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On the Weak Order of Coxeter Groups

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Abstract. This paper provides some evidence for conjectural relations between extensions of (right) weak order on Coxeter groups, closure operators on root systems, and Bruhat order. The conjecture focused upon here refines an earlier question as to whether the set of initial sections of reflection orders, ordered by inclusion, forms a complete lattice. Meet and join in weak order are described in terms of a suitable closure operator. Galois connections are defined from the power set of W to itself, under which maximal subgroups of certain groupoids correspond to certain complete meet subsemilattices of weak order. An analogue of weak order for standard parabolic subsets of any rank of the root system is defined, reducing to the usual weak order in rank zero, and having some analogous properties in rank one (and conjecturally in general).

Introduction

Weak order is a partial order on a Coxeter group W which is of considerable importance in the basic combinatorics of W. For example, the maximal chains in weak order from the identity element to a given element are in natural bijective correspondence with reduced expressions of that element. It is known that weak orders of Coxeter groups are complete meet semilattices.

This paper gives descriptions of meet and join (when existing) of elements in weak order in terms of a closure operator on the root system of W. The closure operator used here is the finest one, *i.e.*, with the most closed sets, for which, given any two roots in a closed set, any root expressible as a non-negative real linear combination of those two roots is also in that closed set, though some of the results hold for other closure operators as well. The paper also introduces two Galois connections from the power set of W to itself, under which maximal subgroups of certain groupoids (generalizing those in [6], and which we shall study in a series of other papers) correspond bijectively to certain complete meet subsemilattices of weak order. It also defines an analogue of weak order associated with standard parabolic subsets of the root system (that are the unions of the positive roots with the root systems of standard parabolic subgroups). The parabolic weak order reduces to the standard weak order in the case of the trivial parabolic subgroup, and it is shown to be a complete meet semilattice (with meet and join given by essentially the same formula in terms of the closure operator as for ordinary weak order) in the case of a rank one parabolic subgroup.

These results have been obtained in attempting to refine some questions and conjectures on the structure of reflection orders of Coxeter groups and their initial sections raised in [14, Remark 2.12] and [16, Remark 2.14]. Reflection orders and their initial sections have applications to the study of Bruhat order, Hecke algebras, and

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Kazhdan–Lusztig polynomials. The original questions are recalled here, and some of the refinements are stated, since they provide some of the principal motivations; additional related results and conjectures will be discussed in other papers. The arrangement of this paper is as follows. Section 1 provides the statements of the main results. Section 2 discusses the conjectures. Section 3 illustrates some of the main results by the example of finite dihedral groups. Sections 4–10 are devoted to the proofs of the main results. Section 11 discusses two other standard closure operators on root systems and the extent to which the results are known to hold for them.

Throughout the paper, it is assumed that the reader is familiar with basic properties of Coxeter groups and their root systems; standard references for this background material are [4, 23]. For the basic properties of weak order, consult [1]. Several of the proofs proceed by reduction to the case of dihedral groups; the easy verifications of the necessary facts in the dihedral case are usually omitted.

1 Statement of Results

1.1 Posets and Lattices

Fix the following standard terminology for partially ordered sets, which are called posets in this paper (see, for example, [9]). A lattice is a non-empty poset $(L, \leq)$ in which any two elements $x, y \in L$ have a least upper bound (join) $x \vee y$ and a greatest lower bound (meet) $x \wedge y$. A lattice L is said to be complete if every subset X of L has a join $\bigvee X$ and a meet $\bigwedge X$ (this implies L has a minimum element and a maximum element).

An ortholattice is a lattice which has a maximum element $\top$, minimum element $\bot$ and is equipped with an order-reversing involution $x \mapsto x^\perp$ such that $x \vee x^\perp = \top$ and $x \wedge x^\perp = \bot$ for all x in the lattice. A complete ortholattice is an ortholattice that is complete as lattice.

A complete meet semilattice is a poset in which every non-empty subset X has a greatest lower bound $\bigwedge X$; in particular, a non-empty complete meet semilattice has a minimum element. By a complete meet subsemilattice of a complete meet semilattice L, we mean a subset X of L such for any non-empty subset Y of X, the meet $\bigwedge Y$ of Y in L is in X (and is therefore the meet of Y in X). If X is non-empty, its minimum element need not coincide with that of L (according to our conventions).

1.2 Coxeter Groups and Root Systems

Let (W, S) be a Coxeter system with standard length function $l = l_W$ and the set of reflections $T = \{wsw^{-1} \mid w \in W, s \in S\}$. Assume without loss of generality that W is the real reflection group associated as in [18] with a based root system Φ in a real vector space V (this allows the standard root system of [4, 23]). Let Π be the standard set of simple roots corresponding to the simple reflections S of W, and let Φ_+ be the set of positive roots corresponding to Π. Abbreviate $\Phi_- := -\Phi_+$. Denote the reflection in a root $\alpha \in \Phi$ by $s_\alpha \in T$. For $z \in W$, let

$$\Phi_z = \Phi_{z,W} := \Phi_+ \cap z(\Phi_-) = \{\alpha \in \Phi_+ \mid l(s_\alpha z) < l(z)\}.$$

It is well known that if $z = s_{\alpha_1} \cdots s_{\alpha_n}$ is any reduced expression, then

$$\Phi_z = \{\alpha_1, s_{\alpha_1}(\alpha_2), \ldots, s_{\alpha_1} \cdots s_{\alpha_{n-1}}(\alpha_n)\}, \quad |\Phi_z| = l(z) = n. \tag{1.1}$$

Write $\Phi'_z := \Phi_+ \setminus \Phi_z$. Note that for $z \in W$, Φ'_z is finite if and only if W is finite, in which case $\Phi'_z = \Phi_{zw_S}$, where w_S is the longest element of W.

1.3 Weak Order

The weak (right) order $\le$ on W is the partial order on W defined by $x \le y$ if and only if $l(y) = l(x) + l(x^{-1}y)$. It is also known that $x \le y$ if and only if $\Phi_x \subseteq \Phi_y$. In particular, for any $x, y \in W$, the equality $\Phi_x = \Phi_y$ holds if and only if $x = y$.

For any fixed $y \in W$, the set $\{x \in W \mid x \le y\}$ is finite. It is known that W in weak order is a complete meet semilattice (see [1, 3.2]).

1.4 The 2-closure Operator on a Root System

Denote the power set of a set X as $\mathscr{P}(X)$. A closure operator on the set X is a function $c\colon \mathscr{P}(X) \to \mathscr{P}(X)$ such that $c(c(A)) = c(A)$ and $A \subseteq c(A)$ for $A \subseteq X$, and $A \subseteq B \subseteq X$ implies $c(A) \subseteq c(B)$. One calls the subsets of X of the form $c(A)$ for $A \subseteq X$, the closed sets (of c). The closure $c(A)$ of A is then the intersection of all closed subsets of X containing A.

Following [14, Remark 2.12], we introduce a closure operator on Φ as follows. Say that a subset Γ of Φ is 2-*closed* if for any $\alpha, \beta \in \Gamma$ we have $\{a\alpha + b\beta \mid a, b \in \mathbb{R}_{\ge 0}\} \cap \Phi \subseteq \Gamma$. The 2-closed sets are the closed sets of the closure operator on Φ for which the closure $\overline{\Gamma}$ of a subset Γ of Φ is the intersection of all 2-closed subsets of Φ which contain Γ. Henceforth, the 2-closed subsets of Φ are usually called closed sets, unless other closure operators are under simultaneous consideration.

It is crucial for the purposes of this paper that Φ_+ is closed and for $x \in W$, that Φ_x and Φ'_x are closed (see Lemma 4.1).

Remarks. The 2-closure operator was called $\mathbb{R}$-closure in [26]; the name 2-closure emphasizes its rank two nature. See also [17] and Section 11 for more about this closure operator.

1.5 Weak Order and 2-closure

The first main result affords a new proof that W in weak order is a semilattice and a description of its meet and join in terms of the closure operator.

Theorem *In weak order $\le$, W is a complete meet semilattice. The join (when existing) and meet of a non-empty subset X of W are given as follows.*

(i) *The join $y := \bigvee X$ exists in $(W, \le)$ if and only if X has an upper bound in W, in which case $\Phi_y = \overline{\bigcup_{x \in X} \Phi_x}$.*
(ii) *If $y := \bigwedge X$, then $\Phi'_y = \overline{\bigcup_{x \in X} \Phi'_x}$.*

Remarks. Subsequent conjectures of this paper would imply that the join in (i) exists if and only if $\overline{\bigcup_{x \in X} \Phi_x}$ is a finite set, but this remains an open question.

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1.6 Order Isomorphisms

The order isomorphism in part (i) of the following corollary is well known [1, Proposition 3.1.6]. Part (ii), which is a simple consequence of Theorem 1.5, shows that the domain of this order isomorphism is closed under formation of those joins that exist in W; it is highlighted since it will play a fundamental role in subsequent papers.

Corollary *Let $x \in W$. Then*

(i) *The map $u \mapsto xu\colon W \to W$ restricts to an order isomorphism between the subposet (actually an order ideal) $\{u \in W \mid l(xu) = l(x) + l(u)\}$ and the subposet (an order coideal) $\{z \in W \mid x \leq z\}$ of $(W, \leq)$.*

(ii) *If $U \subseteq W$ with $l(xu) = l(x) + l(u)$,* i.e., *$\Phi_u \cap \Phi_{x^{-1}} = \varnothing$, for all $u \in U$, and $y := \bigvee U$ exists in W, then $l(xy) = l(x) + l(y)$,* i.e., *$\Phi_y \cap \Phi_{x^{-1}} = \varnothing$.*

1.7 Bruhat Order and 2-closure

Theorem 1.5 (i) and Corollary 1.6 are proved in Section 5, after giving some preliminaries in Section 4. A more general statement (Theorem 7.1) than Theorem 1.5 (ii) is proved in Section 7, after establishing Lemma 1.7 in Section 6. To state the lemma, the following notation will be used. For $n \in \mathbb{Z}$ and $x \in W$, define

$$\Phi_{x,n} := \{\alpha \in \Phi_+ \mid l(s_\alpha x) = l(x) + n\}.$$

Note that $\Phi_{x,n} = \varnothing$ unless n is odd, and that $\Phi_x = \dot\bigcup_{n<0} \Phi_{x,n}$ and $\Phi'_x = \dot\bigcup_{n>0} \Phi_{x,n}$ (here and later, the symbol $\dot\bigcup$ indicates that a union is one of disjoint sets).

The following result is reminiscent of the Krein–Milman theorem, but note that 2-closure is not a "convex", *i.e.*, anti-exchange, closure operator in general (see Section 11).

Lemma *Let $\Gamma \subseteq \Phi_+$ and $x \in W$. Then*

(i) *$\overline{\Gamma} = \Phi_x$ if and only if $\Phi_{x,-1} \subseteq \Gamma \subseteq \Phi_x$.*

(ii) *$\overline{\Gamma} = \Phi'_x$ if and only if $\Phi_{x,+1} \subseteq \Gamma \subseteq \Phi'_x$.*

1.8 Closure After Adjoining a Root

In Section 8, the following is proved using Lemma 1.7 and Theorem 1.5.

Theorem *Let $x \in W$ and $\alpha \in \Phi_+$.*

(i) *Suppose that $l(s_\alpha x) = l(x) + 1$ and also that there is some $v \in W$ with $\Phi_x \cup \{\alpha\} \subseteq \Phi_v$. Then the set $\{z \in W \mid x \leq z, \alpha \in \Phi_z\}$ has a minimum element $y = x \vee s_\alpha x$. One has $\overline{\Phi_x \cup \{\alpha\}} = \Phi_y$ and $\Phi_{s_\alpha y} = \Phi_y \setminus \{\alpha\} = \overline{(\Phi_x \cup \Phi_{s_\alpha x}) \setminus \{\alpha\}}$. In particular, $y > s_\alpha y = y s_\tau$ for some $\tau \in \Pi$.*

(ii) *Suppose that $l(s_\alpha x) = l(x) - 1$. Then there is a maximum element $y = x \wedge s_\alpha x$ in $\{z \in W \mid z \leq x, \alpha \notin \Phi_z\}$. One has $\Phi'_y = \overline{\Phi'_x \cup \{\alpha\}}$ and $\Phi'_{s_\alpha y} = \Phi'_y \setminus \{\alpha\} = \overline{(\Phi'_x \cup \Phi'_{s_\alpha x}) \setminus \{\alpha\}}$. In particular, $y < s_\alpha y = y s_\tau$ for some $\tau \in \Pi$.*

Remarks. The minimum and maximum elements are taken with respect to weak order. Note that the hypotheses of (i) hold for any $x \in W$ and $\alpha \in \Pi \setminus \Phi_x$ for which $x \vee s_\alpha$ exists; then $y = x \vee s_\alpha$. The hypotheses of (ii) hold for any $x \in W$ and $\alpha \in \Pi \cap \Phi_x$.

1.9 Galois Connection Between Subgroups and Subsemilattices

Define a relation R on W by xRz if and only if $z(\Phi_x) = \Phi_x$ for $x, z \in W$. As would any relation on W, R defines a Galois connection (see [9, 24] for general background on Galois connections) from $\mathscr{P}(W)$ to itself as follows. Consider the two maps $X \mapsto X^\dagger$ and $Z \mapsto Z^*$ from $\mathscr{P}(W) \to \mathscr{P}(W)$ defined by

$$X^\dagger := \{z \in W \mid xRz \text{ for all } x \in X\}, \quad Z^* := \{x \in W \mid xRz \text{ for all } z \in Z\}.$$

Ordering $\mathscr{P}(W)$ by inclusion, the maps are order-reversing and satisfy $Z \subseteq X^\dagger$ if and only if $X \subseteq Z^*$, *i.e.*, they give a Galois connection. As with any Galois connection, there are associated families of stable sets (often called closed sets) for the composite maps:

$$\mathscr{W}_* := \{\mathscr{L} \in \mathscr{P}(W) \mid \mathscr{L}^{\dagger *} = \mathscr{L}\} = \{Z^* \mid Z \in \mathscr{P}(W)\},$$
$$\mathscr{W}_\dagger := \{\mathscr{G} \in \mathscr{P}(W) \mid \mathscr{G}^{*\dagger} = \mathscr{G}\} = \{X^\dagger \mid X \in \mathscr{P}(W)\}.$$

Well-known properties of Galois connections imply that the restriction of $*$ to a map $\mathscr{W}_\dagger \to \mathscr{W}_*$ is a bijection with the inverse given by the restriction of $\dagger$ to a map $\mathscr{W}_* \to \mathscr{W}_\dagger$. Also, $\mathscr{W}_\dagger$ and $\mathscr{W}_*$ are complete lattices, dual under the above bijection, with meet given by the intersection of subsets of W. In Section 9, the following is proved and analogous facts are given for the Galois connection determined similarly by the relation R' on W with $xR'z$ if and only if $z(\Phi'_x) = \Phi'_x$.

Theorem (i) *One has*

$$xRz \iff (x \vee z = zx \text{ and } \Phi_x \cap \Phi_z = \varnothing) \iff \Phi_{zx} = \Phi_z \cup \Phi_x$$

(ii) *If xRz, then xRz^{-1} and $l(zx) = l(z) + l(x)$.*
(iii) *The elements of $\mathscr{W}_\dagger$ are subgroups of W.*
(iv) *The elements of $\mathscr{W}_*$ are complete meet subsemilattices of $(W, \le)$ with 1_W as a minimum element.*
(v) *If $L \in \mathscr{W}_*$, then for any subset X of L that has an upper bound in W, its join $x := \bigvee X$ in W is an element of L (and so x is the least upper bound of X in L).*

1.10 Groupoids Associated With the Galois Connections

Adopt here the point of view that a groupoid is a category with a set of objects in which every morphism is an isomorphism. Each element of $\mathscr{W}_\dagger$ is obviously a maximal subgroup of (*i.e.*, the automorphism group of an object of) a groupoid whose objects are I-indexed families $X = (x_i)_{i \in I}$ of elements of W for a suitable index set I, and its morphisms are

$$\mathrm{Hom}(X, Y) \cong \{z \in W \mid z(\Phi_{x_i}) = \Phi_{y_i} \text{ for all } i \in I\}$$

and where composition is induced naturally by multiplication in W, similarly for R'.

Remarks. In the case of the relation R, the full subgroupoids with objects the indexed families of elements of S (more precisely, their variants using subsets of S instead of indexed subfamilies) were studied in [6]. The groupoids defined above will be further generalized and studied in a series of subsequent papers.

1.11 Coclosed and Biclosed Subsets of Roots

For any closed subset Λ of Φ, say that a subset Γ of Λ is coclosed in Λ if $\Lambda \setminus \Gamma$ is closed. Say that $\Gamma \subseteq \Lambda$ is biclosed in Λ if Γ is closed in Λ and $\Lambda \setminus \Gamma$ is closed in Λ. Let $\mathscr{B}(\Lambda)$ denote the set of biclosed subsets of Λ. Order $\mathscr{B}(\Lambda)$ by inclusion. Note that $\mathscr{B}(\Lambda)$ is a complete poset, in the sense that it has a minimum element $\varnothing$ and the union of any directed family of elements of $\mathscr{B}(\Lambda)$ is in $\mathscr{B}(\Lambda)$. For any closure operator a on Φ, define notions of a-coclosed sets, a-biclosed sets, etc., in a similar manner as for 2-closure.

For example, the finite biclosed subsets of Φ_+ are the sets Φ_x for $x \in W$, by Lemma 4.1. Since this paper deals mostly with subsets of Φ_+, the terminology will be slightly abbreviated in that case: by a *coclosed set* (resp., *biclosed set*) is meant a set which is a coclosed subset of Φ_+ (resp., a biclosed subset of Φ_+).

1.12 Parabolic Weak Orders

Standard parabolic subsets of Φ are now defined, following the usual definition for Weyl groups [4, Chapter VI, §1, Proposition 20]. In the general context of this paper, the analogues of the conditions *loc. cit.* are no longer all equivalent, *e.g.*, for an infinite dihedral group; the definition here of a standard parabolic subset is modeled on condition (iii) of *loc. cit.* and the more general definition of quasiparabolic subset in 2.5 is based on condition (i) there.

For $J \subseteq S$, there is a standard parabolic subgroup $W_J := \langle J \rangle$ generated by J, and its root system $\Phi_J = \{\alpha \in \Phi \mid s_\alpha \in W_J\}$. Set $\Lambda := \Lambda_J := \Phi_+ \cup \Phi_J$. Call Λ_J the standard parabolic subset of Φ (of rank $|J|$) associated with J (or associated with Φ_J). Let $\mathscr{L} = \mathscr{L}_J$ denote the set of all finite biclosed subsets Γ of Λ_J. Order $\mathscr{L}_J$ by inclusion, and call the resulting poset the parabolic weak order associated with J. Note that $\mathscr{L}_\varnothing = \{\Phi_x \mid x \in W\}$ naturally identifies with W in weak order via $x \leftrightarrow \Phi_x$. Define $\tau: \mathscr{P}(\Lambda_J) \to \mathscr{P}(\Phi_J)$ by $\tau(\Gamma) := \Gamma \cap \Phi_J$.

