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A MEASURE OF LINEAR INDEPENDENCE FOR SOME EXPONENTIAL FUNCTIONS II

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1. Statement of results. This paper continues the investigations in [**1**] and [**2**] and extends the results of the latter paper to functions of several complex variables. Namely let $\lambda : \mathbf{R}_{>0} \to \mathbf{R}_{>0}$ be monotonically increasing. (If λ is non-constant, we also require that $\log r = O(\lambda(r))$. We write $f = O(g)$ to mean that there is a constant $C > 0$ such that $f(r) \leqq Cg(r)$ except possibly on a set of intervals of finite total length.) Let O_λ denote the set of meromorphic f on $\mathbf{C}^n$ for which the Nevanlinna characteristic function [**5**, p. 174] $T(f, r)$ satisfies $T(f, r) = O(\lambda(r))$.

We shall see below that O_λ is a differential field. Section 2 contains the definitions and a short review of the properties we will use.

THEOREM 1. *Let* $g_1, \ldots, g_l$ *be meromorphic functions. Then* $\exp g_1, \ldots, \exp g_l$ *are linearly dependent over* O_λ *if and only if, for some* $1 \leqq i < j \leqq l$,

$$\exp(g_i - g_j) \in O_\lambda.$$

When the g_i are entire and the $\exp g_i$ are O_λ-linearly *independent*, we obtain the following measure of linear independence, which is our main result:

THEOREM 2. *Let* $g_1, \ldots, g_l$ *be entire such that for all* $1 \leqq i < j \leqq l$, $\exp(g_i - g_j) \notin O_\lambda$. *For* $f_1, \ldots, f_l \in O_\lambda$, *none identically zero, set*

$$G = f_1 \exp g_1 + \ldots + f_l \exp g_l.$$

Then

$$m(G, r) \geqq T(e^g) - l \sum T(f_j, r) + O(\log T(e^g, r)).$$

Unless otherwise indicated, sums and products are to be taken over all $1 \leqq j \leqq l$, $T(e^g, r)$ will denote $\max_i T(\exp g_i, r)$, and $l \geqq 2$.

The example $g_1 = e^z$, $g_2 = \exp g_1, \ldots, g_l = \exp g_{l-1}$, $f_1 = \ldots = f_l$ shows that it is quite possible to have

$$T(G, r) - T(e^g, r) = O(\log T(e^g, r)).$$

(Use Nevanlinna's inequality on the logarithmic derivative as in [**2**, 3.iv] to see that

$$T(g_j, r) \leqq 10 \log T(\exp g_j, r) + O(\log r).)$$

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Thus with respect to the coefficients of the dominant terms $T(G, r)$ and $T(e^g, r)$, Theorem 2 is best possible.

2. Some definitions and results from Nevanlinna theory. Before we proceed with a discussion of the proofs, we recall some of the definitions and properties from Nevanlinna theory for several complex variables [**4**]. We use the notations

$$\begin{aligned} d^c &= (i/4\pi)(\bar{\partial} - \partial), \\ \omega &= dd^c \log ||z||^2, \\ \omega^{n-1} &= \omega \wedge \ldots \wedge \omega \quad (n-1 \text{ factors}), \\ \eta &= d^c \log ||z||^2 \wedge \omega^{n-1}, \\ \mathbf{C}^n[r] &= \{z \in \mathbf{C}^n : ||z|| \leqq r\}, \\ \partial\mathbf{C}^n[r] &= \{z \in \mathbf{C}^n : ||z|| = r\}. \end{aligned}$$

For an analytic divisor D, we set $D[r] = D \cap \mathbf{C}^n[r]$ and define the *counting functions*

$$n(D, t) = \int_{D[t]} \omega^{n-1},$$

$$N(D, r) = \int_0^r (N(D, t) - \mathscr{L}_0(D))dt/t + \mathscr{L}_0(D) \log r,$$

where the Lelong number $\mathscr{L}_0(D)$ is the multiplicity of D at the origin [**5**, pp. 12–14]. For a meromorphic function f we define the *proximity function*

$$m(f, r) = \int_{\partial\mathbf{C}^n[r]} \log^+ |f|.$$

The Nevanlinna characteristic function is defined as

$$T(f, r) = m(f, r) + N(f, r).$$

From the definitions one sees easily that the properties

i)
$$m\left(\prod_1^p f_i, r\right) \leqq \sum_1^p m(f_i, r)$$
$$m\left(\sum_1^p f_i, r\right) \leqq \sum_1^p m(f_i, r) + \log p$$

imply the corresponding properties for $T(\prod f_i, r)$ and $T(\sum f_i, r)$.

