Canad. Math. Bull. Vol. 24 (1), 1981

A GENERALIZATION OF FLOYD'S THEOREM ON UNICOHERENT PEANO CONTINUA WITH INVOLUTION

BY
A. K. GROVER AND J. H. V. HUNT*

ABSTRACT. We generalize a result of E. E. Floyd on unicoherent Peano continua with involution to unicoherent locally connected regular hereditarily Lindelöf spaces. The result has an application in the theory of connectivity functions.

1. **Introduction.** The purpose of this paper is to prove the following theorem.

THEOREM. *Let X be a locally connected regular hereditarily Lindelöf space, T an involution on X, and L a set such that*

(i) *L is invariant under T,*

(*ii*) *L separates x, $T(x)$ for each $x \notin L$.*

Then L contains a closed subset K of X which is irreducible with respect to properties (i), (ii).

If X is also unicoherent, then K is connected.

In [2] E. E. Floyd proved this result when X is a Peano continuum and L is a closed set. In the next section we first present two lemmas, from which the proof of the theorem follows easily.

The results have the following application in the theory of connectivity functions. Lemma (2.1) of [5] is an immediate corollary of the above theorem or, alternatively (as the space in this lemma is a unicoherent Peano continuum), it is a corollary of lemma 2 below and Floyd's result. Lemma (2.1) of [5] was originally proved using the main theorem of [7], but the proof of this theorem is extremely long.

The authors would like to thank the referee for his suggestions on the reorganization of this note.

2. **The Theorem.** A space X is *unicoherent* if it is connected and if for each pair of connected closed sets M, N such that $X = M \cup N$, $M \cap N$ is connected. The other terms we use are well-known.

The following lemma is taken from [3].

Received by the editors December 11, 1978 and in revised form July 4, 1979.

* The second author was partially supported by NRC Grant No. A4749 and the Multinational Program of Mathematics of the O.A.S. when this work was done.

https://doi.org/10.4153/CMB-1981-018-6 Published online by Cambridge University Press

LEMMA 1. *Let X be a locally connected space, T an involution on X, and L a set such that*

(i) *L is closed,*

(ii) *L is invariant under T,*

(iii) *L separates x, T(x) for each $x \notin L$.*

Then L contains a set K which is irreducible with respect to properties (i), (ii), (iii).

Proof. We show that the collection $\{L_\lambda\}_\lambda$ of all subsets of L having properties (i), (ii), (iii) and partially ordered by $\supset$ satisfies the hypotheses of Zorn's lemma. It is a non-empty collection, as it contains L. Let $\{L_{\lambda_\mu}\}_\mu$ be a chain in $\{L_\lambda\}_\lambda$. Then $\bigcap_\mu L_{\lambda_\mu}$ clearly has properties (i), (ii), so it must be shown that it has property (iii).

Suppose on the contrary that x, $T(x)$ are not separated by $\bigcap_\mu L_{\lambda_\mu}$, for some $x \notin \bigcap_\mu L_{\lambda_\mu}$. Since X is locally connected and $X - \bigcap_\mu L_{\lambda_\mu}$ is open, some component C of $X - \bigcap_\mu L_{\lambda_\mu}$ contains both x, $T(x)$. Again, since X is locally connected and the sets $X - L_\lambda$ are open, $\mathcal{U} = \{U \mid U$ is a component of some $X - L_\lambda\}$ is an open covering of C. Since C is connected, $\mathcal{U}$ contains a simple chain $U_1, U_2, \ldots, U_n$ from x to $T(x)$ (see Theorem 3–4, p. 108 of [4]). Let U_i be a component of $X - L_{\lambda_{\mu_i}}$, and let $L_{\lambda_{\mu_m}}$ be the smallest set in the sequence $L_{\lambda_{\mu_1}}, L_{\lambda_{\mu_2}}, \ldots, L_{\lambda_{\mu_n}}$. Then $U_1 \cup U_2 \cup \ldots \cup U_n$ is a connected subset of $X - L_{\lambda_{\mu_m}}$ which contains both x, $T(x)$; i.e., $L_{\lambda_{\mu_m}}$ does not separate x, $T(x)$. The contradiction shows that $\bigcap_\mu L_{\lambda_\mu}$ has property (iii).

By Zorn's lemma, $\{L_\lambda\}_\lambda$ now contains a minimal element K, which is by definition irreducible with respect to properties (i), (ii), (iii).

COROLLARY. *If X is also unicoherent, then any set K which is irreducible with respect to properties (i), (ii), (iii) is connected.*

Except for a use of well-ordering, the proof of this corollary is the same as that of the corresponding part of Floyd's proof.