1.13 2-closure and Rank One Parabolic Weak Order

Using the previous results, the following fact will be proved in Section 10.

Theorem *Suppose above that $|J| = 1$. Then $\mathscr{L}_J$ is a complete meet semilattice of subsets of $\Lambda := \Lambda_J$. Furthermore, we have the following.*

(i) *If $Y \subseteq \mathscr{L}_J$ has an upper bound in $\mathscr{L}_J$, then it has a join $\Delta := \bigvee Y$ in $\mathscr{L}_J$ given by $\Delta = \overline{\bigcup_{\Gamma \in Y} \Gamma}$.*
(ii) *The meet Δ of a subset Y of $\mathscr{L}_J$ is given by $\Lambda \setminus \Delta = \overline{\bigcup_{\Gamma \in Y} (\Lambda \setminus \Gamma)}$.*
(iii) *The map $\Gamma \mapsto \tau(\Gamma)$ maps $\mathscr{L}_J$ into the lattice of finite biclosed subsets of Φ_J (with biclosed sets ordered by inclusion), preserving meets and those joins that exist. Moreover, $\tau(\overline{\Gamma}) = \overline{\tau(\Gamma)}$ for any $\Gamma \subseteq \Lambda$.*

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Remarks. The analogue of Theorem 1.13 with $J = \varnothing$ is essentially equivalent to Theorem 1.5. The analogous statement could be conjectured to hold for any J, but then one could have $\mathscr{L}_J = \{\varnothing\}$, *e.g.* for $J = S$ and W infinite dihedral, and the conjecture in that form is chiefly of interest for finite W (for which it is open if $|J| > 1$). A more general conjecture without this difficulty is formulated in Section 2.

2 Conjectures

The results of this paper have been obtained while investigating an extensive set of questions and conjectures involving generalizations of basic combinatorics of Coxeter groups. In this section, some of the conjectures most closely related to the contents of this paper, and not requiring much additional background, are stated.

The conjectures originated in studying applications of reflection orders, the original motivation for the definition of which was to extend symmetry amongst structure constants of Iwahori–Hecke algebras from the case of finite Coxeter groups to general Coxeter groups; this required a substitute for the reduced expressions of the longest element, which was provided by the reflection orders.

2.1 Reflection Orders and Initial Sections

A reflection order of Φ_+ is defined as a total order $\preceq$ of Φ_+ such that for $\alpha, \beta, \gamma \in \Phi_+$ with $\alpha \prec \gamma$ and $\beta \in \mathbb{R}_{>0}\alpha + \mathbb{R}_{>0}\gamma$, we have $\alpha \prec \beta \prec \gamma$. See [1] for a discussion of them and some applications. Under transport of structure from Φ_+ to T using the natural bijection $\alpha \mapsto s_\alpha$, reflection orders of Φ_+ correspond to the reflection orders of T in the sense of [14] (which are combinatorial, in that they can be defined purely in terms of (W, S)).

Abbreviate the set of biclosed subsets of Φ_+ as $\mathscr{B} := \mathscr{B}(\Phi_+)$. Define an admissible order of $\Gamma \in \mathscr{B}$ to be a total order $\preceq$ of Γ, all of the initial sections of which are in $\mathscr{B}$, where for any totally ordered set P, an initial section of P is by definition an order ideal, *i.e.*, a subset I of P such that $x \leq y$ for all $x \in I$ and $y \in P \smallsetminus I$.

By straightforward reduction to the case of dihedral groups, it follows that a total order $\preceq$ of Φ_+ is an admissible order of Φ_+ if and only if it is a reflection order of Φ_+. Let $\mathscr{A}$ denote the set of all subsets Γ of Φ_+ for which there exists some reflection order $\preceq$ of Φ_+ with Γ as an initial section. It is easily checked from the definitions that $\mathscr{A} \subseteq \mathscr{B}$. Attached to each element of $\mathscr{A}$, there is a *twisted Bruhat order* of $\mathscr{A}$ as in [13]; the definition of these orders can be extended to the elements of $\mathscr{B}$ [20], but the more general orders are not known to have such strong properties as those from elements of $\mathscr{A}$. On the other hand, $\mathscr{B}$ has many useful properties not obviously shared by $\mathscr{A}$; for example, $\mathscr{B}$ is closed under arbitrary directed unions, but this is not known for $\mathscr{A}$.

2.2 Reflection Orders and Maximal Chains of Biclosed Sets

As will be explained in 2.3, the following conjecture is naturally suggested by naive analogy with the most basic facts about combinatorics of Coxeter groups.

Conjecture (i) $\mathscr{A} = \mathscr{B}$

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(ii) *For any $\Gamma \in \mathscr{A}$ and any totally ordered (by inclusion) subset $\mathscr{I}$ of $\mathscr{A} \cap \mathscr{P}(\Gamma)$, there is an admissible order $\preceq$ of Γ such that every element of $\mathscr{I}$ is an initial section of $\preceq$.*

Remarks. Part (i) of the conjecture is equivalent to the conjecture [14, Remark 2.12], and the special case $\Gamma = \Phi_+$ of (ii) is equivalent to a positive answer for a question raised in [16, Remark 2.14]. Given (i), (ii) is equivalent to its own special case with $\Gamma = \Phi_+$. The conjecture is known for finite Coxeter groups (see below) and will be proved in the special case of affine Weyl groups in another paper.

2.3 Admissible Orders as Generalized Reduced Expressions

It follows from [14, Lemma 2.11], [16, Example 2.2], and Lemma 4.1 that

$$\big\{ B \in \mathscr{A} \bigm| |B| \text{ is finite} \big\} = \{ \Phi_w \mid w \in W \} = \big\{ B \in \mathscr{B} \bigm| |B| \text{ is finite} \big\}$$

and that the admissible orders of Φ_w are in natural bijective correspondence with the reduced expressions of w as follows: to a reduced expression $w = s_{\alpha_1} \cdots s_{\alpha_n}$ of w, one attaches an admissible order $\preceq$ of $\Phi_w = \{\beta_1, \dots, \beta_n\}$ given by $\beta_1 \prec \cdots \prec \beta_n$, where $\beta_i := s_{\alpha_1} \cdots s_{\alpha_{i-1}}(\alpha_i)$ (compare (1.1)). Thus, W can be identified with the subset $\{\Phi_w \mid w \in W\}$ of $\mathscr{B}$, the elements of $\mathscr{B}$ can be viewed as generalized elements of W, and the notion of admissible order of an element of $\mathscr{B}$ can be regarded as a generalization of the notion of reduced expression of an element of W. In the case of finite W, the generalized notions are precisely equivalent to the original ones, but this is not true for any infinite Coxeter group. In fact, it can be shown that there are examples of infinite, finitely generated Coxeter groups W for which $\mathscr{A}$ (and hence $\mathscr{B}$) is uncountable. In any case, the partial order of $\mathscr{B}$ by inclusion naturally generalizes the weak order on W, and will be called the extended weak order of W.

Given Conjecture 2.2 (i), Conjecture 2.2 (ii) is equivalent (by Zorn's lemma) to the statement that for $\Gamma \in \mathscr{B}$, the map taking an admissible order of Γ to the set of its initial sections gives a bijection between the admissible orders of Γ and the maximal totally ordered subsets of $\{\Delta \in \mathscr{B} \mid \Delta \subseteq \Gamma\}$. The conjectures together therefore generalize the statements that every element w of W has a reduced expression and that the reduced expressions of an element w of W are in natural bijective correspondence with maximal chains from 1 to w in weak order of W.

2.4 Reflection Subgroups and Closed Sets of Roots

One might ask how much of the standard combinatorics involving elements of finite Coxeter groups and their reduced expressions, when suitably reformulated, extend to elements of $\mathscr{B}$ and their admissible orders. For example, since weak order on W is a lattice if W is finite, it is natural to ask if $\mathscr{B}$, ordered by inclusion, is also a lattice in general; this question was raised in [16, Remark 2.14]. A more precise and more general version of this question is formulated as a conjecture below, extending Theorems 1.5 and 1.13. In order to do this, we first record a simple lemma.

Note that the reflection subgroups W' of W with closed root subsystem $\Phi_{W'} := \{\alpha \in \Phi \mid s_\alpha \in W'\}$ constitute a complete meet subsemilattice of the complete lattice of reflection subgroups. An arbitrary reflection subgroup W' of W need not have closed root system, e.g., in type B_2. However, one does have the following.

Lemma *Let Ξ, Λ be subsets of Φ such that Ξ, Λ are both closed, $\Xi = -\Xi$, and $\Xi \subseteq \Lambda$. Then Ξ is the root system of the reflection subgroup $W' := \langle s_\alpha \mid \alpha \in \Xi \rangle$ and the natural W-action on Φ restricts to a W'-action on Λ.*

Proof For $\alpha \in \Xi$ and $\beta \in \Lambda$, $s_\alpha(\beta) \in (\beta + \mathbb{R}\alpha) \cap \Phi \subseteq \Lambda$ since Λ is closed, $\beta \in \Lambda$, and $\{\alpha, -\alpha\} \subseteq \Xi \subseteq \Lambda$. This proves that Λ is W'-stable. The statement that Ξ is the root system of W' follows by taking $\Lambda = \Xi$. ∎

2.5 Conjecture on Parabolic Weak Order and 2-closure

A subset Λ of Φ will be called quasi-parabolic if Λ is closed and $\Lambda \cup -\Lambda = \Phi$. The standard parabolic subsets are obviously quasi-parabolic. There is a classification of quasi-parabolic subsets (and more generally, of elements of $\mathscr{B}(\Phi)$) in terms of elements of $\mathscr{B}(\Phi_+)$ and additional combinatorial data, which will not be discussed here. In [7], analogues of quasiparabolic sets in (possibly infinite) oriented matroids are called large convex sets. The main examples of quasi-parabolic subsets are the W-conjugates of standard parabolic sets.

Fix a quasi-parabolic subset Λ of Φ. By the preceding lemma, $\Psi := \Psi(\Lambda) = \Lambda \cap -\Lambda$ is the root system of a reflection subgroup $W' = W(\Lambda) := \langle s_\alpha \mid \alpha \in \Psi \rangle$ of W. Note that W' acts on Λ by $(w, \gamma) \mapsto w(\gamma)$, preserving Ψ. This W'-action on Λ (resp., Ψ) obviously induces a W'-action by order automorphisms on $\mathscr{B}(\Lambda)$ (resp., $\mathscr{B}(\Psi)$).

Define the map $\tau\colon \mathscr{P}(\Lambda) \to \mathscr{P}(\Psi)$ by $\tau(\Gamma) := \Gamma \cap \Psi$, for $\Gamma \subseteq \Lambda$. Call $\tau(\Gamma)$ the type of Γ. One clearly has $\tau(w(\Gamma)) = w(\tau(\Gamma))$ for $w \in W$, and $\tau(\Gamma) \in \mathscr{B}(\Psi)$ if $\Gamma \in \mathscr{B}(\Lambda)$.

Conjecture *Let Λ be a standard parabolic subset of Φ, say $\Lambda = \Phi_+ \cup \Phi_J$ for $J \subseteq S$. Set $\Psi := \Psi(\Lambda) = \Psi_J$ and $W' := W(\Lambda) = W_J$.*

(i) *The set $\mathscr{B}(\Lambda)$ of biclosed subsets of Λ is a complete ortholattice. The join of a family X of biclosed subsets of Λ is given by $\bigvee X = \overline{\bigcup_{\Gamma \in X} \Gamma}$, and the ortholattice complement is just set complement in Λ.*

(ii) *The restriction of τ to a map $\mathscr{B}(\Lambda) \to \mathscr{B}(\Psi)$ is a W'-equivariant morphism of complete ortholattices,* i.e., *it preserves W'-action, preserves arbitrary meets and joins, and preserves complements.*

(iii) *If Γ is coclosed in Λ, then $\overline{\Gamma}$ is biclosed in Λ.*

(iv) *If $\Gamma \subseteq \Lambda$, then $\tau(\overline{\Gamma}) = \overline{\tau(\Gamma)}$.*

Remarks. There are several dependencies amongst parts of this conjecture, and reductions of its parts to superficially weaker statements. For example, using that $\mathscr{B}(\Lambda)$ is a complete poset, the conjecture (i) above follows easily from the special case of (iii) according to which $\overline{\Gamma \cup \Delta} \in \mathscr{B}(\Lambda)$ if $\Gamma, \Delta \in \mathscr{B}(\Lambda)$. Most of this paper is devoted to checking parts of the conjecture involving subsets of Λ that are either finite or cofinite in Λ, where Λ is a (rank one or zero) standard parabolic subset. In an earlier version of this manuscript, the conjecture was stated for arbitrary quasi-parabolic subsets, but a counterexample was found by Wang [27, Remark 8.2.9]. On the other hand, we will

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prove elsewhere the special case of the conjecture when $\Lambda = \Phi$. This implies in (ii) that $\mathscr{B}(\Psi)$ is always a complete ortholattice. Except in the case that $\Lambda = \Phi$, the conjecture is open for all infinite irreducible Coxeter groups excluding the infinite dihedral groups.

2.6 Conjectural Structure of Reflection Orders

The set of standard parabolic subsets (and their conjugates) is the largest natural class of subsets of Φ, known to the author, for which Conjecture 2.5 (i) seems plausible; for example, the set $\mathscr{B}(\Lambda)$ is not necessarily a lattice if Λ is the complement, in a finite root system Φ of type A_3, of a rank one standard parabolic subset.

However, for at least some quasi-parabolic sets Λ, the coclosed subsets of Λ may not be the largest natural family of sets, all of which have biclosed closure. Say that a subset Γ of $\Lambda = \Phi_+$ is unipodal if it has the following property: if $\alpha \in \Gamma$ and $W' \in \mathscr{M}_\alpha$ is a maximal dihedral reflection subgroup of W containing s_α, with canonical simple system $\Pi_{W'} = \{\beta, \gamma\}$ with respect to (W, S), then either $\beta \in \Gamma$ or $\gamma \in \Gamma$ (see 6.3 for notation and more details). It is easy to see that coclosed subsets of Φ_+ are unipodal, so the following strengthens the special case $\Lambda = \Phi_+$ of Conjecture 2.5 (iii).

Conjecture *If $\Gamma \subseteq \Phi_+$ is unipodal, then $\overline{\Gamma}$ is biclosed in Φ_+.*

Some evidence for this conjecture will be given in another paper; in particular, it holds for finite Coxeter groups, by an argument using Bruhat order. The conjecture would also imply that an arbitrary biclosed subset Γ of Φ_+ is the directed union of the biclosed sets obtained as the closures of finite unipodal subsets of Γ, and hence that the (conjectured) complete ortholattice $\mathscr{B}(\Phi_+)$ is an algebraic lattice (see [9] for the definition). In conjunction with Conjecture 2.2, the above conjecture would lead to a quite satisfactory description of reflection orders and their initial sections (for example, one could effectively compute with them, in examples or in general, by finite "approximations," in a similar manner as one can work with elements of profinite groups).

Note, however, that if (W, S) is an infinite dihedral Coxeter system, then there are two exceptional quasi-parabolic subsets Λ of Φ such that $\Phi = \Lambda \uplus -\Lambda$ where $\Lambda \cap \Phi_+$ and $\Lambda \cap \Phi_-$ are both infinite; for these, $\mathscr{B}(\Lambda)$ is a complete ortholattice (as conjectured in 2.5) but not an algebraic lattice.

2.7 Conjecture on Biclosed Subsets of Quasi-positive Systems

In the terminology of [17], a subset Λ of Φ is called a quasipositive system if $\Phi = \Lambda \uplus -\Lambda$. Thus, closed quasi-positive systems Λ are special quasi-parabolic subsets of Φ, and Φ_+ is itself a closed quasi-positive system. The following conjecture extends Conjecture 2.2 (ii).

Conjecture *Let Λ be any closed quasi-positive system of Φ. Let M be any maximal (under inclusion) totally ordered subset of $\mathscr{B}(\Lambda)$. Then there is a total order $\leq$ of Λ such that M consists of all initial sections of $\leq$.*

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From the examples in 3.4, one sees that the conjecture does not extend as stated to general quasi-parabolic subsets Λ of Φ.

2.8 Conjecture on Initial Sections and Bruhat Order

Define a function $\tau: \mathscr{P}(\Phi_+) \to \mathscr{P}(W)$ as follows: for any $\Gamma \subseteq \Phi_+$, $\tau(\Gamma)$ is the subset of W consisting of all elements $w \in W$ such that there exist $\alpha_1, \dots, \alpha_n \in \Gamma$ with $w = s_{\alpha_1} \cdots s_{\alpha_n}$ and $0 = l(1_W) < l(1_W s_{\alpha_1}) < \cdots < l(1_W s_{\alpha_1} \cdots s_{\alpha_n})$. The motivation and natural context for the study of this function is in relation to the twisted Bruhat orders of [13], which are not discussed in this paper. Instead, τ is used here to provide another, quite different description of the (conjectural) join in the poset $\mathscr{B}(\Phi_+)$.

Conjecture *If $\Gamma, \Lambda \in \mathscr{B}(\Phi_+)$, then $\{\alpha \in \Phi_+ \mid s_\alpha \in \tau(\Gamma \cup \Lambda)\}$ is the join (least upper bound) of Γ and Λ in the poset $\mathscr{B}(\Phi_+)$*

Remarks. The above conjecture is open even for finite Coxeter groups. As already mentioned, Conjecture 2.5 is also open for finite Coxeter groups in the cases $2 \le |J|$ and $J \ne S$; all other conjectures mentioned are known to hold for finite Coxeter groups. Some other results of this paper, such as Theorem 1.8, are special cases of general conjectures which are not stated here.

2.9 Analogous Questions for Simplicial Oriented Geometries

There is a natural *convex geometric* closure operator $d: \Gamma \mapsto \mathbb{R}_{\ge 0}\Gamma \cap \Phi$ on Φ taking a set of roots to the set of roots in its non-negative real span. It is shown in Section 11 that many of the main results of the paper hold for d just as for 2-closure. However, while every d-biclosed subset of Φ_+ is an initial section of Φ_+, it can be shown that there exist finite rank W for which not every initial section is d-biclosed, and thus the d-analogue of Conjecture 2.2 (i) fails. This is unsurprising since 2-closure and initial sections are combinatorial in nature, but d-closure is not (see Section11.4).

