Can. J. Math., Vol. XXXI, No. 2, 1979, pp. 441–447

RATIO AND STOCHASTIC ERGODIC THEOREMS FOR SUPERADDITIVE PROCESSES

HUMPHREY FONG

1. Introduction. Let $(X, \mathscr{A}, m)$ be a σ-finite measure space and let T be a positive linear operator on $L_1 = L_1(X, \mathscr{A}, m)$. T is called *Markovian* if

(1.1) $\int Tf\, dm = \int f\, dm, \quad f \in L_1.$

T is called *sub-Markovian* if

(1.2) $\int |Tf|\, dm \leqq \int |f|\, dm, \quad f \in L_1.$

All sets and functions are assumed measurable; all relations and statements are assumed to hold modulo sets of measure zero.

For a sequence of L_1^+ functions $(f_0, f_1, f_2, \ldots)$, let

$$s_n = f_0 + f_1 + \ldots + f_{n-1}, \quad n \geqq 1; \quad s_0 = 0.$$

(f_n) is called a *superadditive sequence* or *process*, and (s_n) a *superadditive sum* relative to a positive linear operator T on L_1 if

(1.3) $T^k s_n \leqq s_{n+k} - s_k, \quad k, n \geqq 0,$

and

(1.4) $\gamma = \sup_n (1/n) \int s_n\, dm < \infty.$

(s_n) is said to be *extended superadditive* if (1.3) holds. A sequence of non-negative functions (p_i) is called *T-admissible* if $Tp_i \leqq p_{i+1}$ for $i \geqq 0$. As pointed out in [**1**], the sequence of partial sums $(\sum_0^{n-1} p_i)$ of an admissible sequence (p_i) is extended superadditive. Superadditive sequences relative to a sub-Markovian operator have been studied by Akcoglu and Sucheston in [**1**], in which the theory of subadditive processes of J. F. C. Kingman for invertible measure-preserving transformations is generalized to the operator-theoretic setting of sub-Markovian operators.

Following the terminology of [**1**], an L_1^+-function δ is called an *exact dominant* of a superadditive sequence (f_n) if $\int \delta\, dm = \gamma$ and

(1.5) $\sum_0^{n-1} T^i\delta \geqq s_n, \quad n = 1, 2, \ldots.$

It is proved in [**1**] that if T is Markovian, then a superadditive process admits at least one exact dominant. This result is a generalization of the Kingman decomposition for subadditive processes, and is used in [**1**] to derive ratio ergodic theorems for superadditive sums relative to a sub-Markovian operator.

Received January 6, 1978.

In Section 2, we derive a ratio ergodic theorem for superadditive sums relative to an operator T satisfying the more general boundedness condition (B):

$$(B) \qquad \sup_n \|(1/n) \textstyle\sum_0^{n-1} T^i\|_1 < \infty.$$

The result generalizes the ratio ergodic theorem in [**13**] and extends the ratio ergodic theorem in [**1**].

We also study the 'stochastic convergence' of superadditive sums. Let $B \subseteq X$. We say that a sequence of functions (f_n) *converges stochastically on* B if there exists a function f such that for each $\epsilon > 0$ and each $A \subseteq B$ with $m(A) < \infty$, we have

$$\lim_n m[A \cap \{x: \ |f_n(x) - f(x)| \geqq \epsilon\}] = 0.$$

'Stochastic convergence' is equivalent to 'convergence in measure' if $m(B) < \infty$. It is well-known that for a sub-Markovian operator T on L_1 and $f \in L_1$, the averages $(1/n)(f + Tf + \ldots + T^{n-1}f)$ need not converge a.e. or in L_1 (see [**3**]). However, the following theorem of U. Krengel [**12**] shows that stochastic convergence does hold:

THEOREM A. *If T is a linear contraction operator on $L_1(X, \mathscr{A}, m)$, then for every $f \in L_1$, the averages $(1/n)(f + Tf + \ldots + T^{n-1}f)$ converge stochastically on X.*

In Section 3, we show that Theorem A can be extended to superadditive sums, and also to the case when T is not necessarily sub-Markovian.

Section 4 deals with continuous parameter superadditive processes. We show that most of the results for discrete parameter superadditive processes easily carry over to the continuous parameter case.

