II)}\quad f \in B_0^p; \qquad \text{(III)}\quad \sum_{k=1}^{\infty} |a_k|^p < \infty.$$

By Theorem 1 we can give the answer to the question in the introduction. Let $0 < p < 2$. Then $f(z) = \sum_{n=1}^{\infty} \left(z^{2^n}\right)/\left(n^{1/p}\right) \in VMOA$, but $f \notin B^p$. Let $2 < q < \infty$. Then $g(z) = \sum_{n=1}^{\infty} \left(z^{2^n}\right)/\left(n^{1/2}\right) \in B_0^q$, but $g \notin BMOA$. Hence the inclusions (7) and (8) are strict. Furthermore we know that the following inclusions, for $0 < p < q < \infty$,

$$B^p \subset B^q; \qquad B_0^p \subset B_0^q,$$

are strict.

In order to prove Theorem 1, we need the following two lemmas.

**Lemma 1.** *Let $0 < p < \infty$. If $\{n_k\}$ is an increasing sequence of positive integers satisfying $n_{k+1}/n_k \geqslant \lambda > 1$ for all $k$, then there is a constant $A$ depending only on $p$ and $\lambda$ such that*

$$A^{-1}\left(\sum_{k=1}^{\infty}|a_k|^2\right)^{1/2} \leqslant \left(\frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{k=1}^{\infty}a_k e^{in_k\theta}\right|^p d\theta\right)^{1/p} \leqslant A\left(\sum_{k=1}^{\infty}|a_k|^2\right)^{1/2}$$

*for any number $a_k$ $(k = 1, 2, \ldots)$.*

The above lemma was due to Zygmund [5].

**Lemma 2.** *Let $\alpha > 0$, $p > 0$, $n \geqslant 0$, $a_n \geqslant 0$, $I_n = \{k\colon 2^n \leqslant k < 2^{n+1}, k \in N\}$, $t_n = \sum_{k\in I_n} a_k$ and $f(x) = \sum_{n=1}^{\infty} a_n x^n$. Then there is a constant $K$ depending only on $p$ and $\alpha$ such that*

$$\frac{1}{K}\sum_{n=0}^{\infty}2^{-n\alpha}t_n^p \leqslant \int_0^1 (1-x)^{\alpha-1} f(x)^p dx \leqslant K\sum_{n=0}^{\infty}2^{-n\alpha}t_n^p.$$

The proof of Lemma 2 can be found in [3]. By simple computation we see that the above lemma is still valid for $f(x) = \sum_{n=1}^{\infty} a_n x^{n-1}$, $a_n \geqslant 0$. Let $K$ still denote the constant in Lemma 2 for $f(x) = \sum_{n=1}^{\infty} a_n x^{n-1}$.

For our purpose we will use the following inequalities, which follow immediately from Hölder's inequality. Let $a_n \geqslant 0$ and let $N$ be a positive integer. Then for $0 < p \leqslant 1$,

$$\frac{1}{N^{1-p}}\left(\sum_{n=1}^{N}a_n^p\right) \leqslant \left(\sum_{n=1}^{N}a_n\right)^p \leqslant \left(\sum_{n=1}^{N}a_n^p\right); \tag{9}$$

for $1 \leqslant p < \infty$,

$$\left(\sum_{n=1}^{N}a_n^p\right) \leqslant \left(\sum_{n=1}^{N}a_n\right)^p \leqslant N^{p-1}\left(\sum_{n=1}^{N}a_n^p\right). \tag{10}$$

Before proving Theorem 1 we first prove the following result, which is useful for the proof of Theorem 1 and is of independent interest. We state it as a theorem.

**Theorem 2.** *Let $0 < p < \infty$, $I_n = \{k\colon 2^n \leqslant k < 2^{n+1}, k \in N\}$ and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic on $D$. If*

$$\sum_{n=0}^{\infty}\left(\sum_{k\in I_n}|a_k|\right)^p < \infty,$$

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*then* $f \in B_0^p$.