In Section 11 of this paper, it is shown that the d-closure analogue of Theorem 1.5 holds. We do not know whether, more generally, the analogue of Conjecture 2.5 (i) in the special case $\Lambda = \Phi_+$, but for d-closure instead of 2-closure, holds. However, [2, §5-6] proved an analogue of that conjecture in a different (and quite general) context, namely for the posets of regions of a finite central simplicial hyperplane arrangement in a finite-dimensional real vector space (we consider only essential arrangements, namely those for which the normals to the hyperplanes span the ambient vector space). More generally, such an analogue holds for simplicial oriented geometries, which are oriented matroids with additional properties; see [2, 3] for background. It would be interesting to know if Conjecture 2.5 (iii), for example, also has an analogue in that generality. There are (at least) two natural closure operators which one could use; the natural oriented matroid closure operator d (see [7] or [2, §6]), and an analogue of 2-closure constructed from d in a similar way as the 2-closure on root systems is defined in terms of their geometric d-closure. In view of the results of this paper in the case of finite root systems, it would be particularly interesting to see how the 2-closure behaves in simplicial geometries (it certainly does not have good properties for possibly non-simplicial oriented geometries in general).

Another interesting point for comparison of [2] with the results here is the following. If the poset of regions of a finite central hyperplane arrangement in a real vector space is a lattice, then the base region is simplicial [2, Theorem 3.1]. However, we show in Section 11 that the d-closure analogue of Theorem 1.5 (i) holds even if the simple roots are linearly dependent (and the fundamental chamber therefore not simplicial). These facts are not contradictory, as the Coxeter group case corresponds to an infinite hyperplane arrangement and involves only a subset of the *regions* (those in the W-orbit of the fundamental chamber, *i.e.*, in the Tits cone).

2.10 Questions on Oriented Geometry Root Systems

A final, subtle, point we wish to raise concerning the conjectures in this paper (such as 2.5) is whether 2-closure is the *natural* closure operator for use in their formulation. It may well not be, as it does not induce an antiexchange closure on the positive roots in general (see Section 11).

It is a formal fact (the proof of which is similar to that of [2, Theorem 5.5] and not given here) that if Conjecture 2.5 (i) holds for one closure relation, then the closure operator a on Λ in which the closed sets are intersections of elements of $\mathscr{B}(\Lambda)$ has $\mathscr{B}(\Lambda)$ as its a-biclosed sets and satisfies $\bigvee X = a(\bigcup_{\Gamma\in X}\Gamma)$ for $X \subseteq \mathscr{B}(\Lambda)$. It seems possible that, if the conjectures hold for any closure operator on Φ, there may be several different natural closure operators e on Φ for which they hold (some evidence for this can be seen in Section 11).

One natural candidate closure operator we wish to informally describe requires notions of infinite oriented matroids, one definition of which can be found in [7]. We consider only a natural subclass, which we shall henceforth just call oriented geometries, corresponding to oriented geometries in the case of finite, oriented matroids (see also [3, Exercise 3.13]). We shall not be more precise as we make only vague remarks below.

Take a $W \times \{\pm 1\}$-set $\Psi := T \times \{\pm 1\}$ corresponding abstractly to the $W \times \{\pm 1\}$-set of roots of (W, S) in its standard root systems (see 11.1). Consider oriented geometry structures on this set, preserved by the W-action of (W, S), which restrict to the standard oriented geometry structure on the roots of any dihedral reflection subgroup (the standard structure is the one obtained by transfer of structure from any of their standard root systems) and for which the standard positive system $\Psi_+ := T \times \{1\}$ is closed. Call such a structure an oriented geometry root system of W; any standard root system gives rise to one in this more general sense. Each oriented geometry root system gives (by definition in [7]) a closure operator on Ψ; these include analogues of the d-closures as previously considered, but one might expect there could be many more (not for finite Coxeter groups, but at least for some infinite non-affine Coxeter ones). The closure operator on Ψ of interest (as a possibly natural one for use in the conjectures) is the one which has as closed sets the intersections of closed sets for these oriented geometry root system closure operators. It would also be interesting (and possibly important in relation to the conjectures) to know to what extent the results and conjectures of this paper can be extended to the groupoids introduced in 1.10 or the related and better studied groupoids such as as those in [6, 8, 22] and [21], which is explicitly concerned with weak order on Weyl groupoids.

3 Example: Finite Dihedral Groups

In this section, a number of the objects attached to Coxeter groups in Section 1 are explicitly described in the case of a finite dihedral group W, and, at the end, illustrated even more concretely by the example of the Weyl group of type A_2. The reader is invited to consider the necessary changes for the infinite dihedral group.

3.1 Closed Sets in Finite Dihedral Root Systems

Throughout this section, we consider a finite dihedral Coxeter system (W,S) of order $2m$ with simple roots $\Pi = \{\alpha,\gamma\}$. Then

$$W = \{1 = 1_W, s_\alpha, s_\gamma, s_\alpha s_\gamma, s_\gamma s_\alpha, s_\alpha s_\gamma s_\alpha, \dots, w_S\},$$

where $w_S := (s_\alpha s_\gamma s_\alpha \cdots)_m = (s_\gamma s_\alpha s_\gamma \cdots)_m$ is the longest element, and

$$\Phi_+ = \{\alpha, s_\alpha(\gamma), s_\alpha s_\gamma(\alpha), \dots\}_m = {}_m\{\dots, s_\gamma s_\alpha(\gamma), s_\gamma(\alpha), \gamma\},$$

where each set on the previous line has m elements, and elements in corresponding positions in the two listed sets are equal. The order in which the elements are listed is one of the two (mutually opposite) reflection orders of Φ_+; they are the two possible orders in which a ray sweeping around the origin (beginning and ending with a ray containing a negative root) would pass through the positive roots. For example, when case (W,S) is of type A_2, we have $m = 3$ and $\Phi_+ = \{\alpha,\beta,\gamma\}$, where $\alpha = s_\gamma s_\alpha(\gamma)$, $\beta = \alpha+\gamma = s_\alpha(\gamma) = s_\gamma(\alpha)$, and $\gamma = s_\alpha s_\gamma(\alpha)$.

The closed subsets of Φ_+ (in the general finite dihedral case) are the sets of positive roots that can be obtained by deleting the first j and last k roots from the list

$$\alpha, s_\alpha(\gamma), s_\alpha s_\gamma(\alpha), \dots, s_\gamma s_\alpha(\gamma), s_\gamma(\alpha), \gamma$$

of elements of Φ_+ in the above order, where $j,k \in \mathbb{N}$ with $j+k \le n$. The biclosed sets are the sets Φ_w for $w \in W$; they are the empty set together with the subsets of Φ_+ that are closed and contain a simple root.

3.2 Stable Subgroups and Subsemilattices

In this subsection, the stable subgroups and subsemilattices for the Galois connection associated with R in Section 1.9 are described for the finite dihedral group W (the analogous results for R' can be obtained from this using the fact that $\Phi'_x = \Phi_{xw_S}$). Recall that xRz if and only if $z(\Phi_x) = \Phi_x$. Hence, $1Rw$ and $wR1$ for all $w \in W$. Also, wRw_S if and only if $w = 1$, and $w_S Rw$ if and only if $w = 1$.

Let $z, x \in W \setminus \{1, w_S\}$. We claim that xRz if and only if $z = s_\delta$ and $x = s_\delta w_S$ for some $\delta \in \Phi_+$ with $s_\delta \ne w_S$. Suppose that xRz holds. Then z must be a reflection, *i.e.*, of odd length, since no non-identity rotation can fix any non-empty set of positive roots (such as Φ_x) setwise. Suppose that $z = s_\delta$, where $\delta \in \Phi_+$. There is some root $\epsilon \in \Phi_x \cap \Pi$ since $x \ne 1$. It follows that $\overline{\{\epsilon, s_\delta(\epsilon)\}} \subseteq \Phi_x \subseteq \Phi'_{s_\delta}$. But it is easy to check that the left-hand side is just Φ'_{s_δ}, so equality holds throughout. Hence, $\Phi_x = \Phi'_{s_\delta} = \Phi_{s_\delta w_S}$ and $x = s_\delta w_S$. On the other hand, $s_\delta(\Phi_{s_\delta w_S}) = s_\delta(\Phi'_{s_\delta}) = \Phi'_{s_\delta} = \Phi_{s_\delta w_S}$, and the claim is proved.

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It now follows from the definitions that the pairs (G, G^*) of stable subgroup G of W and the corresponding stable meet subsemilattice G^* of W are $(G = W, G^* = \{1\})$, $(G = \{1_W\}, G^* = W)$, and $(G = \{1, s_\delta\}, G^* = \{1, s_\delta w_S\})$ for $\delta \in \Phi_+$ with $s_\delta \neq w_S$. There are thus $m+1$ stable pairs if m is odd, and $m+2$ if m is even; the map $w \mapsto \{w\}^*$ gives a bijection from the set of elements $w \in W$ with $w^2 = 1_W$ to the set of stable subsemilattices. The lattice of stable subgroups (resp., stable subsemilattices) is a poset with W as the maximum element, $\{1_W\}$ as the minimum element, and all other elements pairwise incomparable.

3.3 Standard Parabolic Weak Orders

This subsection describes the standard parabolic weak orders $\mathscr{L}_J$, where $J \subseteq S$, for the finite dihedral group W.

First, the description of weak order is well known. There is a maximum element w_S, minimum element 1, and exactly two maximal chains from 1 to w_S, namely

$$1 < s_\alpha < s_\alpha s_\gamma < s_\alpha s_\gamma s_\alpha < \cdots < w_S \quad \text{and} \quad 1 < s_\gamma < s_\gamma s_\alpha < s_\gamma s_\alpha s_\gamma < \cdots < w_S,$$

both of length m, *i.e.*, with $m+1$ elements. The poset $\mathscr{L}_\varnothing$ is order isomorphic to $(W, \leq)$ under the map $w \mapsto \Phi_w$.

The poset $\mathscr{L}_J$ for $J = \{s_\alpha\}$ can be described as follows (the description for $J = \{s_\gamma\}$ is obtained by symmetry). In this case, the poset $\mathscr{L}_J$ has a maximum element $\top := \Phi_+ \cup \{-\alpha\}$ and a minimum element $\bot = \varnothing$. The group $\{1, s_\alpha\}$ acts by order automorphisms of the poset by $(w, \Gamma) \mapsto w(\Gamma)$. There are $2n+4$ elements of $\mathscr{L}_J$, namely $\varnothing$, $\Phi_+ \smallsetminus \{\alpha\}$, $\{\alpha, -\alpha\}$, $\top$, and, for each $w \in W$ with $\alpha \in \Phi_w$, Φ_w, and $s_\alpha(\Phi_w)$. There are six maximal chains from $\bot$ to $\top$. Three of the maximal chains are

$$\varnothing < \Phi_{s_\alpha} = \{\alpha\} < \Phi_{s_\alpha s_\gamma} < \Phi_{s_\alpha s_\gamma s_\alpha} < \cdots < \Phi_{w_S} = \Phi_+ < \top$$

(of length $m+1$), $\varnothing < \{\alpha\} < \{\alpha, -\alpha\} < \top$ (of length 3) and, $\varnothing < \Phi_+ \smallsetminus \{\alpha\} < \Phi_+ < \top$ (also of length 3); the other three maximal chains are obtained by acting on these by s_α.

Finally, we describe $\mathscr{L}_S$. The group W acts on $\mathscr{L}(S)$ as a group of order automorphisms. The elements of $\mathscr{L}_S$ are the minimum element $\bot = \varnothing$, maximal element $\top = \Phi$, and elements Ψ_+, $\Psi_+ \smallsetminus \{\delta\}$, $\Psi_+ \cup \{-\delta'\}$, where Ψ_+ runs over positive systems of Φ and for each Ψ, δ and δ' run over the simple roots of Ψ_+. There are $6m+2$ elements of $\mathscr{L}_S$ in all. There are $8m$ maximal chains from $\varnothing$ to Φ, all of length 4. They are exactly the chains of the form $\varnothing < \Psi_+ \smallsetminus \{\delta\} < \Psi_+ < \Psi_+ \cup \{-\delta'\} < \Phi$ for Ψ, δ, δ' as above.

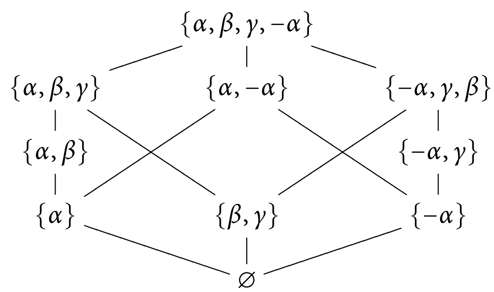
Remarks. It will be shown elsewhere in conjunction with a proof of Conjecture 2.2 for affine Weyl groups that if W is any finite Coxeter group, then $\mathscr{L}_S$ is an ortholattice with maximum element Φ, minimum element $\varnothing$, and in which every maximal chain from $\varnothing$ to Φ has length $2|S|$. However, it has not been checked that the join is as in Conjecture 2.5 (i).

3.4 The Type A_2 Case

In this subsection, we take (W, S) of type A_2 with notation as in Section 3.1, *i.e.*, $\Pi = \{\alpha, \gamma\}$ and $\Phi_+ = \{\alpha, \beta, \gamma\}$, where $\beta = \alpha + \gamma$.

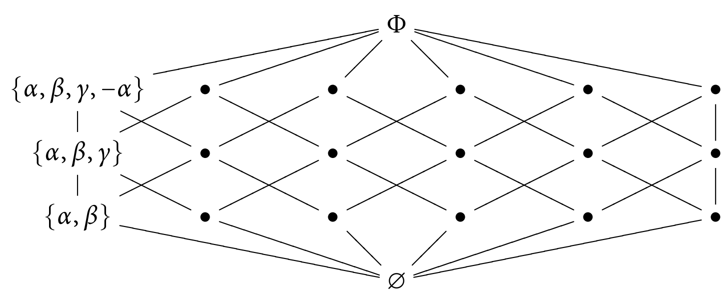
The Hasse diagrams of the stable subgroups ordered by inclusion and corresponding stable subsemilattices ordered by reverse inclusion are as follows:

The Hasse diagram of the parabolic order $\mathscr{L}_J$, where $J = \{s_\alpha\}$, is



Note that $\mathscr{L}_{\{s_\alpha\}}$ is not graded; maximal chains can have different cardinalities.

The Hasse diagram of the parabolic order $\mathscr{L}_S$ is of the form



where we have only explicitly indicated the elements of one maximal chain.

4 Preliminaries

4.1 Finite Biclosed Subsets of the Positive Roots

Recall the terminology concerning biclosed sets from Section 1.11.

Lemma (i) *For $\Gamma \subseteq \Phi_+$ and $x \in W$, set $x \cdot \Gamma := (\Phi_x \setminus x(-\Gamma)) \cup (x(\Gamma) \setminus (-\Phi_x))$. Then $(w, \Gamma) \mapsto w \cdot \Gamma$ gives an action of the group W on the power set $\mathscr{P}(\Phi_+)$.*

(ii) *$x \cdot \Phi_y = \Phi_{xy}$ and $x \cdot \Phi'_y = \Phi'_{xy}$ for $x, y \in W$.*

(iii) *If $\Gamma \subseteq \Phi_+$ is biclosed, then $w \cdot \Gamma$ is biclosed for all $w \in W$.*

(iv) *A finite subset Γ of Φ_+ is biclosed if and only if it is of the form $\Gamma = \Phi_w$ for some $w \in W$.*

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(v) $\Phi_{xy} = (\Phi_x \setminus (-x(\Phi_y))) \uplus (x(\Phi_y) \setminus (-\Phi_x))$, *where* $\Phi_x \cap x(\Phi_y) = \varnothing$, *for any* $x, y \in W$.

(vi) *For* $x, y \in W$, *we have* $x \le xy$ *if and only if* $l(xy) = l(x) + l(y)$ *if and only if* $\Phi_{x^{-1}} \cap \Phi_y = \varnothing$ *if and only if* $x(\Phi_y) \subseteq \Phi_+$ *if and only if* $\Phi_{xy} = \Phi_x \uplus x(\Phi_y)$ *if and only if* $\Phi_x \subseteq \Phi_{xy}$ *if and only if* $x(\Phi_y) \subseteq \Phi_{xy}$.

Proof There is a W-action $(w, A) \mapsto w \cdot A = N(w) + wAw^{-1}$ on $\mathscr{P}(T)$, where

$$N(w) = N_{(W,S)}(w) := \{s_\alpha \mid \alpha \in \Phi_w\} = \{t \in T \mid l(tw) < l(w)\}$$

and $+$ denotes symmetric difference. This action was used in [13,14], for instance; the fact that the formula gives an action follows from the cocycle property

$$N(xy) = N(x) + xN(y),$$

for $x, y \in W$. The W-action on $\mathscr{P}(\Phi_+)$ in (i) is easily seen to be obtained from this action by transfer of structure using the bijection $\alpha \mapsto s_\alpha : \Phi_+ \to T$. For further discussion of this action and of its geometric interpretation, see [16, 1.1–1.2].

The formula in (i) immediately shows that $\Phi_y = y \cdot \varnothing$ and then $\Phi_{xy} = (xy) \cdot \varnothing = x \cdot (y \cdot \varnothing) = x \cdot \Phi_y$, proving the first part of (ii). The second part of (ii) follows since the formula in (i) implies that $x \cdot (\Phi_+ \setminus \Gamma) = \Phi_+ \setminus (x \cdot \Gamma)$. Part (iii) is proved by reducing to the easily checked case of dihedral groups by considering the intersections of Γ with the maximal dihedral reflection subgroups (see 6.3) of W; see [20, Proposition 2.6]. Part (iv) was proved in [26] for the standard reflection representation of [4, 23] by a straightforward modification of a well-known argument for finite Weyl groups. Exactly the same argument as in [26] applies to the class of root systems we consider here to establish (iv). Another proof of (iv) is as follows. Note that since $\varnothing$ is clearly biclosed, so is $\Phi_x = x \cdot \varnothing$ for $x \in W$ by (ii). The reverse implication will be proved using the easily checked fact that for any non-empty biclosed set Δ', a root $\alpha \in \Delta'$ with s_α of minimal length must be simple; then $|s_\alpha \cdot \Delta'| + 1 = |\Delta'|$. Let Γ be any finite biclosed set and chose Δ of minimal cardinality in the orbit $W \cdot \Gamma$. If $\Delta \neq \varnothing$, applying the above with $\Delta' = \Delta$ gives a contradiction to minimality of $|\Delta|$. Hence $\varnothing \in W \cdot \Gamma$, $\Gamma \in W \cdot \varnothing$, and, say $\Gamma = x \cdot \varnothing = \Phi_x$ as required.

For (v)–(vi), note that the definitions give $\Phi_{x^{-1}} = -x^{-1}(\Phi_x)$. Part (v) is a straightforward consequence of (i)–(ii) and the fact that $x^{-1}(\Phi_x \cap x(\Phi_y)) = x^{-1}(\Phi_x) \cap \Phi_y = -\Phi_{x^{-1}} \cap \Phi_y \subseteq \Phi_+ \cap \Phi_- = \varnothing$. Then (vi) follows easily from (v) on recalling that $l(z) = |\Phi_z|$ for any $z \in W$. ∎

4.2 A More General Order Isomorphism

The following lemma will be used in Section 10 (*cf.* Corollary 1.6).