2. A ratio ergodic theorem. In this section, we assume that T is a positive linear operator on $L_1(X, \mathscr{A}, m)$ satisfying condition (B). The 'Sucheston decomposition' states that the space X decomposes into a 'remaining' part Y and a 'disappearing' part Z, with the properties that Z is T-closed and that there exists a function $e \in L_\infty^+$ such that $e > 0$ on Y and $T^* e = e$ ([**13**], also [**6**]).

THEOREM 2.1. *Assume condition (B). Let (s_n) be superadditive and (s_n') be extended superadditive relative to T. Then the ratios s_n/s_n' converge a.e. on the set $\{s_n' > 0 \ \text{ for some } n\} \cap Y$.*

Proof. The operator V defined by the relation

$$(2.1) \qquad Vf = e \cdot T(f \cdot 1_Y/e), \quad f \in L_1,$$

is a Markovian operator on $L_1(Y)$ (see [**13**]). Since Z is T-closed, we have that for $k \geqq 0$,

$$(2.2) \qquad V^k f = e \cdot T^k(f/e), \quad f \in L_1(Y).$$

Set $u_n = e \cdot s_n$, $u_n' = e \cdot s_n'$. (2.2) and (1.3) imply that for $k, n \geqq 0$,

(2.3) $\quad V^k u_n = e \cdot T^k s_n \leqq e(s_{n+k} - s_k) = u_{n+k} - u_k.$

It also follows from (1.4) that for $n \geqq 1$,

$$(1/n) \cdot \int u_n \, dm \leqq (||e||_\infty / n) \cdot \int s_n \, dm \leqq \gamma \cdot ||e||_\infty < \infty.$$

Hence (u_n) is superadditive relative to the Markovian operator V. Similarly (u_n') is extended superadditive relative to V. By Theorem 3.3 of [**1**], the ratios u_n/u_n' converge a.e. on the set $\{u_n' > 0 \quad \text{for some } n\} = \{s_n' > 0 \quad \text{for some } n\} \cap Y$. The conclusion of the theorem follows since $(u_n/u_n') = (s_n/s_n')$ on the set $\{s_n' > 0 \quad \text{for some } n\} \cap Y$.

Remark. It is known that if there is a function $g \in L_1^+$ such that $\{\sum_0^\infty T^i g = \infty\} \cap Z \neq \emptyset$, then the ratios $\sum_0^{n-1} T^i f / \sum_0^{n-1} T^i g$ need not converge a.e. on Z for $f \in L_1$ (see [**8**], [**7**]). The trivial example $T \equiv 0$ (in this case, $X = Z$) shows that in general the ratios s_n/s_n' of Theorem 2.1 need not converge on the set $\{s_n' > 0 \text{ for some } n\} \cap Z$ even if $\sum_0^\infty T^i g < \infty$ a.e. for every $g \in L_1^+$.

3. Stochastic convergence. We consider in this section the stochastic convergence of the sequence s_n/n. The definition of 'stochastic convergence' is given in Section 1.

We first recall some known facts about sub-Markovian operators: For a sub-Markovian operator T, the space X decomposes into the *conservative part* C and the *dissipative part* D such that for any $f \in L_1^+$, $\sum_0^\infty T^i f = \infty \quad \text{or} \quad 0$ on C, and $\sum_0^\infty T^i f < \infty \quad$ on D.

THEOREM 3.1. *If T is Markovian, and (s_n) is superadditive, then s_n/n converges stochastically on X.*

Proof. By Theorems 2.1 and 3.1 of [**1**], the sequence (s_n) has an exact dominant δ such that

(3.1) $\quad \lim_n s_n / \sum_0^{n-1} T^i \delta = 1 \quad \text{a.e.}$

on $C \cap E$, where $E = \{\sum_0^{n-1} T^i \delta > 0 \quad \text{for some } n\}$. On $C \cap E$, the stochastic convergence of s_n/n follows from Theorem A and (3.1) since

$$s_n/n = (s_n / \textstyle\sum_0^{n-1} T^i \delta) \cdot (\sum_0^{n-1} T^i \delta / n);$$

on $D \cap E$, $0 \leqq s_n/n \leqq \sum_0^{n-1} T^i \delta / n$ which tends to 0 a.e. on D; on E^c, $s_n/n = 0$.