PROOF: By the following identity:

$$1-|\varphi_\lambda(z)|^2 = \frac{\left(1-|\lambda|^2\right)\left(1-|z|^2\right)}{\left|1-\overline{\lambda}z\right|^2}, \qquad (\lambda,\ z \in D),$$

we have

$$\begin{aligned}
&\int_D |f'(z)|^p\left(1-|z|^2\right)^{p-2}\left(1-|\varphi_\lambda(z)|^2\right)dA(z)\\
&\leqslant \int_D\left(\sum_{n=1}^{\infty} n\,|a_n|\,|z|^{n-1}\right)^p \frac{\left(1-|z|^2\right)^{p-1}\left(1-|\lambda|^2\right)}{\left|1-\overline{\lambda}z\right|^2}dA(z)\\
&= \int_0^1\left(\sum_{n=1}^{\infty} n\,|a_n|\,r^{n-1}\right)^p(1-r^2)^{p-1}\left(1-|\lambda|^2\right)\left(\int_0^{2\pi}\frac{d\theta}{\left|1-\overline{\lambda}re^{i\theta}\right|^2}\right)r\,dr\\
&= 2\pi\int_0^1\left(\sum_{n=1}^{\infty} n\,|a_n|\,r^{n-1}\right)^p(1-r^2)^{p-1}\frac{1-|\lambda|^2}{1-|\lambda|^2r^2}r\,dr\\
&\leqslant 2^p\pi\int_0^1\left(\sum_{n=1}^{\infty} n\,|a_n|\,r^{n-1}\right)^p(1-r)^{p-1}dr\\
&\leqslant 2^p\pi K\sum_{n=0}^{\infty}2^{-np}t_n^p,
\end{aligned}$$

because of Lemma 2, where

$$t_n = \sum_{k\in I_n} k\,|a_k| < 2^{n+1}\sum_{k\in I_n}|a_k|.$$

Then we get

$$\begin{aligned}
\|f\|_{B^p}^p &= \sup_{\lambda\in D}\int_D |f'(z)|^p\left(1-|z|^2\right)^{p-1}\frac{1-|\lambda|^2}{\left|1-\overline{\lambda}z\right|^2}dA(z)\\
&\leqslant 4^p\pi K\sum_{n=0}^{\infty}\left(\sum_{k\in I_n}|a_k|\right)^p < \infty,
\end{aligned}$$

that is, $f \in B^p$. To prove that $f \in B_0^p \subset B^p$, we note that the integral $\int_0^1\left(\sum_{n=1}^{\infty} n\,|a_n|\,r^{n-1}\right)^p(1-r^2)^{p-1}dr$ is convergent, for $\sum_{n=0}^{\infty}\left(\sum_{k\in I_n}|a_k|\right)^p < \infty$. Hence for any $\varepsilon > 0$, there is a $\delta \in (0,1)$ such that

$$\int_\delta^1\left(\sum_{n=1}^{\infty} n\,|a_n|\,r^{n-1}\right)^p(1-r^2)^p dr < \varepsilon.$$

Then

$$\int_D |f'(z)|^p \left(1-|z|^2\right)^{p-1} \frac{1-|\lambda|^2}{\left|1-\bar{\lambda}z\right|^2} dA(z)$$

$$\leqslant 2\pi \int_0^1 \left(\sum_{n=1}^{\infty} n\,|a_n|\,r^{n-1}\right)^p \left(1-r^2\right)^{p-1} \frac{1-|\lambda|^2}{1-|\lambda|^2 r^2}\,dr$$

$$< 2\pi \int_0^{\delta} \left(\sum_{n=1}^{\infty} n\,|a_n|\,r^{n-1}\right)^p \left(1-r^2\right)^{p-1} \frac{1-|\lambda|^2}{1-|\lambda|^2 r^2}\,dr + 2\pi\varepsilon$$

$$< 2\pi \frac{1-|\lambda|^2}{1-\delta^2} \int_0^1 \left(\sum_{n=1}^{\infty} n\,|a_n|\,r^{n-1}\right)^p \left(1-r^2\right)^{p-1} dr + 2\pi\varepsilon.$$