Lemma (i) *For any* $x \in W$, *the map* $\Gamma \mapsto x \cdot \Gamma$ *induces an order isomorphism*

$$\{\Gamma \in \mathscr{B}(\Phi_+) \mid \Gamma \cap \Phi_{x^{-1}} = \varnothing\} \xrightarrow{\cong} \{\Delta \in \mathscr{B}(\Phi_+) \mid \Phi_x \subseteq \Delta\}.$$

(ii) *If a non-empty subset* X *of* $\mathscr{B}(\Phi_+)$ *is such that* $\Gamma \cap \Phi_{x^{-1}} = \varnothing$ *for all* $\Gamma \in X$, *and* X *has a join* $\Lambda = \bigvee X$ *in* $\mathscr{B}(\Lambda)$, *then* $\Lambda \cap \Phi_{x^{-1}} = \varnothing$.

Proof The map in (i) is the restriction of a similar order isomorphism with $\mathscr{B}$ replaced by $\mathscr{P}$. Explicitly, the inverse bijections are $\Gamma \mapsto \Delta = \Phi_x \cup x(\Gamma)$ and $\Delta \mapsto \Gamma = x^{-1}(\Gamma \setminus \Phi_x)$. One can put $\mathscr{B}$ in place of $\mathscr{P}$ since $\mathscr{B}(\Phi_+) \subseteq \mathscr{P}(\Phi_+)$ is W-stable.

For (ii), one notes that for all $\Gamma \in X$, one has $\Gamma \subseteq \Phi'_{x^{-1}} \in \mathscr{B}(\Phi_+)$, so $\Phi'_{x^{-1}}$ is an upper bound for X and therefore $\bigvee X \subseteq \Phi'_{x^{-1}}$ *i.e.*, $\Lambda \cap \Phi_{x^{-1}} = \varnothing$. ∎

4.3 Trivial Properties of Closure

The following assorted simple facts are stated for future reference, omitting the proofs.

Lemma *Let $\Gamma, \Delta \subseteq \Phi$ and $w \in W$.*

(i) *Recursively define $\Gamma_0 := \Gamma$ and $\Gamma_{n+1} = \bigcup_{\alpha,\beta\in\Gamma_n} \overline{\{\alpha,\beta\}}$ for $n \in \mathbb{N}$. Then we have $\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \cdots$ and $\overline{\Gamma} = \bigcup_{n\in\mathbb{N}} \Gamma_n$.*

(ii) $\overline{w(\Gamma)} = w(\overline{\Gamma})$.

(iii) $\overline{\Gamma \cup \Delta} \supseteq \overline{\Gamma} \cup \Delta$ *and* $\overline{\overline{\Gamma} \cup \Delta} = \overline{\Gamma \cup \Delta}$.

5 Closure and Joins

Denote the join of a family of elements $\{x_i\}$ of W in weak order as $\bigvee_i x_i$ when it exists (which it may not). Similarly, denote the meet as $\bigwedge_i x_i$ when it exists. Also write $\bigvee X$ and $\bigwedge X$ for the join and meet of $X \subseteq W$ when they exist. This section will prove Theorem 1.5 (i), describing joins in weak order in terms of 2-closure, and show that $(W, \leq)$ is a complete meet semilattice.

5.1 Joins in Cosets of Rank Two Standard Parabolic Subgroups

The proof of Theorem 1.5 (i) begins with the following observation.

Lemma *Let $x \in W$ and $\alpha, \beta \in \Pi$ with $x < y := xs_\alpha$ and $x < z := xs_\beta$. Then y and z have an upper bound in $(W, \leq)$ if and only if the standard parabolic subgroup $W' := \langle s_\alpha, s_\beta \rangle$ of W is finite if and only if $\overline{\{\alpha,\beta\}}$ is finite. In that case, let w denote the longest element of W'. Then*

(i) $y \vee z = xw$,

(ii) $\Phi_{xw} = \Phi_x \uplus x(\Phi_w) = \Phi_x \uplus \overline{\{x(\alpha), x(\beta)\}}$,

(iii) $\Phi_{y\vee z} = \Phi_x \uplus \overline{(\Phi_y \setminus \Phi_x) \cup (\Phi_z \setminus \Phi_x)} = \overline{\Phi_y \cup \Phi_z}$.

Proof Since $x < xs_\alpha$ and $x < xs_\beta$, it follows that x is the (unique) element of minimal length in the coset xW'. Hence

$$l(xw') = l(x) + l(w'), \quad \Phi_{xw'} = \Phi_x \uplus x(\Phi_{w'}), \quad x \leq xw',$$

for all $w' \in W$. In particular, $\Phi_y = \Phi_x \uplus \{x(\alpha)\}$ and $\Phi_z = \Phi_x \uplus \{x(\beta)\}$. Also note that the set of positive roots of W' is $\overline{\{\alpha,\beta\}}$, so W' is finite if and only if $\overline{\{\alpha,\beta\}}$ is finite, and in that case the longest element w of W' satisfies $\Phi_w = \overline{\{\alpha,\beta\}}$ and

$$\Phi_{xw} = \Phi_x \uplus x\overline{\{\alpha,\beta\}} = \Phi_x \uplus \overline{\{x(\alpha), x(\beta)\}}.$$

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Finally, $u \in W$ is an upper bound of y and z if and only if $\Phi_y \subseteq \Phi_u$ and $\Phi_z \subseteq \Phi_u$. This holds if and only if $\Phi_y \cup \Phi_z \subseteq \Phi_u$ or, equivalently, if and only if $\overline{\Phi_y \cup \Phi_z} \subseteq \Phi_u$, since Φ_u is closed.

Now suppose that an upper bound of y and z exists, and let u be any such upper bound. From above, it follows that $\Phi_x \uplus \overline{\{x(\alpha), x(\beta)\}} \subseteq \overline{\Phi_y \cup \Phi_z} \subseteq \Phi_u$. In particular, $\overline{\{x(\alpha), x(\beta)\}}$ is finite, so W' is finite, $\Phi_{xw} \subseteq \Phi_u$, and $xw \leq u$. Hence if an upper bound for y and z exists, then W' is finite, and any upper bound u satisfies $u \geq xw$. On the other hand, if W' is finite, then the above proves that $\Phi_y, \Phi_z \subseteq \Phi_{xw}$, so xw is an upper bound of y and z. The first part of this paragraph with $u = xw$ shows that $y \vee z = xw$ and $\Phi_{xw} = \Phi_x \uplus \overline{\{x(\alpha), x(\beta)\}} \subseteq \overline{\Phi_y \cup \Phi_z} \subseteq \Phi_{xw}$. This proves the first assertion of the lemma, and its parts (i)–(ii). Part (iii) follows from (i)–(ii) and what has just been proved, using $\Phi_y \setminus \Phi_x = \{x(\alpha)\}$ and $\Phi_z \setminus \Phi_x = \{x(\beta)\}$. ■

5.2 Joins in Weak Order

The following proof is quite similar to that of [2, Lemma 2.1], taking account of the extra structure of concern here.

Proposition *Suppose that $x, y, z, u \in W$ with $x \leq y \leq u$ and $x \leq z \leq u$. Then $y \vee z$ exists and $\Phi_{y \vee z} = \Phi_x \uplus \overline{(\Phi_y \setminus \Phi_x) \cup (\Phi_z \setminus \Phi_x)} = \overline{\Phi_y \cup \Phi_z}$.*

Remarks. The more complicated statement here as compared to the case $|X| = 2$ of Theorem 1.5 (i) is only to facilitate the proof. The two statements are equivalent, using Corollary 1.6 (i). It can be shown that Conjecture 2.5 implies, for example, that for $\Lambda, \Gamma, \Delta \in \mathscr{B}(\Phi_+)$ with $\Lambda \subseteq \Gamma \cap \Delta$, one has $\overline{\Gamma \cup \Delta} = \Lambda \cup \overline{(\Gamma \setminus \Lambda) \cup (\Delta \setminus \Lambda)}$.

Proof Observe that the union $\Phi_x \cup \overline{(\Phi_y \setminus \Phi_x) \cup (\Phi_z \setminus \Phi_x)}$ is one of disjoint sets since $\overline{(\Phi_y \setminus \Phi_x) \cup (\Phi_z \setminus \Phi_x)} \subseteq \overline{\Phi'_x} = \Phi'_x$. Note that if $x = y$, then $y \vee z = x \vee z = z$ and the result is trivial, since Φ_z is closed. Similarly, the result is trivial if $x = z$. In particular, the result holds if $l(x) = l(u)$ (in which case $x = y = z = u$). The proposition will be proved by induction on $N := l(u) - l(x)$. Assume inductively that $N \in \mathbb{N}_{>0}$ and the assertion of the proposition holds for all x, y, z, u satisfying the hypotheses of the proposition with $l(u) - l(x) < N$. Let x, y, z, u satisfy the hypotheses with $l(u) - l(x) = N$. As above, assume without loss of generality that $x \neq y$ and $x \neq z$.

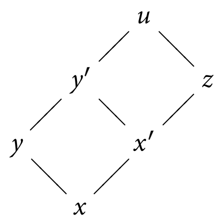
Fix a simple reflection r satisfying $x < x' := xr \leq z$. Consider the following hypothesis (H):

(H) $\quad y' := y \vee x'$ exists and $\Phi_{y'} = \Phi_x \uplus \overline{(\Phi_y \setminus \Phi_x) \cup (\Phi_{x'} \setminus \Phi_x)} = \overline{\Phi_y \cup \Phi_{x'}}$.

We claim that (H) implies the conclusion (A) of the proposition:

(A) $\quad y \vee z$ exists and $\Phi_{y \vee z} = \Phi_x \uplus \overline{(\Phi_y \setminus \Phi_x) \cup (\Phi_z \setminus \Phi_x)} = \overline{\Phi_y \cup \Phi_z}$.

Assume for the proof of the claim that (H) holds. Since $y \leq u$ and $x' \leq z \leq u$, it follows that $y' = y \vee x' \leq u$. Hence there is a diagram indicating some of the order relations in $(W, \leq)$ as follows:



Compute

$$\begin{aligned}
(5.1)\qquad \Gamma := \overline{\Phi_y \cup \Phi_z} &\supseteq \overline{(\Phi_y \smallsetminus \Phi_x) \cup (\Phi_z \smallsetminus \Phi_x)} \cup \Phi_x \\
&= \overline{(\Phi_y \smallsetminus \Phi_x) \cup (\Phi_{x'} \smallsetminus \Phi_x) \cup (\Phi_z \smallsetminus \Phi_{x'})} \cup \Phi_x \\
&= \overline{(\Phi_{y'} \smallsetminus \Phi_{x'}) \cup (\Phi_{x'} \smallsetminus \Phi_x) \cup (\Phi_z \smallsetminus \Phi_{x'})} \cup \Phi_x \\
&\supseteq \overline{(\Phi_{y'} \smallsetminus \Phi_{x'}) \cup (\Phi_z \smallsetminus \Phi_{x'})} \cup (\Phi_{x'} \smallsetminus \Phi_x) \cup \Phi_x \\
&= \overline{(\Phi_{y'} \smallsetminus \Phi_{x'}) \cup (\Phi_z \smallsetminus \Phi_{x'})} \cup \Phi_{x'} =: \Delta,
\end{aligned}$$

where we use (H) and Lemma 4.3 (iii) (resp., Lemma 4.3 (iii)) to get the third (resp., first and fourth) line. Since $l(u) - l(x') = N - 1 < N$, the inductive hypothesis implies that $y' \vee z$ exists and $\Delta = \Phi_{y' \vee z}$. Now $\Delta \supseteq \Phi_{y'} \cup \Phi_z \supseteq \Phi_y \cup \Phi_z$. Hence $\Delta = \overline{\Delta} \supseteq \overline{\Phi_y \cup \Phi_z} = \Gamma$ and so the containments in (5.1) are all equalities. Clearly, $y' \vee z$ is an upper bound for $\{y, z\}$. On the other hand, if v is any upper bound for $\{y, z\}$, then $\Phi_y \cup \Phi_z \subseteq \Phi_v$, so $\Phi_{y' \vee z} = \Gamma = \overline{\Phi_y \cup \Phi_z} \subseteq \Phi_v$, which implies $y' \vee z \leq v$. This shows that $y \vee z$ exists and in fact that $y \vee z = y' \vee z$. From (5.1) and $\Gamma = \Delta$, it follows that (A) holds. This proves the claim that (H) implies (A).

Hence we are reduced to proving (H). Fix a simple reflection s with $x < xs \leq y$. Consider the following hypothesis (H′):

(H′) $\quad xs \vee xr$ exists and $\Phi_{xs \vee xr} = \Phi_x \mathbin{\dot\cup} \overline{(\Phi_{xr} \smallsetminus \Phi_x) \cup (\Phi_{xs} \smallsetminus \Phi_x)} = \overline{\Phi_{xr} \cup \Phi_{xs}}$.

Since $x < xr \leq u$, $x < y \leq u$, and $l(u) - l(x) \leq N$, replacing (x, y, z, u, r) by (x, xr, y, u, s) in the above proof that (H) implies (A), shows that (H′) implies (H). Hence it remains only to prove (H′). But (H′) follows from Lemma 5.1, and so the proof of the proposition is complete. ∎

5.3 Meets From Joins

The following well-known, simple facts are used in the argument to show that the weak order on W is a complete meet semilattice.

Lemma *Let $(\Lambda, \leq)$ be a poset with a minimum element, denoted $\perp$, such that for every $x \in \Lambda$, $\{w \in W \mid w \leq x\}$ is finite. Assume that any two elements of Λ with an upper bound have a least upper bound. Then Λ is a complete meet semi-lattice,* i.e., *any non-empty subset of Λ has a greatest lower bound. Furthermore, any non-empty subset with an upper bound has a least upper bound.*

Proof The assumptions imply (by induction on $|B|$) that the join $\bigvee B$ exists for any finite non-empty subset B of Λ with an upper bound. Since any non-empty subset with an upper bound is finite, it has a least upper bound, proving the last assertion. Consider now any non-empty subset A of Λ. We need to show that the greatest lower bound $\bigwedge A$ exists. Consider the set B of lower bounds of A. For any $a \in A$, we have $b \leq a$ for all $b \in B$, so B is finite, non-empty (it contains $\perp$), and bounded above. It follows B has a join $b := \bigvee B$. For any $a \in A$, we have that a is an upper bound of B and so $b \leq a$. Hence b is a lower bound of A, *i.e.*, $b \in B$. This implies that b is the maximum element of B, *i.e.*, $b = \bigwedge A$ as required. ■

5.4 The Semi-lattice Property of Weak Order

The corollary below summarizes the main results of this section.

Corollary (i) *$(W, \leq)$ is a complete meet semilattice.*

(ii) *If a subset of W has an upper bound in W, then it has a join in $(W, \leq)$ given by the formula in Theorem 1.5* (i)*.*

(iii) *If W is finite, then the meet in W is given by the formula in Theorem 1.5* (ii)*.*

Proof The join of a family of two elements (with an upper bound) is given by Theorem 1.5 (i). Then the formula in 1.5 (i) follows for finite subsets X by induction on $|X|$. The formula in 1.5 (i) then applies to any non-empty subset X with an upper bound, since such a set X is finite, proving (ii). Part (i) now follows from Lemma 5.3. To prove (iii), recall that (assuming W finite) the longest element w_S of W satisfies $w_S\Phi_+ = -\Phi_+$, $w_S^2 = 1_W$, and $\Phi'_x = \Phi_{xw_S}$ for all $x \in W$. Further, the map $x \mapsto xw_S$ is an order-reversing bijection of W with itself. (See [1, 2.3.1, 2.3.2, 3.2.2] for these well-known facts). Now Theorem 1.5 (ii) for finite W follows easily from Theorem 1.5 (i) using these facts. ■

Remarks. If $X = \{x, y\} \subseteq W$ and $\bigvee X$ exists, a similar (essentially, dual) argument to the proof of Proposition 5.2 shows that the meet $\bigwedge X$ is given by the formula in Theorem 1.5 (ii); in particular, this argument can be extended to give another proof of Corollary 5.4 (iii). However, X need not have a join, so one needs a different argument to prove Theorem 1.5 (ii) in general.

5.5 Proof of Corollary 1.6

Corollary 1.6 (i) is an easy consequence of the definitions [1, Proposition 3.1.6]. Using Lemma 4.1 (vi), Corollary 1.6 (ii) says that the domain D_x of the order isomorphism in (i) is closed under taking those joins that exist in W. This holds since

$$D_x := \{u \in W \mid \Phi_u \subseteq \Phi'_{x^{-1}}\},$$

so if $U \subseteq D_x$, then $\Phi_y = \overline{\bigcup_{u\in U} \Phi_u} \subseteq \overline{\bigcup_{u\in U} \Phi'_{x^{-1}}} = \Phi'_{x^{-1}}$ by Theorem 1.5 (i) and therefore $y \in D_x$.

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6 Closure and Chevalley–Bruhat Order

This section proves Lemma 1.7 after giving requisite background on the Bruhat graph and reflection subgroups.

6.1 Bruhat Graph

Define an edge-labelled, directed graph called the Bruhat graph $\Omega = \Omega_{(W,S)}$ of (W,S) as follows [12]. The vertex set of Ω is W. There is an edge (x,y) from x to y for each $x,y \in W$ such that $l(x) < l(y)$ and $y = s_\alpha x$ for some (necessarily unique) $\alpha \in \Phi_+$. Let $E = E_{(W,S)}$ denote the set of edges of Ω. Endow Ω with an edge labelling by attaching to the edge $(x,y) \in E$ as above the label $L_{x,y} := \alpha$ where $\alpha \in \Phi_+$ with $y = s_\alpha x$. For any subset V of W, let $\Omega(V)$ denote the full edge-labelled subgraph of Ω on vertex set V, *i.e.*, with edge set $E \cap (V \times V)$).

6.2 Bruhat Order

The Chevalley–Bruhat order, which is denoted here either as $\leq_\varnothing$ to distinguish it from weak order or as $\leq_{W,\varnothing}$ to indicate dependence on W, is the partial order on W defined by the condition that $x \leq_\varnothing y$ if and only if there is $n \in \mathbb{N}$ and a path of length n from x to y in E, *i.e.*, there exist $x = x_0, x_1, \dots, x_n = y$ in W such that $(x_{i-1}, x_i) \in E$ for $i = 1, \dots, n$. Write $[x,y]_\varnothing = [x,y]_{W,\varnothing} := \{z \in W \mid x \leq_\varnothing z \leq_\varnothing y\}$.