Notice that this corollary can also be stated in terms of n-coherence (see [1] for the definition); viz., *if X is also n-coherent, then any set K which is irreducible with respect to properties* (i), (ii), (iii) *has at most n-components.* This result is used in [6].

The following lemma is partly due to E. D. Tymchatyn.

LEMMA 2. *Let X be a regular hereditarily Lindelöf space, T an involution on X, and L a set such that*

(i) *L is invariant under T,*

(ii) *L separates x, T(x) for each $x \notin L$.*

Then L contains a closed set L′ satisfying properties (i), (ii).

Proof. For each point $x \notin L$ there are separated sets P_x, Q_x containing x, $T(x)$, respectively, such that $X - L = P_x \cup Q_x$. Since X is completely normal (being a regular hereditarily Lindelöf space), there are disjoint open sets U_x, V_x

https://doi.org/10.4153/CMB-1981-018-6 Published online by Cambridge University Press

containing P_x, Q_x, respectively. Let $W_x = U_x \cap T(V_x)$. Since X is an hereditarily Lindelöf space, the open covering $\{W_x \cup T(W_x) \mid x \notin L\}$ of $X-L$ contains a countable subcovering $W_{x_1} \cup T(W_{x_1})$, $W_{x_2} \cup T(W_{x_2}), \ldots$. Now define the open sets

$$G_1 = W_{x_1},$$

$$G_n = W_{x_n} - \left(\bigcup_{i=1}^{n-1} \bar{W}_{x_i} \cup T(\bar{W}_{x_i})\right),$$

for $n > 1$, and let

$$L' = X - \left(\bigcup_{n=1}^{\infty} G_n \cup T(G_n)\right),$$

which is a closed set.

To see that $L' \subset L$, observe that L is invariant under T and Fr U_{x_n}, Fr $V_{x_n} \subset L$, for each n. This implies that Fr W_{x_n}, Fr $(T(W_{x_n})) \subset L$. It follows by induction that $(\bigcup_{i=1}^{n} G_i \cup T(G_i)) - L$, $(\bigcup_{i=1}^{n} W_{x_i} \cup T(W_{x_i})) - L$ are equal. Hence $G_1 \cup T(G_1)$, $G_2 \cup T(G_2), \ldots$ is also a covering of $X-L$. That is, $L' \subset L$.

By definition L' satisfies (i). However, since $W_{x_n} \cup T(W_{x_n}) = \varnothing$ for each n, G_1, $T(G_1)$, G_2, $T(G_2), \ldots$ is a sequence of mutually disjoint open sets, and so L' also satisfies (ii).

Proof of theorem. We use the notation in the statement of the theorem. By Lemma 2, L contains a set L' satisfying properties (i), (ii), (iii) of Lemma 1. By Lemma 1, L' contains a set K which is irreducible with respect to properties (i), (ii), (iii) of Lemma 1; i.e., K is a closed set which is irreducible with respect to properties (i), (ii) of the theorem. That K is connected when X is unicoherent follows from the corollary to Lemma 1.

Notice that a Peano space (i.e., a locally compact connected locally connected metric space) has a countable basis by p. 75 of [8], and so satisfies the hypotheses of the theorem.

References

1. S. Eilenberg, *Sur les espaces multicohérents I*, Fund. Math. **27** (1936) 153–190.

2. E. E. Floyd, *Real-valued mappings of spheres*, Proc. Amer. Math. Soc. **6** (1955) 957–959.

3. A. K. Grover, *Involutions on n-coherent spaces*, M. Sc. Thesis, University of Saskatchewan, Saskatoon, 1972.

4. J. G. Hocking and G. S. Young, *Topology*, Addison Wesley, Reading, Mass., 1961.

5. J. H. V. Hunt, *A connectivity map $f: S^n \to S^{n-1}$ does not commute with the antipodal map*, Bol. Soc. Mat. Mex. **16** (1971) 43–45.

6. ——, *A proof of Dyson's theorem on real-valued mappings of the sphere*, to appear.

7. —— and E. D. Tymchatyn, *A theorem on involutions on unicoherent spaces*, to appear.

8. M. H. A. Newman, *Elements of the topology of plane sets of points*, 2nd ed., Cambridge University Press, Cambridge, 1951.

University of Saskatchewan,
Saskatoon, Sask.,
Centro de Investigacion y de Estudios Avanzados del IPN,
Mexico.

https://doi.org/10.4153/CMB-1981-018-6 Published online by Cambridge University Press