THEOREM 3.2. *If T is sub-Markovian, (s_n) superadditive, and if on D, $s_n = \sum_0^{n-1} T^i \delta$ for some $\delta \in L_1^+$, then s_n/n converges stochastically on X.*

Proof. Since $T(1_C \cdot s_n) \leqq 1_C(Ts_n) \leqq 1_C(s_{n+k} - s_k)$, the sequence $(1_C \cdot s_n)$ is superadditive relative to the conservative (hence Markovian) operator $T_C = 1_C T 1_C$. By Theorem 3.1, s_n/n converges stochastically on C. By assumption,

https://doi.org/10.4153/CJM-1979-048-2 Published online by Cambridge University Press

$s_n/n = \sum_0^{n-1} T^i\delta/n$ on D. Thus s_n/n also converges stochastically on D by Theorem A.

We next relax the norm condition on T. For an operator T satisfying condition (B), $X = Y + Z$ is the 'Sucheston decomposition' discussed at the beginning of Section 2.

THEOREM 3.3. *Assume condition* (B), *and let* (s_n) *be superadditive. Then* s_n/n *converges stochastically on* Y.

Proof. Let the sequence $(u_n) = (e \cdot s_n)$ and the operator V be as in the proof of Theorem 2.1. Thus (u_n) is superadditive relative to the operator V, which is Markovian on $L_1(Y)$. It follows from Theorem 3.1 that u_n/n converges stochastically on X. Since $\{e > 0\} = Y$ and $u_n/n = e \cdot s_n/n$ for $n \geqq 1$, s_n/n converges stochastically on Y.

Example. The following example shows that s_n/n need not converge stochastically on Z even if s_n is additive, i.e., $s_n = \sum_0^{n-1} T^i f$ for some $f \in L_1^+$, $n \geqq 1$.

Let $X = \{0, 1, 2, \ldots\}$ and let m be counting measure on X. Thus $L_1 = l_1$. Let $A = \{n \geqq 1 : \ 2^{2i} \leqq n < 2^{2i+1}$ for some $i \geqq 0\}$. For $f = (f(j)) \in l_1$, define

$$Tf(j) = \begin{cases} \sum_{i \in A} f(i), & j = 0 \\ 0 , & j = 1 \\ f(j-1) , & j > 1. \end{cases}$$

It follows that for $n \geqq 1$,

$$T^n f(j) = \begin{cases} \sum_{i+n-1 \in A} f(i), & j = 0 \\ 0 , & 1 \leqq j \leqq n \\ f(j-n) , & j > n. \end{cases}$$

Thus

$$||T^n f||_1 \leqq \sum_{i+n-1 \in A} |f(i)| + \sum_{j>n} |f(j-n)| \leqq 2||f||_1.$$

Hence $||T^n|| \leqq 2$ for $n \geqq 1$, $Y = \{1, 2, \ldots\}$, and $Z = \{0\}$. Let $f = 1_{\{1\}}$. Then

$$(1/2^{2k+1}) \sum_{n=0}^{2^{2k+1}-1} T^n f(0) = \frac{2^{2k+2} - 1}{3(2^{2k+1})} \to \tfrac{2}{3},$$

and

$$(1/2^{2k}) \sum_{n=0}^{2^{2k}-1} T^n f(0) = (2^{2k} - 1)/3(2^{2k}) \to \tfrac{1}{3}.$$

Hence $(1/n) \sum_0^{n-1} T^i f$ does not converge pointwise or stochastically on the set $Z = \{0\}$.

4. Continuous parameter. In this section, we deal with continuous parameter superadditive processes. We first state several lemmas which are simple consequences of the results in [**1**].

Let C and D be respectively the conservative and dissipative parts of a sub-Markovian operator T on L_1.

LEMMA 4.1. *Let T be Markovian, and let (s_n) be superadditive with exact dominant δ. Then for any fixed integer k,*

$$\lim_n s_{n+k}/\sum_0^{n-1} T^i\delta = 1$$

a.e. on $C \cap E$, where $E = \{\sum_0^{n-1} T^i\delta > 0 \quad \text{for some } n\}$.

Proof. For fixed k and large n,

$$(4.1) \quad s_{n+k}/\sum_0^{n-1} T^i\delta = (s_{n+k}/\sum_0^{n+k-1} T^i\delta)\cdot(\sum_0^{n+k-1} T^i\delta/\sum_0^{n-1} T^i\delta) \quad \text{on } E.$$

The conclusion of the lemma follows since $s_{n+k}/\sum_0^{n+k-1} T^i\delta$ converges to 1 a.e. on $C \cap E$ according to Theorem 3.1 in [**1**], and $\sum_0^{n+k-1} T^i\delta/\sum_0^{n-1} T^i\delta$ converges to 1 a.e. on E by a lemma of Chacon and Ornstein [**3**].