Note that the Hasse diagram of the Chevalley–Bruhat order, when regarded as a directed graph with edges (x,y) for $x,y \in W$ with $x <_\varnothing y$ and $l(y) = l(x)+1$, is a subgraph of Ω. Then $\Phi_{x,1}$ (resp., $\Phi_{x,-1}$) as defined in Section 1.7 is the set of labels in Ω of edges of the directed Hasse diagram with x as initial (resp., terminal) vertex, corresponding to the elements which cover (resp., which are covered by) x in the order $\leq_\varnothing$.

6.3 Maximal Dihedral Reflection Subgroups

From [11], any reflection subgroup W' of W, *i.e.*, a subgroup $W' = \langle W' \cap T \rangle$, has a canonical set of Coxeter generators $\chi(W') = \{t \in T \mid N(t) \cap W' = \{t\}\}$ (with respect to the simple reflections S of W). Always consider W' as a Coxeter group with simple reflections $\chi(W')$, unless otherwise stated. Recall that W' has a root system $\Phi_{W'} := \{\alpha \in \Phi \mid s_\alpha \in W\}$ (in the class of root systems considered in [18]) with positive roots $\Phi_+ \cap \Phi_{W'}$ and simple roots $\Pi_{W'} := \{\alpha \in \Phi_+ \mid s_\alpha \in \chi(W')\}$. The maximal dihedral reflection subgroups of W are the dihedral reflection subgroups, *i.e.*, those generated by two distinct reflections) which are maximal under inclusion amongst the dihedral reflection subgroups; equivalently, they are the reflection subgroups W' with $|\Pi_{W'}| = 2$ and $\Phi_{W'} = \Phi \cap \mathbb{R}\Pi_{W'}$ [12, Remark 3.2]. Let $\mathscr{M}$ be the set of all maximal dihedral reflection subgroups. For $\alpha \in \Phi$, let $\mathscr{M}_\alpha := \{W' \in \mathscr{M} \mid s_\alpha \in W'\}$.

6.4 Reflection Subgroups and the Bruhat Graph

The following lemma collects some basic facts about cosets of reflection subgroups in relation to the Bruhat graph.

Lemma (i) *Let W' be a reflection subgroup of W, $S' := \chi(W')$. For any $x \in W$, there is a unique element $u = x'_{W'}$ of $W'x$ of minimal length $l(u)$. For any $w \in W'$, one has $N_{(W,S)}(wu) \cap W' = N_{(W',S')}(w)$. The map $z \mapsto zu$ induces an isomorphism of edge-labelled directed graphs $\Omega_{(W',S')} \xrightarrow{\cong} \Omega_{(W,S)}(W'x)$.*

(ii) *Let $(x,y) \in E$ and $\alpha = L(x,y)$. For any $W' \in \mathscr{M}_\alpha$, let $u = x'_{W'}$ denote the element of minimal length in $W'x = W'y$, $x_{W'} := xu^{-1} \in W'$, and $y_W := yu^{-1} = s_\alpha x_{W'} \in W'$. Then $x_{W'} \leq_{W',\varnothing} y_{W'}$ and the map $z \mapsto zu$ induces an isomorphism of edge-labelled directed graphs $\Omega_{W'}([x_{W'}, y_{W'}]_{W',\varnothing}) \to \Omega_W([x,y]_{W,\varnothing} \cap xW')$.*

(iii) *If $(x,y) \in E$, $L(x,y) = \alpha$, and $l(y) - l(x) \geq 3$, there is some $W' \in \mathscr{M}$ such that $l_{W'}(y_{W'}) - l_{W'}(x_{W'}) \geq 3$, where $l_{W'}$ is the length function on $(W', \chi(W'))$.*

Proof Part (i) is from [11, 12], and (ii) follows from (i) (see also [13, (1.4)]). An ad hoc proof of (iii) is given in [12]. A more natural argument for (iii) is to note that in the identity

$$l(y) - l(x) - 1 = \sum_{W' \in \mathscr{M}_\alpha} (l_{W'}(y_{W'}) - l_{W'}(x_{W'}) - 1), \tag{6.1}$$

which holds for $(x,y) \in E$ with $y = s_\alpha x$, $\alpha \in \Phi_+$, the left-hand side and terms on the right are even elements of $\mathbb{N}$, since

$$y_{W'} = s_\alpha x_{W'} >_{W',\varnothing} x_{W'}.$$

This identity and argument are given in a more general context as [13, (1.2.1), (2.7)–(2.8)]. A simple direct proof of (6.1) in the special situation here can be given as follows. Since $\alpha \in \Phi_+$, it follows that $\Phi_+ \setminus \{\alpha\} = \dot\bigcup_{W' \in \mathscr{M}_\alpha} (\Phi_{W',+} \setminus \{\alpha\})$. Since $\alpha \in \Phi_y$, (i) implies that

$$\begin{aligned} l(y) - 1 = |\Phi_y \setminus \{\alpha\}| &= \sum_{W' \in \mathscr{M}_\alpha} |(\Phi_y \setminus \{\alpha\}) \cap \Phi_{W'<+}| \\ &= \sum_{W' \in \mathscr{M}_\alpha} |\Phi_{y_{W'}, W'} \setminus \{\alpha\}| = \sum_{W' \in \mathscr{M}_\alpha} (l_{W'}(y_{W'}) - 1). \end{aligned}$$

Similarly, since $\alpha \notin \Phi_x$,

$$l(x) = |\Phi_x| = \sum_{W' \in \mathscr{M}_\alpha} |\Phi_x \cap \Phi_{W',+}| = \sum_{W' \in \mathscr{M}_\alpha} |\Phi_{x_{W'}, W'}| = \sum_{W' \in \mathscr{M}_\alpha} l_{W'}(x_{W'})$$

and (6.1) follows on subtracting. ∎

6.5 Closure and the Bruhat Graph (Proof of Lemma 1.7).

For any $\Gamma \subseteq \Phi_+$, write $\widehat{\Gamma} = \bigcup_{\alpha,\beta \in \Gamma} \overline{\{\alpha, \beta\}}$. Recall that $\Phi_{x,n} = \varnothing$ for even n. By Lemma 4.3 (i), it will suffice to prove the following two assertions for any $x \in W$ and $n \in \mathbb{N}$.

(i) Let $\Gamma \subseteq \Phi'_x$. Then $\widehat{\Gamma} \cap \Phi_{x,1} = \Gamma \cap \Phi_{x,1}$, and if $\Gamma \supseteq \bigcup_{j \in \mathbb{N}_{\leq n}} \Phi_{x,2j+1}$, then

$$\widehat{\Gamma} \supseteq \bigcup_{j \in \mathbb{N}_{\leq n+1}} \Phi_{x,2j+1}.$$

(ii) Let $\Gamma \subseteq \Phi_x$. Then $\widehat{\Gamma} \cap \Phi_{x,-1} = \Gamma \cap \Phi_{x,-1}$, and if $\Gamma \supseteq \bigcup_{j \in \mathbb{N}_{\leq n}} \Phi_{x,-(2j+1)}$, then

$$\widehat{\Gamma} \supseteq \bigcup_{j \in \mathbb{N}_{\leq n+1}} \Phi_{x,-(2j+1)}.$$

The proof of both parts is by reduction to the case of dihedral groups using Lemma 6.4; the proof is given only for (i), since that of (ii) is entirely similar.

Let $\Gamma \subseteq \Phi'_x$. Obviously, $\Gamma \subseteq \widehat{\Gamma}$ implies that $\Gamma \cap \Phi_{x,1} \subseteq \widehat{\Gamma} \cap \Phi_{x,1}$. For the reverse inclusion, suppose $\alpha \in (\widehat{\Gamma} \setminus \Gamma) \cap \Phi_{x,1}$. Then $\alpha = c\beta + d\gamma$, where $c, d \in \mathbb{R}_{\geq 0}$ and $\beta, \gamma \in \Gamma$. Since $\alpha \notin \Gamma$, it follows that $c, d > 0$ and $\beta \neq \gamma$. There is $W' \in \mathscr{M}$ with $\Phi_{W'} = \Phi \cap (\mathbb{R}\beta + \mathbb{R}\gamma)$. In fact, $W' \in \mathscr{M}_\alpha$. For $\delta \in \{\alpha, \beta, \gamma\}$, we have $l(s_\delta x) > l(x)$, and so from Lemma 6.4 (ii), $l_{W'}(s_\delta xu) > l_{W'}(xu)$, where $u := x'^{-1}_{W'}$. Suppose the notation is chosen so that $l_{W'}(s_\beta xu) \leq l_{W'}(s_\gamma xu)$. From the well-known descriptions of dihedral groups and their root systems, one checks that the above conditions imply that $l_{W'}(s_\alpha xu) > l_{W'}(s_\beta xu)$, and hence there is a path of non-zero length from $s_\beta xu$ to $s_\alpha xu$ in $\Omega_{W'}$. From 6.4 (ii) again, it follows that there is a path of non-zero length from $s_\beta x$ to $s_\alpha x$ in Ω and so $l(x) + 1 = l(s_\alpha x) > l(s_\beta x) \geq l(x) + 1$, a contradiction which completes the proof that $\Gamma \cap \Phi_{x,1} = \widehat{\Gamma} \cap \Phi_{x,1}$.

To prove the second part of (i), take $\Gamma \subseteq \Phi'_x$ with $\Gamma \supseteq \bigcup_{j \in \mathbb{N}_{\leq n}} \Phi_{x,2j+1}$. Let $\alpha \in \Phi_{x,2n+3}$ *i.e.*, $\alpha \in \Phi_+$ with $l(s_\alpha x) = l(x) + (2n+3)$. It will suffice to show that $\alpha \in \widehat{\Gamma}$. Since $l(s_\alpha x) - l(x) \geq 3$, Lemma 6.4 (iii) implies that there exists $W' \in \mathscr{M}_\alpha$ such that $l_{W'}(s_\alpha xu) - l_{W'}(xu) \geq 3$, where $u := x'^{-1}_{W'}$. Now there are distinct roots $\beta, \gamma \in \Phi_{W',+}$ such that $l_{W'}(s_\beta xu) = l_{W'}(s_\gamma xu) = l_{W'}(xu) + 1$. Again using the descriptions of dihedral groups and their root systems, one checks that $\alpha \in \mathbb{R}_{>0}\beta + \mathbb{R}_{>0}\gamma$ and that there are paths of non-zero lengths in $\Omega_{W'}$ from $s_\beta xu$ to $s_\alpha xu$ and from $s_\gamma xu$ to $s_\alpha xu$. By Lemma 6.4 (ii) again, there are paths in Ω of non-zero length from $s_\beta x$ to $s_\alpha x$ and from $s_\gamma x$ to $s_\alpha x$. Hence, $l(x) < l(s_\beta x) < l(s_\alpha x) = l(x) + 2n + 3$ and $l(x) < l(s_\gamma x) < l(s_\alpha x) = l(x) + 2n + 3$. This implies that $\beta, \gamma \in \bigcup_{j \in \mathbb{N}_{\leq n}} \Phi_{x,2j+1} \subseteq \Gamma$ and so $\alpha \in \overline{\{\beta, \gamma\}} \subseteq \widehat{\Gamma}$ as claimed.

7 Closure and Meets

In this section, Theorem 1.5 (ii) is deduced from the more general statement Theorem 7.1, which is itself a special case of Conjecture 2.5.

7.1 Cofinite Closures of Coclosed Sets Are Biclosed

Theorem *Let Γ be any coclosed subset of Φ_+ such that $\overline{\Gamma}$ has finite complement in Φ_+. Then $\overline{\Gamma}$ is biclosed,* i.e., $\overline{\Gamma} = \Phi'_x$ *for some $x \in W$.*

Proof The following trivial fact is used below. If v is a non-minimum, non-maximum element of a dihedral reflection subgroup W' in its Chevalley–Bruhat order $\leq_{W',\varnothing}$, then one can write $\Pi_{W'} = \{\alpha', \beta'\}$, where $s_{\alpha'} v <_{W',\varnothing} v <_{W',\varnothing} s_{\beta'} v$.

The theorem is proved by induction on $n := |\Phi_+ \setminus \overline{\Gamma}|$. If $n = 0$, then $\overline{\Gamma} = \Phi_+$ is obviously biclosed. Suppose next that $n > 0$, *i.e.* $\overline{\Gamma} \neq \Phi_+$. By Lemma 1.7 (ii), there is $\alpha \in \Pi \setminus \Gamma$. Abbreviate $s := s_\alpha$. We claim that $\Gamma' = s \cdot \Gamma$ is coclosed; this follows by a similar reduction to the dihedral case as that for Lemma 4.1 (iii) (note $\alpha \notin \Gamma$ is essential this time, since it is so in the dihedral case). Since $\alpha \notin \Gamma \subseteq \Phi_+$, it follows that $\alpha \notin \overline{\Gamma}$. Hence, $\overline{\Gamma'} = \overline{\{\alpha\} \cup s(\Gamma)} \supseteq \{\alpha\} \cup \overline{s(\Gamma)} = \{\alpha\} \cup s(\overline{\Gamma})$. This shows that the map $\beta \mapsto s(\beta)$ indexes an injection $\Phi_+ \setminus \overline{\Gamma'} \to \Phi_+ \setminus (\overline{\Gamma} \cup \{\alpha\})$, and hence that $|\Phi_+ \setminus \overline{\Gamma'}| < |\Phi_+ \setminus \overline{\Gamma}|$.

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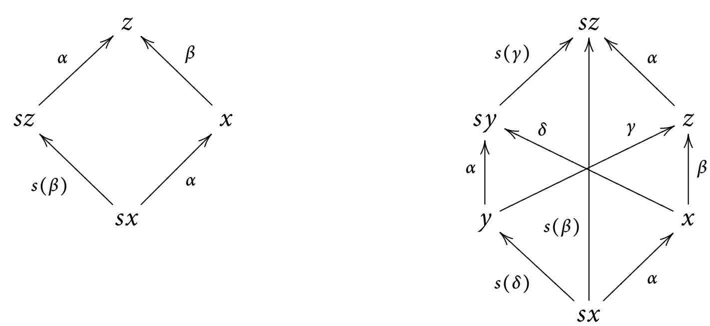
By induction, there exists $x \in W$ such that $\overline{\Gamma'} = \Phi'_{sx}$. Since $\alpha \in \overline{\Gamma'}$, this implies that $\alpha \in \Phi'_{sx}$, $\alpha \notin \Phi_{sx}$, and $\alpha \in \Phi_x$, *i.e.*, $l(sx) < l(x)$. To prove the theorem, it will be shown that $\overline{\Gamma} = \Phi'_x$.

By Lemma 1.7 (ii), $\Gamma' \supseteq \Phi_{sx,1}$ and it will suffice to show that $\Gamma \supseteq \Phi_{x,1}$. Let $\beta \in \Phi_{x,1}$, *i.e.*, $\beta \in \Phi_+$ with $l(s_\beta x) = l(x) + 1$. Let $z := s_\beta x$. By the Z-property of the Bruhat order [13] and the fact that each length two Bruhat interval has four elements, the situation is as in one of the following two diagrams that show vertices, edges, and edge labels appearing in paths (all paths from sx to z if $l(sz) < l(z)$, or from sx to sz if $l(sz) > l(z)$) in the Bruhat graph Ω.



Consider first the case that $l(sz) < l(z)$. Then $s(\beta) \in \Phi_{sx,1} \setminus \{\alpha\} \subseteq \Gamma' \setminus \{\alpha\} = s(\Gamma)$, so $\beta \in \Gamma$ as desired in this case. Consider now the contrary case that $l(sz) > l(z)$. Let $W' \in \mathscr{M}$ with $\Phi_{W'} = \Phi \cap (\mathbb{R}\alpha + \mathbb{R}\beta)$. Multiplying the vertex labels of this diagram by $u := x'^{-1}_{W'}$ on the right gives a corresponding diagram in $\Omega_{W'}$. One necessarily has $l_{W'}(szu) = l_{W'}(sxu) + 3$.

Since $\alpha \in \Pi_{W'}$, inspecting the vertex xu of the resulting diagram and using the trivial fact at the start of the proof shows that either $\Pi_{W'} = \{\alpha, \delta\}$ or $\Pi_{W'} = \{\alpha, \beta\}$. The first case cannot occur, since then syu would be the longest element of W', which it is not, since it is the initial vertex of an edge (syu, szu). Therefore, $\Pi_{W'} = \{\alpha, \beta\}$ and $\delta \notin \Pi_{W'}$. Now $\Gamma \cap \Phi_{W',+}$ is coclosed in $\Phi_{W',+}$, and from the first case, it follows that $\delta \in \Gamma \cap \Phi_{W',+}$. By examining the possible coclosed sets in dihedral groups, one sees that this implies that either $\alpha \in \Gamma \cap \Phi_{W',+}$ (which is false here, since $\alpha \notin \Gamma$) or $\beta \in \Gamma \cap \Phi_{W',+}$. Hence, $\beta \in \Gamma$ whether $l(sz) < l(z)$ or $l(sz) > l(z)$. This shows that $\Phi_{x,1} \subseteq \Gamma$ and completes the proof. ∎

7.2 Closure and Meets (Proof of Theorem 1.5 (ii))

To prove Theorem 1.5 (ii), consider $\overline{\Delta}$, where X is a non-empty subset of W and $\Delta := \bigcup_{x \in X} \Phi'_x$. Note that $\overline{\Delta}$ has finite complement in Φ_+ (as Δ itself does, since $|X| \geq 1$). Also, $\bigcup_{x \in X} \Phi'_x$ is a union of biclosed sets, so it is coclosed, and therefore $\overline{\Delta} = \Phi'_y$ for some $y \in W$ by Theorem 7.1. One can check that $y = \bigwedge X$ in weak order as follows. For any $x \in X$, $\Phi'_x \subseteq \overline{\Delta} = \Phi'_y$ implies $\Phi_y \subseteq \Phi_x$, so y is a lower bound for X. On the other hand, if z is any lower bound of X, then $\Phi_z \subseteq \Phi_x$ for all $x \in X$, so $\Delta \subseteq \Phi'_z$. Taking closures, $\Phi'_y = \overline{\Delta} \subseteq \Phi'_z$ so $\Phi_z \subseteq \Phi_y$ and $z \leq y$. Hence $y = \bigwedge X$ and 1.5 (ii) follows.

7.3 Closure of the Union of Finite and Cofinite Biclosed Sets

The following corollary is another special case of Conjecture 2.5 (i).

Corollary *Let $x, y \in W$. Then $\overline{\Phi_x \cup \Phi'_y} = \Phi'_z$ for some $z \in W$. The element z is the maximum element in weak order of the set $\{w \in W \mid w \le y, l(x^{-1}w) = l(x) + l(w)\}$.*

Proof This follows easily from the theorem by taking $\Gamma := \Phi_x \cup \Phi'_y$. The details are omitted. ■

8 Closure of the Union of a Biclosed Set and a Root

In general, if Γ is biclosed and $\alpha \in \Phi_+$, then $\overline{\Gamma \cup \{\alpha\}}$ need not be biclosed. There need not even be a unique inclusion-minimal biclosed set containing $\overline{\Gamma \cup \{\alpha\}}$; for example, consider $\Gamma = \varnothing$, (W, S) of type A_2 and α the highest root. Theorem 1.8, which is proved in this section, gives sufficient (but far from necessary) conditions to ensure that such a closure is biclosed when either Γ or $\Phi_+ \smallsetminus \Gamma$ finite.