Jensen's theorem [**5**, p. 174] shows that

ii) $T(1/f, r) = T(f, r) + 0(1).$

Thus O_λ is a field.

In higher dimensional value distribution theory one also studies a generalization $T_1(f, r)$ of the Ahlfors-Shimizu order function. $T_1(f, r)$ can be interpreted as the average over all of the projective space $\mathbf{P}^{n-1}$ of complex lines ξ through

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the origin of $\mathbf{C}^n$ of the one-dimensional Ahlfors-Shimizu characteristic function on the lines ξ [**4**, pp. 18, 19, 25]. The two functions $T(f, r)$ and $T_1(f, r)$ are related as in the one variable case [**5**, p. 190] by

$$T(f, r) = T_1(f, r) + O(1).$$

Thus one can say that $T(f, r)$ tends to ∞ unless f is a constant and even [**4**, p. 24] that

iii) $T(f, r) = O(\log r)$ if and only if f is a rational function.

Quite recently A. Vitter [**7**] has established a remarkable generalization of Nevanlinna's fundamental lemma on the logarithmic derivative: For any meromorphic function f on $\mathbf{C}^n$,

$$m(f_z/f) = O(\log T(f, r) + \log r),$$

where f_z denotes any partial derivative $\partial f/\partial z_j$, $j = 1, \ldots, n$.

Let us write $f = g/h$, where g and h are entire functions whose divisors of zero share no common analytic hypersurface in $\mathbf{C}^n$, as in the theorem of Weierstrass and Cousin. Then

$$f_z = g_z/h - gh_z/h^2.$$

So

$$N(f_z, r) \leqq 2N(f, r)$$

and similarly

$$N(\partial^i f, r) \leqq (i + 1)N(f, r),$$

where ∂^i denotes any i^{th} order partial derivative. In particular Vitter's theorem shows that O_λ is a differential field with derivations $\partial/\partial z_j$, $j = 1, \ldots, n$, since

$$m(f_z, r) \leqq m(f, r) + O(\log T(f, r) + \log r).$$

3. The Derivation D. As in the one variable case [**2**] our proof will be an adaptation of Nevanlinna's celebrated generalization of the Picard-Borel theorem. Apparently we now have n likely derivations to play the role that ordinary derivation played in the original proof. But rather than trying to decide which partial derivative to use, we introduce a new derivation $\mathrm{D} = z \cdot \mathrm{grad}$, i.e. for meromorphic f,

$$\mathrm{D}f = z_1\partial f/\partial z_1 + \ldots + z_n\partial f/\partial z_n.$$

It is easy to check that D is a derivation, i.e. is a $\mathbf{C}$-linear map on the field of meromorphic functions satisfying the product rule.

It is clear that if f is entire, then so is $\mathrm{D}f$. From local considerations it is straight-forward to check that

iv) $\ker D = \mathbf{C}$.

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Moreover if f is entire, then

$$De^f = (Df)e^f.$$

Vitter's result shows that for entire f

v) $m(Df, r) \leqq O(\log T(e^f, r) + \log r)$

and that D carries O_λ into itself.

As above for f_z, one sees easily that

$$N(D^{(k)}f, r) \leqq (k + 1)N(f, r).$$

Since Vitter's result shows directly that

vi) $m(Df/f, r) \leqq O(\log T(f, r) + \log r)$

and thus

$$m(Df, r) \leqq m(f, r) + O(\log T(f, r) + \log r),$$

one sees inductively that

$$T(D^{(k)}f, r) = O(T(f, r) + \log r).$$

Applying this fact along with i) and v) to

$$\frac{D^{(k)}f}{f} = \frac{D^{(k)}f}{D^{(k-1)}f} \cdots \frac{Df}{f},$$

we find that for fixed k

vii) $m(D^{(k)}f/f, r) = O\left(\sum_0^{k-1} \log T(D^{(j)}f, r) + \log r\right) = O(\log T(f, r) + \log r).$

4. Proof of theorem 2. Theorem 1 follows from Theorem 2 on dividing any dependence relation by exp g_l, setting $G = -f_l$, and applying Theorem 2 with l replaced by $l - 1$. Since the details are practically the same as in [**2**, pp. 165–166], we will not dwell on them, except to recall the fact that in a differential field, linear independence over the constant subfield is equivalent to the non-vanishing of the Wronskian.