LEMMA 4.2. *Let T be sub-Markovian, and let (s_n), (s_n') be superadditive. Then for any fixed integer k,*

$$\lim_n s_{n+k}/s_n' = \lim_n s_n/s_n'$$

a.e. on $C \cap E$, where $E = \{s_n' > 0 \quad \text{for some } n\}$. If either (a) T is Markovian, or (b) $s_n = \sum_0^{n-1} T^i\delta$ on $D \cap E$ for some $\delta \in L_1^+$, then the conclusion also holds on $D \cap E$.

Proof. Let δ' be the exact dominant of (s_n') relative to the conservative operator $T_C = 1_C\cdot T\cdot 1_C$ on C. For fixed k,

$$(4.2) \quad s_{n+k}/s_n' = (s_{n+k}/s_{n+k}')\cdot(s_{n+k}'/\sum_0^{n-1} T^i\delta')\cdot(\sum_0^{n-1} T^i\delta'/s_n')$$

on the set $C \cap E$. By Theorem 3.2 of [**1**], $\lim_n s_n/s_n'$ exists on $C \cap E$. By Lemma 4.1, the ratios $s_{n+k}'/\sum_0^{n-1} T^i\delta'$ and $s_n'/\sum_0^{n-1} T^i\delta'$ converge to 1 a.e. on $C \cap E$. Thus the first assertion of the lemma follows.

If (*a*) holds, then

$$(4.3) \quad 0 \leqq s_n \leqq \sum_0^{n-1} T^i\delta \leqq \sum_0^{\infty} T^i\delta < \infty$$

a.e. on $D \cap E$ where δ is the exact dominant of (s_n). (4.3) is also valid if (*b*) holds. In either case, $\lim s_n$ exists and is finite a.e. on $D \cap E$. Hence on $D \cap E$, $\lim s_{n+k}/s_n' = \lim s_n/\lim s_n'$.

LEMMA 4.3. *Let T be sub-Markovian, (s_n) superadditive, (s_n') extended superadditive. Then for any fixed integer k,*

$$\lim_n s_{n+k}/s_n' = \lim_n s_n/s_n'$$

a.e. on $C \cap E'$, where $E' = \{s_n' > 0 \quad \text{for some } n\}$. The conclusion holds also *on* $D \cap E'$ *if either* (a) T *is Markovian, or* (b) $s_n = \sum_0^{n-1} T^i\delta$ *for some* $\delta \in L_1^+$ *on* $D \cap E'$.

Proof. Let $E = \{s_n > 0 \text{ for some } n\}$. For fixed k and large n, we have

$$(4.5)\quad s_{n+k}/s_n' = \begin{cases} (s_{n+k}/s_n)\cdot(s_n/s_n') & \text{on } E \cap E', \\ 0 & \text{on } E^c \cap E'. \end{cases}$$

By Lemma 4.2, $\lim s_{n+k}/s_n = 1$ a.e. on $E \cap C$, and also on $E \cap D$ if either (a) or (b) holds. According to Theorem 3.3 of [**1**], $\lim s_n/s_n'$ exists and is finite on $E' \cap C$, and also on $E' \cap D$ if either (a) or (b) holds. The conclusion of the lemma now follows from (4.5).

Let $\{S_t : t \geqq 0\}$ be a family of L_1^+ functions such that $S_s \leqq S_t$ for $0 \leqq s \leqq t$. $\{S_t : t \geqq 0\}$ is said to be *superadditive* (resp. *extended superadditive*) relative to a positive linear operator T on L_1 if for some $\alpha > 0$, the sequence $\{S_{n\alpha} : n \geqq 0\}$ is superadditive (resp. extended superadditive). We may and do assume that $\alpha = 1$; otherwise we consider instead the process $U_t = S_{t\alpha}$, $t \geqq 0$.