8.1 Closures in Dihedral Groups

The proof of Theorem 1.8 is by reduction to the case of dihedral groups. The following trivial lemma isolates some relevant properties of dihedral groups for the proof of Theorem 1.8 (i).

Lemma *Assume that (W, S) is dihedral. Let $\alpha \in \Phi_+$.*

(i) *For $w \in W$ with $l(s_\alpha w) = l(w) + 1$, exactly one of the following three possibilities occurs.*
   (a) *$\alpha \in \Pi$, $w = 1_W$, and $\overline{(\Phi_w \cup \Phi_{s_\alpha w}) \smallsetminus \{\alpha\}} = \varnothing$.*
   (b) *$\alpha \in \Pi$, $w \neq 1_W$, and $\overline{(\Phi_w \cup \Phi_{s_\alpha w}) \smallsetminus \{\alpha\}} = \Phi_+ \smallsetminus \{\alpha\}$.*
   (c) *$\alpha \notin \Pi$, $\Phi_{s_\alpha w} = \Phi_w \uplus \{\alpha\}$, and $\overline{(\Phi_w \cup \Phi_{s_\alpha w}) \smallsetminus \{\alpha\}} = \Phi_w$.*

(ii) *Assume that $\alpha \notin \Pi$. If W is infinite, there is a unique $w \in W$ satisfying* (i) (c)*. If W is finite, there are exactly two elements $w \in W$ satisfying* (i) (c)*; denoting them as w' and w'', one has $\Phi_+ = \Phi_{w'} \uplus \{\alpha\} \uplus \Phi_{w''}$.*

8.2 Closure After Adjoining a Root (Proof of Theorem 1.8).

We prove (i). As in Lemma 6.4, for any $W' \in \mathscr{M}_\alpha$ and $p \in W$, write $p = p_{W'} p'_{W'}$, where $p_{W'} \in W'$ and $p'_{W'}$ is the element of minimal length in $W'p$. Also write

$$\Phi_{p,W'} := \Phi_p \cap \Phi_{W'} = \Phi_{W',+} \cap p'_{W'}(\Phi_{W',+}),$$

where the right-hand equality is by Lemma 6.4 (i). Note $\alpha \notin \Phi_x$. Set $z = s_\alpha x$. Then $z'_{W'} = x'_{W'}$ and $z_{W'} = s_\alpha x_{W'}$. From 6.4 (ii), it follows that $l_{W'}(s_\alpha x_{W'}) = l_{W'}(z_{W'}) = l_{W'}(x_{W'}) + 1$. Lemma 8.1 implies that $\overline{\Phi_{x_{W'},W'} \cup \{\alpha\}} = \overline{\Phi_{x_{W'},W'} \cup \Phi_{s_\alpha x_{W'},W'}}$. So

$$\overline{\Phi_x \cup \{\alpha\}} = \overline{\bigcup_{W' \in \mathscr{M}_\alpha} \Phi_{x_{W'},W'} \cup \{\alpha\}} = \overline{\bigcup_{W' \in \mathscr{M}_\alpha} (\Phi_{x_{W'},W'} \cup \Phi_{s_\alpha x_{W'},W'})} = \overline{\Phi_x \cup \Phi_{s_\alpha x}}.$$

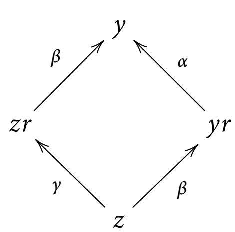
Since $\Phi_x \cup \{\alpha\} \subseteq \Phi_v$, it follows that $\overline{\Phi_x \cup \Phi_{s_\alpha x}} = \overline{\Phi_x \cup \{\alpha\}} \subseteq \Phi_v$, *i.e.*, v is an upper bound for x and $s_\alpha x$ in weak order. Let $y := x \vee s_\alpha x$, so by Theorem 1.5 (i), $\overline{\Phi_x \cup \Phi_{s_\alpha x}} = \overline{\Phi_x \cup \{\alpha\}} = \Phi_y \subseteq \Phi_v$. If $z \in W$ with $x \le z$ and $\alpha \in \Phi_z$, then $\Phi_y = \overline{\Phi_x \cup \{\alpha\}} \subseteq \Phi_z$. Hence, y is the minimal element of $\{z \in W \mid x \le z, \alpha \in \Phi_z\}$.

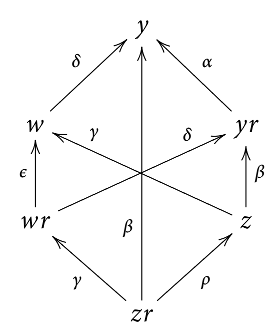
Clearly, $\alpha \in \Phi_y$, so $l(s_\alpha y) < l(y)$. Since $x \le y$, it is possible to write $y = xu$, where $l(y) = l(x) + l(u)$. Note that $u \ne 1_W$, since $\alpha \in \Phi_y \setminus \Phi_x$. Choose $\tau \in \Pi$ such that $l(us_\tau) < l(u)$. Then $x \le ys_\tau < y$. Since $\Phi_y = \overline{\Phi_x \cup \{\alpha\}}$, it follows that $\alpha \notin \Phi_{ys_\tau}$ and therefore $\alpha \in \Phi_y \setminus \Phi_{ys_\tau} = \{-y(\tau)\}$. In particular, $s_\alpha y = ys_\tau < y$ and $\Phi_{s_\alpha y} = \Phi_y \setminus \{\alpha\}$. Obviously, $\overline{(\Phi_x \cup \Phi_{s_\alpha x}) \setminus \{\alpha\}} \subseteq \Phi_y \setminus \{\alpha\} = \Phi_{s_\alpha y}$ since $\Phi_y \setminus \{\alpha\}$ is closed. To complete the proof of (i), it will suffice to verify the following claim:

$$\Phi_{s_\alpha y} \subseteq \overline{(\Phi_x \cup \Phi_{s_\alpha x}) \setminus \{\alpha\}}.$$

From Lemma 1.7, it follows both that $\Phi_x \cup \{\alpha\} \supseteq \Phi_{y,-1}$ and that it will suffice to show that $(\Phi_x \cup \Phi_{s_\alpha x}) \setminus \{\alpha\} \supseteq \Phi_{s_\alpha y,-1}$.

Abbreviate $s_\alpha = s$, $s_\tau = r$ so $sy = yr$ and $r \in S$. Let $\beta \in \Phi_{sy,-1}$, *i.e.*, $\beta \in \Phi_+$ with $l(s_\beta sy) = l(sy) - 1$. Set $z := s_\beta sy$. Similarly as in Section 7.1, we have one of the possibilities indicated by the following diagrams of vertices, edges, and edge labels appearing in paths (all paths from z to y if $l(zr) > l(z)$, or from zr to y if $l(zr) < l(z)$) in the Bruhat graph Ω.





Consider first the case that $l(zr) > l(z)$. Then $\beta \in \Phi_{y,-1} \setminus \{\alpha\} \subseteq \Phi_x$, so $\beta \in (\Phi_x \cup \Phi_{sx}) \setminus \{\alpha\}$ as desired in this case. Consider now the contrary case that $l(zr) < l(z)$. Let $W' \in \mathscr{M}$ with $\Phi_{W'} = \Phi \cap (\mathbb{R}\alpha + \mathbb{R}\beta)$. Multiplying the vertex labels of the second diagram by $u := y'^{-1}_{W'}$ on the right, gives a corresponding diagram in $\Omega_{W'}$. One necessarily has $l_{W'}(yu) = l_{W'}(zru) + 3$. Note that $s_\alpha(x_{W'}) = (s_\alpha x)_{W'}$ and $s_\alpha(y_{W'}) = (s_\alpha y)_{W'}$. By the argument in the first case, $\delta \in \Phi_x$. Hence ,$\delta \in \Phi_{x,W'}$ and $x_{W'} \ne 1_{W'}$. If $\alpha \in \Pi_{W'}$, then Lemma 8.1 (i) gives

$$\beta \in \Phi_{W',+} \setminus \{\alpha\} = \overline{(\Phi_{x,W'} \cup \Phi_{s_\alpha x,W'}) \setminus \{\alpha\}} \subseteq \overline{(\Phi_x \cup \Phi_{s_\alpha x}) \setminus \{\alpha\}}$$

as required. So we can assume that $\alpha \notin \Pi_{W'}$. Lemma 8.1 (i) implies that $\delta \in \Phi_{x,W'} = \Phi_{s_\alpha x,W'} \setminus \{\alpha\}$. Also, since $\Phi_{s_\alpha y} = \Phi_y \setminus \{\alpha\}$, we have $\delta \in \Phi_{s_\alpha y,W'} = \Phi_{y,W'} \setminus \{\alpha\}$. Thus, Lemma 8.1 (ii) implies that $x_{W'} = s_\alpha y_{W'}$. From the above diagram, $\beta \in \Phi_y$ so

$$\beta \in \Phi_{y,W'} \setminus \{\alpha\} = \Phi_{s_\alpha y,W'} = \Phi_{x,W'} \subseteq \overline{(\Phi_{x,W'} \cup \Phi_{s_\alpha x,W'}) \setminus \{\alpha\}} \subseteq \overline{(\Phi_x \cup \Phi_{s_\alpha x}) \setminus \{\alpha\}}$$

as required.

The proof of Theorem 1.8 (ii) is very similar to that of Theorem 1.8 (i) and is omitted.

8.3 Fibering the Closure Over a Newly Adjoined Root

Theorem 1.8 implies that Φ_y in (i) (resp., Φ'_y in (ii)) is well fibered over α in the following sense: for any $W' \in \mathscr{M}_\alpha$, the intersection of Φ_y (resp., Φ'_y) with the plane spanned by the roots of W' consists of all positive roots lying in a fixed one of the two closed half-planes in that plane bounded by the line spanned by α; for all but the finitely many $W' \in \mathscr{M}_\alpha$ for which $s_\alpha \notin \Pi_{W'}$, this intersection is either $\Phi_{W',+}$ or $\{\alpha\}$.

Higher-dimensional analogues of this phenomenon, involving closures of sets obtained by adjoining all roots of a suitable reflection subgroup, may be expected but remain conjectural in general. The corollary below is the simplest result of this type.

Corollary *Let W_J be a finite parabolic subgroup of W with longest element w_J. Let $x \in W$ such that $l(w_J x) = l(w_J) + l(x)$ and $y := w_J \vee x$ exists in weak order on W. Write $y = w_J z$.*

(i) *$u \vee z = uz$ and $\Phi_{u\vee z} = \Phi_u \uplus \Phi_z$ for all $u \in W_J$.*
(ii) *For any $\alpha \in \Phi_{W_J,+}$ and any $W' \in \mathscr{M}_\alpha$, either $\Phi_{W',+} \subseteq \Phi_y$ or $\Phi_{W',+} \cap \Phi_y = \{\alpha\}$.*

Proof Note that x and z are the minimal length elements of their cosets $W_J x$ and $W_J z$, respectively, so for any $u \in W_J$, $l(ux) = l(u) + l(x)$ and $l(uz) = l(u) + l(z)$.

It is well known that the map $s \mapsto w_J s w_J$ defines a bijection $\theta: J \to J$. Now for any $s \in J$, $w_J z = w_J \vee x = (s \vee w_{J\setminus\{s\}}) \vee x = s \vee (w_{J\setminus\{s\}} \vee x)$. Since $\Phi_{w_{J\setminus\{s\}}} \cap \Phi_s = \varnothing = \Phi_x \cap \Phi_s$, Corollary 1.6 implies that $\Phi_{w_{J\setminus\{s\}}\vee x} \cap \Phi_s = \varnothing$. From Theorem 1.8, it follows that $w_J z = s \vee s w_J z$. Using that θ is a bijection,

$$(w_J s)z = \theta(s) w_J z \le \theta(s) \vee \theta(s) w_J z = w_J z = (w_J s)(sz).$$

Since $l((w_J s)z) = l(w_J s) + l(z)$ and $l((w_J s))(sz)) = l(w_J s) + l(sz)$, this implies that $z \le sz$. Write $sz = zs'$, where $s' \in S$. Varying s gives a bijection $s \mapsto s': J \to K$ for some subset K of S. This bijection extends to a group isomorphism $W_J \to W_K$ that will be denoted as $u \mapsto u'$. Note that $uz = zu'$ for all $u \in W_J$. Now z is the minimal length element in $W_J z$ and in zW_K. Let $u \in W_J$. Then $l(uz) = l(zu') = l(z) + l(u) = l(z) + l(u')$. This implies that $z \le uz$, $u \le uz$, and therefore $z \vee u \le uz$. Since

$$l(z \vee u) = |\Phi_{z\vee u}| \ge |\Phi_z \cup \Phi_u| = |\Phi_z \uplus \Phi_u| = |\Phi_z| + |\Phi_u| = l(u) + l(z) = l(uz),$$

it follows that $z \vee u = uz$ and $\Phi_{u\vee z} = \Phi_u \uplus \Phi_z$, proving (i).

Now let α, W' be as in (ii). If $W' \subseteq W_J$, then $\Phi_{W',+} \subseteq \Phi_{W_J,+} = \Phi_{w_J,+} \subseteq \Phi_y$. Otherwise, $\Phi_{W_J} \cap \Phi_{W',+} = \{\alpha\}$ (since W_J finite-parabolic implies $\Phi_{W_J} = \Phi \cap \mathbb{R}\Phi_{W_J}$). This implies that $\alpha \in \Pi_{W'}$. Choose $u \in W_J$, $r \in J$ so that $s_\alpha = uru^{-1}$ and $l(s_\alpha) = 2l(u) + 1$. Then $s_\alpha u = ur > u$, so $\overline{\{\alpha\} \cup \Phi_u} = \Phi_{s_\alpha u}$. Also, $s_\alpha uz = urz = uzr' > uz$ so $\overline{\{\alpha\} \cup \Phi_{uz}} = \Phi_{s_\alpha uz}$. From the remarks at the start of this subsection, either $\Phi_{W'} \cap \Phi_{s_\alpha uz} = \Phi_{W',+}$ or $\Phi_{W'} \cap \Phi_{s_\alpha uz} = \{\alpha\}$. Since $\Phi_{s_\alpha uz} = \Phi_{s_{\alpha u}} \uplus \Phi_z$ and $\Phi_y = \Phi_{w_J z} = \Phi_{w_J} \uplus \Phi_z$ with $\Phi_{s_\alpha u} \cap \Phi_{W',+} = \{\alpha\} = \Phi_{w_J} \cap \Phi_{W',+}$, (ii) follows. ∎

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9 Galois Connections

In this section, Theorem 1.9 and its dual are proved.

9.1 First Galois Connection (Proof of Theorem 1.9)

Assume that $x, z \in W$ with xRz, *i.e.*, $z(\Phi_x) = \Phi_x$. Then $\Phi_x \cap \Phi_{z^{-1}} = \varnothing$ and $l(zx) = l(z) + l(x)$. Also, xRz^{-1} holds since $z^{-1}(\Phi_x) = \Phi_x$. This proves (ii). Note that $\Phi_x \cup \Phi_z = \Phi_z \uplus z(\Phi_x) = \Phi_{zx}$, so clearly $z \vee x = zx$. This proves "$\Rightarrow$" for the left implication in (i). For the reverse implication, suppose that $x \vee z = zx$ and $\Phi_x \cap \Phi_z = \varnothing$. From the first equation, $z \leq zx$, so $l(zx) = l(z) + l(x)$. Also, the two equations imply that $\Phi_x \uplus \Phi_z \subseteq \Phi_{zx}$, where the left-hand side has cardinality $l(x)+l(z)$ and the right-hand side has cardinality $l(zx) = l(z)+l(x)$. This implies that $\Phi_x \uplus \Phi_z = \Phi_{zx} = \Phi_z \uplus z(\Phi_x)$ and so $z(\Phi_x) = \Phi_x$. It remains to prove the equivalence on the right-hand side of (i). The implication "$\Leftarrow$" is trivial, and the converse follows since, if $\Phi_x \cap \Phi_z = \varnothing$, then

$$|\Phi_{x\vee z}| \geq |\overline{\Phi_x \uplus \Phi_z}| \geq |\Phi_x| + |\Phi_z| = l(z) + l(x) \geq l(zx) = |\Phi_{zx}|.$$

This completes the proof of (i).

For $X \subseteq W$, $X^{\dagger} = \{z \in W \mid z(\Phi_x) = \Phi_x \text{ for all } x \in X\}$ is clearly a subgroup of W, proving (iii). To prove (iv), consider a subset Z of W and let

$$\mathscr{L} := Z^{*} = \{x \in W \mid x \vee z = zx, \Phi_x \cap \Phi_z = \varnothing \quad \text{for all } z \in Z\}.$$

Obviously $1_W \in \mathscr{L}$. Suppose, given a non-empty subset A of $\mathscr{L}$, say with $a' \in A$, let $a'' := \bigwedge A$ in $(W, \leq)$. Let $z \in Z$. Note that $a'' \leq a'$ implies that $\Phi_z \cap \Phi_{a''} \subseteq \Phi_z \cap \Phi_{a'} = \varnothing$ for all $z \in Z$. Furthermore, $l(za) = l(z) + l(a)$ for all $a \in A$; in particular, since $a'' \leq a' \in A$, it follows that $l(za'') = l(z) + l(a'')$ and

$$z \vee a'' = z \vee \big(\bigwedge A\big) \leq \bigwedge_{a\in A} (z \vee a) = \bigwedge_{a\in A} za = z\big(\bigwedge_{a\in A} a\big) = za'' \tag{9.1}$$

using Corollary 1.6 (i). Since $\Phi_z \cap \Phi_{a''} = \varnothing$, we have

$$l(z \vee a'') = |\Phi_{z \vee a''}| \geq |\Phi_z \uplus \Phi_{a''}| = |\Phi_z| + |\Phi_{a''}| = l(z) + l(a'') = l(za''),$$

and the equality holds throughout (9.1). Hence $a'' \in \mathscr{L}$ and so a'' is obviously the greatest lower bound of A in $\mathscr{L}$. This proves (iv). It remains to prove (v). Maintain the notation in the proof of (iv), but suppose now that A has a join b in $(W, \leq)$. From $\Phi_z \cap \Phi_a = \varnothing$ and $l(za) = l(z) + l(a)$ for all $a \in A$, it follows that $\Phi_b \cap \Phi_z = \varnothing$ and $l(zb) = l(z) + l(b)$ by Corollary 1.6 (ii). Hence,

$$z \vee b = z \vee \big(\bigvee A\big) = \bigvee_{a\in A} (z \vee a) = \bigvee_{a\in A} za = z\big(\bigvee_{a\in A} a\big) = zb$$

by Corollary 1.6 (i). This completes the proof of (v) and of the theorem.