If $|\lambda|$ is chosen appropriately so $1-|\lambda|$ may be sufficiently small, then the above quantity can be less than $4\pi\varepsilon$. Hence

$$\lim_{|\lambda|\to 1-0} \int_D |f'(z)|^p \left(1-|z|^2\right)^{p-2} \left(1-|\varphi_\lambda(z)|^2\right) dA(z) = 0.$$

According to definition (6), it follows that $f \in B_0^p$. This completes the proof. ∎

Proof of Theorem 1: It is clear that (II) implies (I). We first prove that (III) follows from (I). Applying Lemma 1 and Lemma 2 we get

$$\|f\|_{B^p}^p \geqslant \int_D |f'(z)|^p \left(1-|z|^2\right)^{p-1} dA(z)$$

$$= \int_D \left|\sum_{k=1}^{\infty} n_k a_k z^{n_k-1}\right|^p \left(1-|z|^2\right)^{p-1} dA(z)$$

$$\geqslant \frac{2\pi}{A^p} \int_0^1 (1-r^2)^{p-1} \left(\sum n_k^2 |a_k|^2 r^{2(n_k-1)}\right)^{p/2} r\,dr$$

$$\geqslant \frac{\pi}{A^p} \int_0^1 (1-x)^{p-1} \left(\sum_{k=1}^{\infty} n_k^2 |a_k|^2 x^{n_k}\right)^{p/2} dx$$

$$\geqslant \frac{\pi}{KA^p} \sum_{k=0}^{\infty} 2^{-kp} t_k^{p/2},$$

where

$$t_k = \sum_{n_j \in I_k} n_j^2 |a_j|^2.$$

Because $n_{k+1}/n_k \geqslant \lambda > 1$ for all $k$, the number of Taylor coefficients $a_j$ is at most $[\log_\lambda 2] + 1$ when $n_j \in I_k$, for $k = 1, 2, \ldots$. Then

$$t_k^{p/2} \geqslant 2^{kp} C_p \sum_{n_j \in I_k} |a_j|^p,$$

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where $C_p = 1$ for $p/2 \geqslant 1$ and $C_p = 1/([\log_\lambda 2] + 1)^{1-p/2}$ for $p/2 < 1$, by (9) and (10). Combining the above inequalities yields that (III) holds.

By Theorem 2 it is easy to prove that (II) follows from (III). Assuming that $\sum_{k=1}^{\infty} |a_k|^p < \infty$ and $n_{k+1}/n_k \geqslant \lambda > 1$ for all $k$, we have

$$\sum_{n=0}^{\infty} \left( \sum_{n_k \in I_n} |a_k| \right)^p \leqslant ([\log_\lambda 2] + 1)^p \sum_{k=1}^{\infty} |a_k|^p < \infty,$$

by (9) and (10). Thus $f \in B_0^p$, and the proof is complete. □

Theorem 1 should be compared with the following result (see [1]):

**Theorem A.** *Let $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$ be analytic on $D$. If $f$ has Hadamard gaps, then $f \in \mathcal{B}$ if and only if $a_k = 0(1)$ $(k \to \infty)$; and $f \in \mathcal{B}_0$ if and only if $a_k \to 0$ $(k \to \infty)$.*

Setting $p = 2$ in Theorem 2, we have

**Corollary.** *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic on $D$. If*

$$\sum_{n=0}^{\infty} \left( \sum_{j \in I_n} |a_j| \right)^2 < \infty,$$

*then $f \in VMOA$.*

Remark. By (10) we have

$$\left( \sum_{j \in I_n} |a_j| \right)^2 \leqslant 2^n \sum_{j \in I_n} |a_j|^2 \leqslant \sum_{j \in I_n} j\,|a_j|^2,$$

thus

$$\sum_{n=0}^{\infty} \left( \sum_{j \in I_n} |a_j| \right)^2 \leqslant \sum_{n=1}^{\infty} n\,|a_n|^2 = \frac{1}{\pi} \int_D |f'(z)|^2 \, dA(z).$$

Hence the condition in the Corollary is weaker than

$$\int_D |f'(z)|^2 \, dA(z) < \infty.$$

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