THEOREM 4.4. *Let* T *be sub-Markovian,* $\{S_t : t \geqq 0\}$ *superadditive,* $\{S_t' : t \geqq 0\}$ *extended superadditive. Then*

$$\lim_{t\to\infty} S_t/S_t' = \lim_{n\to\infty} S_n/S_n'$$

a.e. on the set $C \cap E'$, *where* $E' = \{S_t' > 0 \text{ for some } t > 0\}$. *The conclusion holds also on* $D \cap E'$ *if* T *is Markovian.*

Proof. For $n \leqq t < n + 1$,

$$(4.6)\quad S_n/S_{n+1}' \leqq S_t/S_t' \leqq S_{n+1}/S_n'$$

on the set E'. By Lemma 4.3, the ratios S_n/S_{n+1}', S_{n+1}/S_n' and S_n/S_n' all have the same limit on $C \cap E'$, and also on $D \cap E'$ if T is Markovian. Thus the theorem follows from (4.6).

We next prove a continuous parameter analogue of Theorem 2.1. For an operator T satisfying the boundedness condition (B), $X = Y + Z$ is the 'Sucheston decomposition' discussed in Section 2.

THEOREM 4.5. *Let* T *be a positive linear operator satisfying condition* (B). *Let* $\{S_t : t \geqq 0\}$ *be superadditive and* $\{S_t' : t \geqq 0\}$ *extended superadditive. Then the ratios* S_t/S_t' *converge to a finite limit a.e. on the set* $Y \cap \{S_t' > 0$ for some $t > 0\}$.

Proof. The proof is analogous to the proof of Theorem 2.1, except that here we apply Theorem 4.4 above instead of Theorem 3.3 of [**1**].

The last two theorems are continuous parameter analogues of Theorems 3.2 and 3.3. Their proofs follow immediately from Theorems 3.2 and 3.3 and the obvious inequality

(4.7) $S_n/(n+1) \leqq S_t/t \leqq S_{n+1}/n.$

THEOREM 4.6. *If T is sub-Markovian, $\{S_t\}$ superadditive, and if on the dissipative part D, $S_n = \sum^{n-1} T^i\delta$ for some $\delta \in L_1^+$, $n \geqq 1$, then the ratios S_t/t converge stochastically on X.*

THEOREM 4.7. *Assume condition (B), and let $\{S_t\}$ be superadditive. Then the ratios S_t/t converge stochastically on Y.*

Acknowledgement. The author wishes to thank Professor Louis Sucheston for his helpful suggestions and comments.

REFERENCES

1. M. A. Akcoglu and L. Sucheston, *A ratio ergodic theorem for super-additive processes*, preprint. For a résumé, see C. R. Acad. Sci., Paris, *285* (1977), 637–639.

2. R. V. Chacon, *Convergence of operator averages*, in Ergodic theory (Academic Press, New York, 1963, 89–120.

3. ——— *A class of linear transformations*, Proc. Amer. Math. Soc. *15* (1964), 560–564.

4. R. V. Chacon and D. S. Ornstein, *A general ergodic theorem*, Illinois J. Math. *4* (1960), 153–160.

5. Y. Derriennic, *Sur le theorem ergodique sous-additif*, C. R. Acad. Sci., Paris, *281* (1975), 985–988.

6. Y. Derriennic and M. Lin, *On invariant measures and ergodic theorems for positive operators*, J. Functional Anal. *13* (1973), 252–267.

7. H. Fong, *On invariant functions for positive operators*, Colloq. Math. *22* (1970), 75–84.

8. A. Ionescu-Tulcea and M. Moretz, *Ergodic properties of semi-Markovian operators on the Z-part*, Z. Wahrscheinlichkeitstheorie verw. Geb. *13* (1969), 119–122.

9. J. F. C. Kingman, *The ergodic theory of subadditive stochastic processes*, J. Royal Statist. Soc. B *30* (1968), 499–510.

10. ——— *Subadditive ergodic theory*, Ann. Prob. *1* (1973), 883–905.

11. ——— *Subadditive processes*, Ecole d'été des probabilités de Saint-Flour, Springer Verlag Lecture Notes in Mathematics, Vol. *539* (1976), 168–223.

12. U. Krengel, *On the global limit behaviour of Markov chains and of general nonsingular Markov processes*, Z. Wahrscheinlichkeitstheorie verw. Geb. *6* (1966), 302–316.

13. L. Sucheston, *On the ergodic theorem for positive operators*, *I*, Z. Wahrscheinlichkeitstheorie verw. Geb. *8* (1967), 1–11.

Bowling Green State University,
Bowling Green, Ohio