9.2 Second Galois Connection

The dual result to Theorem 1.9 will now be formulated and its proof sketched. Define a relation R' on W by $xR'z$ if and only if $z(\Phi'_x) = \Phi'_x$ for $x, z \in W$. Define the two maps $X \mapsto X^{\dagger'}$ and $Z \mapsto Z^{*'}$ from $\mathscr{P}(W) \to \mathscr{P}(W)$ by replacing R by R' in the

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analogous definition in Section 1.9. Also define the corresponding families of stable subsets

$$\mathscr{W}'_* := \{\mathscr{L} \in \mathscr{P}(W) \mid \mathscr{L}^{\dagger'*'} = \mathscr{L}\} = \{Z^{*'} \mid Z \in \mathscr{P}(W)\},$$
$$\mathscr{W}'_\dagger := \{\mathscr{G} \in \mathscr{P}(W) \mid \mathscr{G}^{*'\dagger'} = \mathscr{G}\} = \{X^{\dagger'} \mid X \in \mathscr{P}(W)\}.$$

Theorem (i) *One has*

$$xR'z \iff (\overline{\Phi'_x \cup \Phi_z} = \Phi'_{zx} \text{ and } \Phi'_x \cap \Phi_z = \varnothing) \iff \Phi_z \uplus \Phi'_x = \Phi'_{zx}.$$

(ii) *If $xR'z$, then $z \leq x$, $l(x) - l(z) = l(z^{-1}x)$ and $xR'z^{-1}$.*

(iii) *The elements of $\mathscr{W}'_\dagger$ (other than, perhaps, W) are finite subgroups of W.*

(iv) *The elements of $\mathscr{W}'_*$ are (possibly empty) complete meet subsemilattices of $(W, \leq)$. One has $W^{*'} = \varnothing$ if W is infinite, and otherwise $W^{*'} = \{w_S\}$, where w_S is the longest element of W.*

(v) *If $L \in \mathscr{W}'_*$, then for any subset X of L, which has an upper bound in W, its join $x = \bigvee X$ in W is an element of L (and so x is the least upper bound of X in L).*

(vi) *Let $w \in \mathscr{I} = \mathscr{I}(W) := \{w \in W \mid w^2 = 1_W\}$. Then $wR'w$. The stable subgroup $\{w\}^{*'\dagger'}$ is a subgroup of W contained in $\{x \in W \mid x \leq w\}$ and containing w. The corresponding stable subsemilattice $\{w\}^{*'}$ is contained in $\{x \in W \mid x \geq w\}$ and has w as minimum element. The map $w \to \{w\}^{*'}$ gives an injection*

$$i\colon \mathscr{I}(W) \longrightarrow \mathscr{W}'_* \smallsetminus \{\varnothing\}.$$

Proof The proofs of (i)–(v) are similar to those of the corresponding parts of Theorem 1.9 and most of the details are omitted, except to remark that finiteness of the stable subgroups in (iii) and the statement about $W^{*'}$ in (iv) follow using that $xR'z$ implies $z \leq x$. We prove (vi). Let $w \in W$. Then

$$w^{-1}(\Phi'_w) = w^{-1}(\Phi_+ \cap w(\Phi_+)) = \Phi_+ \cap w^{-1}(\Phi_+) = \Phi'_{w^{-1}}.$$

Hence, if $w^2 = 1$, then $w(\Phi'_w) = \Phi'_w$ *i.e.*, $wR'w$. This implies that w is the minimum element of $\{w\}^{*'}$ and the maximum element of $\{w\}^{\dagger'}$ in weak order. All statements of (vi) follow readily. ∎

Remarks.

• The map i in (vi) is not a bijection in general. Let W be of type A_4 with simple reflections $S = \{r, s, t, u\}$ and Coxeter graph

$$r \text{ ——— } s \text{ ——— } t \text{ ——— } u\ .$$

One checks that $\langle s \rangle^\dagger = \langle u, rstsr \rangle$, $\langle t \rangle^\dagger = \langle r, stuts \rangle$ and $\langle s, t \rangle^\dagger = \langle rstutsr \rangle$ (by direct calculation, or by using results of [6]), where $\langle A \rangle$ is the subgroup generated by A. This gives three stable subgroups in which $rstutsr$ is an element of maximal length. But if i is a bijection, these stable subgroups would be uniquely determined by their maximum element in weak order (which would be an involution), by (f).

• There is a partial order $\preceq$ defined on the set $\mathscr{I}$ by $v \preceq w$ if and only if

$$\{v\}^{*'\dagger'} \subseteq \{w\}^{*'\dagger'}.$$

Clearly, $v \preceq w$ implies $v \leq w$, but the reverse implication fails by Section 3.2.

- For finite W, the stable subgroups $X^{\dagger'}$ or $X^{\dagger}$ are not necessarily subsemilattices of W in weak order; they also need not be reflection subgroups of W. Also, the stable subsemilattices Z^* or $Z^{*\prime}$ need not be subgroups.
- The stable subgroups $X^{\dagger}$ need not be Coxeter groups for infinite W, as the groups $\{s\}^{\dagger}$ for $s \in S$ may include non-trivial free groups by [5]. We do not know (for infinite or finite W) if the (necessarily finite) stable subgroups $X^{\dagger'}$ for $X \neq \varnothing$ are always Coxeter groups.

10 Rank One Parabolic Weak Orders

This section gives a proof of Theorem 1.13.

10.1 Joins With a Simple Reflection

The following lemma collects special cases and consequences of the main results already proved, for use in the proof of Theorem 1.13

Lemma *Let $s \in S$, say $s = s_\alpha$, where $\alpha \in \Pi$.*

(i) *For $x \in W$, we have $x \in \{s\}^*$ if and only if $s(\Phi_x) = \Phi_x$ if and only if $\Phi_{sx} = \Phi_x \uplus \{\alpha\}$.*
(ii) $\{s\}^* = \{x \in W \mid sx > x\}$.
(iii) *$\{s\}^*$ is closed under taking meets, and those joins that exist, in W.*
(iv) *If $x \in W$, $l(sx) = l(x) + 1$, and $\{s, x\}$ has an upper bound, then $s \vee x = sy$ for some $y \in \{s\}^*$.*

Proof Part (i) follows from Theorem 1.9 (i). For (ii), note first that if $x \in \{s\}^*$, then $\Phi_x \subseteq \Phi_{sx}$ by (i), so $x < sx$. On the other hand, if $sx > x$, then $\alpha \notin \Phi_x$, so $\Phi_{sx} \supseteq \Phi_x \uplus \{\alpha\}$. But $|\Phi_{sx}| = l(sx) = l(x) + 1 = |\Phi_x \uplus \{\alpha\}|$, so equality holds, proving (ii). Part (iii) is a special case of Theorem 1.9 (iv)–(v). Part (iv) follows from Theorem 1.8(i), Remark 1.8, and (ii). ■

10.2 Notation for a Rank One Parabolic Weak Order

For the remainder of this section, fix $s \in S$, say $s = s_\alpha$, where $\alpha \in \Pi$, and set $J = \{s\}$. Let $\Lambda := \Lambda_J = \Phi_+ \cup \{-\alpha\} = \Phi_+ \cup s(\Phi_+)$ and $\mathscr{L} = \mathscr{L}_s := \mathscr{L}_{\{s\}}$. Then $\Phi(\mathscr{L}) = \{\alpha, -\alpha\}$ and $W(\mathscr{L}) = \{1, s\}$. In particular, every subset of $\Phi(\mathscr{L})$ is biclosed in both Λ and $\Phi(\mathscr{L})$, the biclosed subsets of $\Phi(\mathscr{L})$ form a complete lattice and $\tau(\overline{\Gamma}) = \tau(\Gamma) = \overline{\tau(\Gamma)}$ is biclosed in $\Phi(\mathscr{L})$ for any $\Gamma \subseteq \Lambda_J$. Recall that $W(\mathscr{L})$ acts on $\mathscr{L}$ as a group of order automorphisms by $(w, \Lambda) \mapsto w(\Lambda)$ satisfying $\tau(w(\Gamma)) = w(\tau(\Gamma))$ for biclosed subsets Γ of Λ and $w \in W(\mathscr{L})$.

10.3 Description of Elements of the Weak Order by Type

The proof of Theorem 1.13 uses the following lemma describing the elements Γ of $\mathscr{L}_s$ according to their type $\tau(\Gamma) \subseteq \{\alpha, -\alpha\}$.

Lemma (i) $\{\Gamma \in \mathscr{L}_s \mid \tau(\Gamma) = \varnothing\} = \{\Phi_x \mid x \in W, x < sx\}$,

(ii) $\{\Gamma \in \mathscr{L}_s \mid \tau(\Gamma) = \{\alpha\}\} = \{\Phi_x \mid x \in W, \alpha \in \Phi_x\}$,
(iii) $\{\Gamma \in \mathscr{L}_s \mid \tau(\Gamma) = \{-\alpha\}\} = \{s(\Phi_x) \mid x \in W, \alpha \in \Phi_x\}$,
(iv) $\{\Gamma \in \mathscr{L}_s \mid \tau(\Gamma) = \{\alpha, -\alpha\}\} = \{\Phi_x \cup \{-\alpha\} \mid x \in W, sx < x\}$.

Proof Note first that $\Psi_+ := s(\Phi_+) = (\Phi_+ \smallsetminus \{\alpha\}) \cup \{-\alpha\}$ is another positive system for Φ, with simple roots $s(\Pi)$. Also, $\Lambda = \Phi_+ \cup \Psi_+$. It follows readily that if $\Gamma \subseteq \Lambda$, then Γ is closed (resp., biclosed) in Λ, if and only if $\Gamma \cap \Phi_+$ is closed (resp., biclosed) in Φ_+ and $\Gamma \cap \Psi_+$ is closed (resp., biclosed) in Ψ_+. So by Lemma 4.1 (iv), for any $\Gamma \subseteq \Lambda$, one has $\Gamma \in \mathscr{L}_s$ if and only if there are $x, z \in W$ with $\Gamma \cap \Phi_+ = \Phi_x$ and $\Gamma \cap \Psi_+ = s_\alpha(\Phi_z)$. In that case, $\alpha \in \Phi_x$ (resp., $\alpha \in \Phi_z$) if and only if $\alpha \in \tau(\Gamma)$ (resp., $-\alpha \in \tau(\Gamma)$).

The inclusions "$\subseteq$" will be proved first. Let $\Gamma \in \mathscr{L}_s$ and let x, z be as above. If $\tau(\Gamma) = \{\alpha\}$ (resp., $\tau(\Gamma) = \{-\alpha\}$), the above immediately shows $\Gamma = \Phi_x \ni \alpha$ (resp., $\Gamma = s(\Phi_z)$ with $\alpha \in \Phi_z$) which gives the inclusion "$\subseteq$" in (ii)–(iii). For "$\subseteq$" in (i), assume that $\Gamma \in \mathscr{L}_s$ with $\tau(\Gamma) = \varnothing$. Then $\alpha \notin \Phi_x$, $\alpha \notin \Phi_z$, and $\Phi_x = s(\Phi_z)$. Hence $\Phi_{sz} = s(\Phi_z) \uplus \{\alpha\} = \Phi_x \uplus \{\alpha\}$. Therefore $\Phi_{x \vee s} = \overline{\Phi_x \cup \{\alpha\}} = \Phi_{sz}$. Lemma 10.1 gives $z < sz$ and $\Phi_{sz} = \Phi_z \uplus \{\alpha\}$, so $x = z$ and $x < sx$, proving "$\subseteq$" in (i). Next, the proof of the inclusion "$\subseteq$" in (iv) is given. In this case, $\alpha \in \Phi_x$, $\alpha \in \Phi_z$, and $\Phi_x \smallsetminus \{\alpha\} = \Gamma \cap \Phi_+ \cap \Psi_+ = s(\Phi_z) \smallsetminus \{-\alpha\} = \Phi_{sz}$. This gives $sz < x$, $sz \vee s = x$, and $\Phi_x \smallsetminus \{\alpha\} = \Phi_{sx}$, so $sx = sz$, $x = z$, $sx < x$ and $\Gamma = \Phi_x \cup \{-\alpha\}$.

Next, the reverse inclusions "$\supseteq$" are proved, again using the characterization of elements of $\mathscr{L}_s$ in the first paragraph of the proof. For (i), suppose $\Gamma = \Phi_x$, where $x < sx$. By Lemma 10.1, $\Phi_x = s(\Phi_x)$. So $\Gamma \cap \Phi_+ = \Phi_x$, and $\Gamma \cap \Psi_+ = s(\Phi_x)$. For (ii), suppose $\Gamma = \Phi_x$, where $\alpha \in \Phi_x$. Then $\Gamma \cap \Phi_+ = \Phi_x$ and $\Gamma \cap \Psi_+ = \Phi_x \smallsetminus \{\alpha\} = s(\Phi_{sx})$. For (iii), suppose $\Gamma = s(\Phi_x)$, where $\alpha \in \Phi_x$. Then $\Gamma \cap \Psi_+ = s(\Phi_x)$ and $\Gamma \cap \Phi_+ = s(\Phi_x) \smallsetminus \{-\alpha\}) = \Phi_{sx}$. Finally, for (iv) suppose that $\Gamma = \Phi_x \cup \{-\alpha\}$, where $sx < x$. From Lemma 10.1, it follows that $\Phi_{sx} = s(\Phi_{sx}) = \Phi_x \smallsetminus \{\alpha\}$. Hence, $\Gamma \cap \Phi_+ = \Phi_x$ and $\Gamma \cap \Psi_+ = (\Phi_x \smallsetminus \{\alpha\}) \cup \{-\alpha\} = s(\Phi_x)$. This completes the proof in all cases. ■

10.4 Proof of Theorem 1.13

Part (i) will be proved first. Note for any $\Gamma \in \mathscr{L}$, there are only finitely many $\Delta \in \mathscr{L}$ with $\Delta \subseteq \Gamma$ (since Γ is finite). Hence any subset X of $\mathscr{L}$ with an upper bound is finite. So by induction, one is reduced to proving the result for joins $\Gamma \vee \Delta$ of two elements (when the join exists). More precisely, it needs to be shown that

$$\Gamma \vee \Delta = \overline{\Gamma \cup \Delta} \tag{10.1}$$

whenever Γ, Δ have an upper bound Σ in $\mathscr{L}$. Note that obviously $\Sigma \supseteq \overline{\Gamma \cup \Delta}$ for any such upper bound (in particular, for $\Sigma = \Gamma \vee \Delta$ if the join exists). To prove (10.1), there are sixteen cases depending on the types $\tau(\Gamma)$ and $\tau(\Delta)$. To facilitate reductions of some of the cases to others, (10.1) will first be proved in the special case that $\Delta = \{\alpha\}$. The symmetry $\Gamma \longleftrightarrow \Delta$ given by interchanging Γ and Δ and the symmetry given by the $W(\mathscr{L})$-action will also be used to reduce the number of cases to be considered.

Suppose then that $\Delta = \{\alpha\}$. If $\alpha \in \Gamma$, then $\Gamma \vee \Delta = \Gamma = \overline{\Gamma \cup \Delta}$; it can be assumed that $\tau(\Gamma) \subseteq \{-\alpha\}$. If $\tau(\Gamma) = \varnothing$, then $\Gamma = \Phi_x$, where $x < sx$. Then $\overline{\Gamma \cup \Delta} = \overline{\Phi_x \cup \{\alpha\}} = \Phi_x \cup \{\alpha\} = \Phi_{sx}$. Since $\Phi_{sx} \in \mathscr{L}$, it follows that $\overline{\Gamma \cup \Delta} = \Phi_{sx} = \Gamma \vee \Delta$ in this case. Next, consider the case that $\tau(\Gamma) = \{-\alpha\}$. In this case, $\Gamma = s(\Phi_x)$, where $\alpha \in \Phi_x$. By

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assumption, there is is some $\Sigma \in \mathscr{L}$ with $\Gamma \cup \Delta \subseteq \Sigma$. Necessarily, $\tau(\Sigma) = \{\alpha, -\alpha\}$, so $\Sigma = \Phi_z \cup \{-\alpha\}$, where $sz < z$. Now $\Gamma = s(\Phi_x) = \Phi_{sx} \cup \{-\alpha\}$. Hence,

$$\overline{\Gamma \cup \Delta} = \overline{\Phi_{sx} \cup \{\alpha\} \cup \{-\alpha\}} \subseteq \Phi_z \cup \{-\alpha\},$$

which implies that $\overline{\Phi_{sx} \cup \{\alpha\}} \subseteq \Phi_z$. Therefore, $sx \vee s$ exists and $\Phi_{sx \vee s} = \overline{\Phi_{sx} \cup \{\alpha\}}$. By Lemma 10.1, it follows that $sx \vee s = y$, where $sy < y$. So

$$\overline{\Gamma \cup \Delta} = \overline{\Phi_{sx} \cup \{\alpha\} \cup \{-\alpha\}} \supseteq \overline{\Phi_{sx} \cup \{\alpha\}} \cup \{-\alpha\} = \Phi_y \cup \{-\alpha\}.$$

But $\Gamma \cup \Delta = \Phi_{sx} \cup \{\alpha, -\alpha\} \subseteq \Phi_y \cup \{-\alpha\} \in \mathscr{L}$. Hence, the join of Γ and Δ exists and is given by $\Gamma \vee \Delta = \overline{\Gamma \cup \Delta} = \Phi_y \cup \{-\alpha\}$. This completes the proof when $\Delta = \{\alpha\}$. Using the above-mentioned symmetry, (10.1) also holds when $\Delta = \{-\alpha\}$, $\Gamma = \{\alpha\}$ or $\Gamma = \{-\alpha\}$.

Next observe the following. Suppose that for $i = 1, 2$, $\Delta_i \in \mathscr{L}$ is such that for all $\Gamma \in \mathscr{L}$ such that Γ, Δ_i have an upper bound in $\mathscr{L}$, they have a join $\Gamma \vee \Delta_i = \overline{\Gamma \cup \Delta_i}$. Assume also that Δ_1, Δ_2 have an upper bound, so $\Delta_1 \vee \Delta_2 = \overline{\Delta_1 \cup \Delta_2}$. For any $\Gamma \in \mathscr{L}$ for which Γ and $\Delta_1 \vee \Delta_2$ have an upper bound Σ, Lemma 4.3 (iii) implies that

$$\overline{\Gamma \cup (\Delta_1 \vee \Delta_2)} = \overline{\Gamma \cup \Delta_1 \cup \Delta_2} = \overline{(\Gamma \vee \Delta_1) \cup \Delta_2} = (\Gamma \vee \Delta_1) \vee \Delta_2 = \Gamma \vee (\Delta_1 \vee \Delta_2),$$

noting that Σ is also an upper bound for Γ, Δ_1 and for $\Gamma \vee \Delta_1, \Delta_2$.

Using the previous two paragraphs, it follows that (10.1) holds if $\Delta \subseteq \{\alpha, -\alpha\}$ or $\Gamma \subseteq \{\alpha, -\alpha\}$. The next step is to reduce to the case that $\tau(\Gamma) = \tau(\Delta)$. Suppose in general that $\Gamma, \Delta \in \mathscr{L}$ have an upper bound Σ. Set $\Xi := \overline{\tau(\Gamma) \cup \tau(\Delta)} \subseteq \{\alpha, -\alpha\}$. Note $\Xi \in \mathscr{L}$. Then Σ is an upper bound for Γ, Δ, and Ξ. Set $\Gamma' := \Gamma \vee \Xi = \overline{\Gamma \cup \Xi}$ and $\Delta' = \Delta \vee \Xi = \overline{\Delta \cup \Xi}$. Then $\tau(\Gamma') = \tau(\Delta') = \Xi$, $\overline{\Gamma \cup \Delta} = \overline{\Gamma' \cup \Delta'}$, and Σ is an upper bound for Γ', Δ'. If it is known that Γ', Δ' have a least upper bound $\Gamma' \vee \Delta' = \overline{\Gamma' \cup \Delta'}$, it follows that Γ, Δ have the join $\Gamma \vee \Delta = \overline{\Gamma \cup \Delta}$. That is, the proof of (10.1) is reduced to its special case in which $\tau(\Gamma) = \tau(\Delta)$.

So now suppose $\Gamma, \Delta \in \mathscr{L}$ have an upper bound Σ and are arbitrary except that $\tau(\Gamma) = \tau(\Delta) = \Xi$. Write $\Sigma \cap \Phi_+ = \Phi_z$, where $z \in W$. It is necessary to consider four cases, according to the value of Ξ.

Case 1: $\Xi = \varnothing$. Write $\Gamma = \Phi_x$ and $\Delta = \Phi_y$, where $x < sx$ and $y < sy$. Then $x \in \{s\}^*$ and $y \in \{s\}^*$. Observe that $\Phi_x, \Phi_y \subseteq \Phi_z$. Hence $w := x \vee y$ exists in $(W, \leq)$, with $\Phi_w = \overline{\Phi_x \cup \Phi_y}$. From Lemma 1.9, it follows that $w \in \{s\}^*$, so $w < sw$ and $\Phi_w \in \mathscr{L}$. Hence, $\overline{\Gamma \cup \Delta} = \Phi_w = \Gamma \vee \Delta$, completing the proof in this case.

Case 2: $\Xi = \{\alpha\}$. Here we can write $\Gamma = \Phi_x$, $\Delta = \Phi_y$, where $\alpha \in \Phi_x \cap \Phi_y$. Then $x, y \leq z$, so $w := x \vee y$ exists with $\Phi_w = \overline{\Phi_x \cup \Phi_y}$. Since $\alpha \in \Phi_w$, this immediately implies that $\Phi_w \in \mathscr{L}$ and $\overline{\Gamma \cup \Delta} = \Phi_w = \Gamma \vee \Delta$ as required.

Case 3: $\Xi = \{-\alpha\}$. The third case reduces immediately to the second case by using the symmetry given by the action of s on $\mathscr{L}$.

Case 4: $\Xi = \{\alpha, -\alpha\}$. Here write $\Gamma = \Phi_x \cup \{-\alpha\}$ and $\Delta = \Phi_y \cup \{-\alpha\}$, where $sx < x$ and $sy < y$. Then $sx \in \{s\}^*$ and $sy \in \{s\}^*$. Also, $\Phi_{sx}, \Phi_{sy} \subseteq \Phi_z$, so $sx \vee sy$ exists. Write $sx \vee sy = sw$, so $\Phi_{sw} = \overline{\Phi_{sx} \cup \Phi_{sy}}$ and $sw \in \{s\}^*$ by Lemma 1.9. In particular, $sw < w$, so $\Phi_w \cup \{-\alpha\} \in \mathscr{L}$. Now

$$\overline{\Gamma \cup \Delta} = \overline{\Phi_{sx} \cup \Phi_{sy} \cup \{\alpha\} \cup \{-\alpha\}} \supseteq \Phi_{sw} \cup \{\alpha, -\alpha\} = \Phi_w \cup \{-\alpha\} \in \mathscr{L}.$$

On the other hand, $sx, sy \leq sw$ imply $x, y \leq w$ and $\Gamma \cup \Delta \subseteq \Phi_w \cup \{-\alpha\}$. It follows that $\overline{\Gamma \cup \Delta} = \Gamma \vee \Delta$.

This completes the proof of (i).

Using Lemma 5.3, it follows that $\mathscr{L}$ is a complete meet semilattice. The proof of (ii) involves a few additional facts supplementing those of Lemma 10.1. First, one checks using Lemma 10.1 (i) that

$$x \in s^* \iff s(\Phi'_{sx}) = \Phi'_{sx} \iff \Phi'_x = \Phi'_{sx} \cup \{\alpha\}.$$

Using this and Corollary 7.3, it follows that for $x, y, z \in W$, if $\overline{\Phi_x \cup \Phi'_y} = \Phi'_z$, $x \in s^*$, and $sy \in s^*$, then $sz \in s^*$. To prove (ii), note first that the meet of any non-empty subset X of $\mathscr{L}$ is equal to the (directed) intersection of the family of meets of its finite subsets. Since $\mathscr{L}$ has a minimum element and finite intervals, the proof of (ii) easily reduces to that of its special case for meets of finite subsets, and then by induction it reduces to the case of meets of pairs of elements of $\mathscr{L}$. The proof of this is very similar to that of (i). It is required to show that for $\Gamma, \Delta \in \mathscr{L}$, we have

$$\Lambda \setminus (\Gamma \wedge \Delta) = \overline{(\Lambda \setminus \Gamma) \cup (\Lambda \setminus \Delta)}. \tag{10.2}$$

Again, there are 16 possible cases initially. To reduce the number of cases, one first shows that for $\Gamma \in \mathscr{L}$ and $\Xi \subseteq \{\alpha, -\alpha\}$ one has

$$\overline{(\Lambda \setminus \Gamma) \cup \Xi} = \Lambda \setminus \Delta, \tag{10.3}$$

for some $\Delta \in \Lambda$. This is trivial for $\Xi = \varnothing$. One checks (10.3) first for $\Xi = \{\alpha\}$ using Lemma 10.2, as in (i). Next, (10.3) follows using the $W(\Lambda)$-action for $\Xi = \{-\alpha\}$, and finally it follows for $\Xi = \{\alpha, -\alpha\}$ on writing

$$\overline{(\Lambda \setminus \Gamma) \cup \{\alpha, -\alpha\}} = \overline{\overline{(\Lambda \setminus \Gamma) \cup \{\alpha\}} \cup \{-\alpha\}} = \overline{(\Lambda \setminus \Delta_1) \cup \{-\alpha\}} = \Lambda \setminus \Delta,$$

for some Δ_1, Δ in $\mathscr{L}$. Finally, using (10.3) one reduces (10.2) to the case $\tau(\Gamma) = \tau(\Delta)$ (as in (i)) which one checks (also as in (i), but using Lemma 4.2 instead of Corollary 1.6, where necessary). The proof of (ii) is in fact slightly simpler than that of (i) since the meets in W involved in the proof of (ii) automatically exist, whereas the existence of the necessary joins in W had to be checked. Further details are omitted.

Finally, it remains to prove (iii). Suppose that $X \subseteq \mathscr{L}$ has an upper bound. Then

$$\tau(\bigvee X) = \tau(\overline{\bigcup_{\Gamma \in X} \Gamma}) = \overline{\tau(\bigcup_{\Gamma \in X} \Gamma)} = \overline{\bigcup_{\Gamma \in X} \tau(\Gamma)} = \bigvee_{\Gamma \in X} \tau(\Gamma)$$

The proof for meets in (iii) is similar and is omitted.

10.5 Join of a Finite and Cofinite Biclosed Set

The following additional special case of Conjecture 2.5 is proved along similar lines as the proof of Theorem 1.13 (in fact, a critical special case of it was already required in the proof of 1.13 (ii)), and details of its proof are omitted.

Proposition *Let notation be as in Theorem 1.13. If $\Gamma, \Xi \in \mathscr{L}$, then $\overline{(\Lambda \setminus \Gamma) \cup \Xi} = \Lambda \setminus \Delta$ for some $\Delta \in \mathscr{L}$.*

11 Variants for Other Closure Operators

Given a closure operator c on Φ, one can define c-coclosed, c-biclosed subsets of Φ_+, etc. in the same way as for 2-closure, and ask whether the analogues of the main results (Section 1) hold with all $\overline{\Gamma}$ replaced by $c(\Gamma)$, "closed" replaced by "c-closed," "biclosed" replaced by "c-biclosed", etc. This section indicates, somewhat informally, what can be proved about two other such closure relations by simple modifications of the arguments in earlier sections. We first introduce some more terminology concerning closure operators.

11.1 Terminology for Closure Operators

A closure operator c on a set X is said to be of *finite character* if, for all $A \subseteq X$,

$$c(A) = \bigcup_{\substack{A_0 \subseteq A \\ |A_0| < \aleph_0}} c(A_0).$$

Say that a closure operator c on X is an *antiexchange* closure operator if, for $A \subseteq X$ and all $x, y \in X \setminus c(A)$, $x \in c(A \cup \{y\})$ implies $y \notin c(A \cup \{x\})$ (this terminology is often restricted to the case of finite X [19], but we shall not do so here).

Consider a family of root systems of a Coxeter group (W, S) for each one Ψ of which there is an associated closure operator on Ψ. Say the family of closure operators is *combinatorial* if the closure operator on $T \times \{\pm 1\}$ defined by transport of structure using the canonical bijection $\Psi \xrightarrow{\cong} T \times \{\pm 1\}$ given by $\epsilon\alpha \mapsto (s_\alpha, \epsilon)$ for $\alpha \in \Psi_+, \epsilon \in \{\pm 1\}$, is independent of the choice of root system Ψ in the family. Of course, other, quite different definitions of combinatorial closure operators could be made; the above is convenient for our purposes here.

11.2 $\mathbb{Z}$-closure on Finite Crystallographic Root Systems

Let Ψ be a (reduced) crystallographic root system of a finite Weyl group as in [4]. There is a standard closure operator on Ψ, which we call $\mathbb{Z}$-closure to avoid confusion with 2-closure, for which the $\mathbb{Z}$-closed sets Γ are those for which $\alpha, \beta \in \Psi$ and $\alpha + \beta \in \Phi$ imply $\alpha + \beta \in \Psi$. Equivalently, Ψ is $\mathbb{Z}$-closed if $\alpha, \beta \in \Psi$ and $m\alpha + n\beta \in \Phi$ with $m, n \in \mathbb{N}$ implies $m\alpha + n\beta \in \Psi$ (as one sees by reduction to rank two; see also [26]). The $\mathbb{Z}$-closure has well-known natural interpretations in the context of semisimple complex Lie algebras, for instance.

Both 2-closure and $\mathbb{Z}$-closure are closure operators of finite character, trivially. However, 2-closure differs from $\mathbb{Z}$-closure in some significant respects (aside from its obvious applicability to more general classes of Coxeter groups). For example, it was shown in [26] that $\mathbb{Z}$-closure restricted to the set Ψ_+ of positive roots of Ψ is an anti-exchange closure operator whereas 2-closure on Φ_+ is not anti-exchange for (W, S) of type F_4, H_3, or H_4. Also, 2-closure is easily seen to be combinatorial, but $\mathbb{Z}$-closure is not combinatorial in general, *e.g.*, in type B_2.

11.3 Analogues for $\mathbb{Z}$-closure of Some of the Results for 2-closure

This subsection discusses the extent to which the main results and conjectures of this paper are known to apply to finite Weyl groups with $\mathbb{Z}$-closure on Ψ in place of 2-closure.

Proposition *The $\mathbb{Z}$-closure analogue of* Theorem 1.5 *is true.*

Proof According to [4, Chapter VI, §1, Example 16], a subset of Ψ_+ is $\mathbb{Z}$-biclosed if and only if it is of the form $\Psi_w := \Psi_+ \cap w(-\Psi_+)$ for some $w \in W$. The proof of Theorem 1.5 (i) applies *mutatis mutandis* to establish the analogue for $\mathbb{Z}$-closure of (i); then the $\mathbb{Z}$-closure analogue (ii) holds by the $\mathbb{Z}$-closure analogue of the argument for the proof of Corollary 5.4 (iii) (or by using Remark 5.4). ■

The analogues for $\mathbb{Z}$-closure of Lemma 1.7 (i)–(ii) fail for W of type B_2. It was asserted in [25] that the $\mathbb{Z}$-closure analogue of Conjecture 2.5 (iii) in the case $\Lambda = \Psi$ holds (though there is a gap in the published proof; see [10]. I thank Eugene Karolinsky for these references). It is not immediately clear if the $\mathbb{Z}$-closure analogues of Theorems 7.1, 1.8, or 1.13 hold.

11.4 The Convex Geometric Closure Operator d

Another natural closure operator d on Φ is given by $d(\Gamma) := \Phi \cap \mathbb{R}_{\geq 0}\Gamma$. The operator d is a closure operator of finite character and restricts to an antiexchange closure on Φ_+, but it is not combinatorial in general. In fact, for infinite W of rank four with no braid relations, *i.e.*, all entries of its Coxeter matrix are either 1 or ∞, there are many possible root systems (in the class [18], with the inner product normalized so $\langle \alpha, \alpha \rangle = 1$ for all $\alpha \in \Pi$) with linearly independent simple roots Π, determined by arbitrary choices of inner products $\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle \leq -1$ for distinct $\alpha, \beta \in \Pi$. It is easy to check that the closure operators on $T \times \{\pm 1\}$ corresponding to the resulting closures d as above, genuinely depend on the choice of root system.

Remarks. The operators d for root systems of a Coxeter system (W, S) of rank three are combinatorially invariant, by an argument involving homotopies of root systems and the combinatorial nature of d-closure restricted to the maximal dihedral root subsystems. Informally, any two root systems in the class are connected by a homotopy, determined by a suitable homotopy from one matrix $(\langle \alpha, \beta \rangle)$ to another in the space of such matrices attached to root systems. As the root system varies in such a homotopy, a root can never enter or leave a plane spanned by two other roots (since the reflection in roots of this plane generate a maximal dihedral reflection subgroup, and the sets of reflections of such subgroups are completely determined by the Coxeter system). More precisely, in the rank three case, the oriented matroid closure operators on $T \times \{\pm 1\}$ obtained by transfer of structure from the various root systems all coincide; this argument uses a characterization of finite rank (possibly infinite) oriented matroids by their basis orientations [3, Exercise 3.13].

An obvious question is whether combinatorial invariance in this sense extends to oriented geometry root systems of rank three Coxeter systems (with definition as suggested in Section 2.10).

11.5 Coverings in Bruhat Order, and Polyhedral Cones

The following fact from [15, Proposition 3.6] will be used below.

Lemma *For any $x \in W$, $\Phi_{x,1}$ (resp., $\Phi_{x,-1}$) is a set of representatives of the extreme rays of a pointed polyhedral cone $\mathbb{R}_{\geq 0}\Phi'_x$ (resp., $\mathbb{R}_{\geq 0}\Phi_x$) spanned by Φ'_x (resp., Φ_x). Furthermore, $\mathbb{R}_{\geq 0}\Phi_x \cap \mathbb{R}_{\geq 0}\Phi'_x = \{0\}$.*

11.6 Analogues for d of Some Results for 2-closure

Recall that d-coclosed sets and d-biclosed subsets are defined in the obvious way.

Proposition *The d-closure analogues of* Theorem 1.5, Lemma 1.7, Theorem 1.8, *and* Proposition 7.3 *are all true.*

Proof Any d-closed set is obviously closed. Hence, for any $\Gamma \subseteq \Phi$, we have $\overline{\Gamma} \subseteq d(\Gamma)$. Also, Φ_w and Φ'_w are d-biclosed for $w \in W$; this holds since Φ_+ and $-\Phi_+$ can be strictly separated by a (linear) hyperplane, and hence so can

$$w(\Phi_+) \supseteq \Phi'_w \quad \text{and} \quad w(-\Phi_+) \supseteq \Phi_w.$$

Hence, if $\Gamma \subseteq \Phi$ is such that $\overline{\Gamma}$ is d-closed, *e.g.*, $\overline{\Gamma} = \Phi_w$ or $\overline{\Gamma} = \Phi'_w$, then $\overline{\Gamma} = d(\Gamma)$.

Using the previous paragraph, one sees that the analogues for d of Theorems 1.5 and 1.8 and Proposition 7.3 hold *mutatis mutandis*. In each case, the d-analogue follows from the corresponding statement for 2-closure. For example, suppose $X \subseteq W$ has an upper bound. Then by Theorem 1.5 (i), $\overline{\bigcup_{x\in X}\Phi_x} = \Phi_y$, where $y = \bigvee X$, and so by above, $d(\bigcup_{x\in X}\Phi_x) = \Phi_y$, *i.e.*, we have proved the d-analogue of 1.5 (i).

The d-analogue of Lemma 1.7 is proved using Lemma 11.5. In fact, that lemma implies that for $\alpha \in \Phi_{x,1}$ (resp., $\alpha \in \Phi_{x,-1}$), one has $\alpha \notin d(\Phi'_x \smallsetminus \{\alpha\})$ (resp., $\alpha \notin d(\Phi_x \smallsetminus \{\alpha\})$). On the other hand, $d(\Phi_{x,1}) = \Phi'_x$ (resp., $c(\Phi_{x,-1}) = \Phi_x$) holds by the lemma and the first paragraph above. These remarks easily imply that the d-analogue of Lemma 1.7 holds. ■

Remarks. • One can show that $\mathscr{L}_s$ in Theorem 1.13 is the set of all finite d-biclosed subsets of Λ (one uses simple arguments in convex geometry beginning with Lemma 10.3 and the possibility of separating Φ_w and Φ'_w by a linear hyperplane). Then the d-closure analogues of Theorem 1.13 and Proposition 10.5 follow from the corresponding results for 2-closure exactly as above.

• The analogue for d of Theorem 7.1 cannot be deduced from Theorem 7.1 in the same way as the other results above (since if $\Gamma \subseteq \Phi_+$ is d-coclosed and $d(\Gamma)$ has finite complement in Φ_+, one can not deduce that $\overline{\Gamma}$ has finite complement and apply Theorem 7.1; one only knows $\overline{\Gamma} \subseteq d(\Gamma)$). However, a simple argument involving convex geometry shows that if Γ is d-coclosed and $\alpha \in \Pi \smallsetminus \Gamma$, then $s_a \cdot \Gamma$ is d-coclosed; using this and the d-analogue of Lemma 1.7, one sees that Theorem 7.1 and its proof also hold *mutatis mutandis* for d.

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