As mentioned above, the proof of Theorem 2 retains the outline established by Nevanlinna himself [**6**, Chapt. V, 1]. We set $F_{0,i} = f_i, \quad i = 1, \ldots, l$, and define for $1 \leqq k < l$,

$$F_{k,i} = F_{k-1,i} + D(F_{k-1,i}).$$

Then for $0 \leqq k < l$,

(1) $F_{k,1}e^{g_1} + \ldots + F_{k,l}e^{g_l} = D^{(k)}G.$

Since $f_1 \exp g_1, \ldots, f_l \exp g_l$ are **C**-linearly independent, their Wronskian with respect to D,

$$\det (F_{k,i}) \exp (g_1 + \ldots + g_l)$$

is non-zero. Consequently $\varphi = \det (F_{k,i})$ is also non-zero.

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Consequently we can solve for each exp g_i in (1) by Cramer's rule

$$\varphi e^{g_i} = \varphi_i.$$

We also have

$$f_i \delta e^{g_i} = G\delta_i,$$

where $\delta = \det(F_{k,j}/f_j)$ and δ_i is the determinant of the matrix obtained by replacing the i^{th} column of $(F_{k,j}/f_j)$ by the transpose of

$$(1, \mathrm{D}G/G, \ldots, \mathrm{D}^{(l-1)}G/G).$$

From (2) and i), we see that

$$m(e^{g_i}, r) \leqq m(\varphi_i, r) + m(1/\varphi, r).$$

From ii) one obtains that

$$m(1/\varphi, r) \leqq T(\varphi, r) - N(1/\varphi, r) + O(1);$$

from i) that

$$m(\varphi_i, r) \leqq m(G, r) + m(\textstyle\prod_{j\neq i} f_j, r) + m(\delta_i, r)$$

and that

$$T(\varphi, r) \leqq m(\textstyle\prod f_j, r) + m(\delta, r) + N(\varphi, r).$$

From v), vii) and the usual expansion of the determinant, we have that

$$m(\delta, r) = O(\textstyle\sum \log T(f_j, r) + \log T(e^{g_j}, r) + \log r).$$

From the obvious upper bound on $T(G, r)$, we see similarly that

$$m(\delta_i, r) = O(\textstyle\sum \log T(f_j, r) + \log T(e^{g_j}, r) + \log r).$$

We must now bound $N(\varphi, r)$ from above. Since.

$$N(\mathrm{D}^{(k)} f, r) \leqq (k + 1)N(f, r),$$

it follows that

$$N(F_{k,j}, r) \leqq (k + 1)N(f_j, r).$$

Thus

$$N(\varphi, r) \leqq l \textstyle\sum N(f_j, r).$$

5. Remarks. One could also introduce the notion of $\bar{N}$ as in the one variable case and obtain sharpenings of Theorem 2 corresponding to Theorems 3 and 4 of [**2**].

The first version of this manuscript was written before the appearance of Vitter's result. The proofs were obtained by verifying vii) more directly, following the lines of the proof of Lemma 2 of P. Gauthier and W. Hengartner

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in [**3**]. The idea is to use the step in Nevanlinna's proof of his fundamental lemma which bounds $m(f'/f, r)$ in terms of $T(f, R)$, $\log^+ R$, $\log^+(R - r)^{-1}$, and $\log^+ r^{-1}$, for *every* $0 < r < R$. This explicit bound is applied on each complex line ξ in $\mathbf{P}^{n-1}$ and integrated over $\mathbf{P}^{n-1}$. Now the convexity of logarithm and Borel's lemma [**4**, p. 29] are invoked.

After the manuscript was in its final form, it was brought to my attention that P. Bonneau has independently established an analogue in terms of maximum modulus of our Theorem 1 in Comptes Rendus 285(1977), 111–113 and his thesis, Toulouse, 1977.

References

1. W. D. Brownawell, *The algebraic independence of certain exponential functions*, Can. J. Math. *28* (1976), 968–976.

2. ——— *A measure of linear independence for some exponential functions*, in A. Baker and D. W. Masser, eds, Transcendence Theory: Advances and Applications (Academic Press, London and New York, 1977).

3. P. M. Gauthier and W. Hengartner, *The value distribution of most functions of one or more complex variables*, Annals of Math. *96* (1972), 31–52.

4. P. A. Griffiths, Entire Holomorphic Mappings in One and Several Complex Variables, (Princeton Univ. Press, Princeton, N.J., 1976).

5. P. A. Griffiths and J. King, *Nevanlinna theory and holomorphic mappings between algebraic varieties*, Acta Math. *130* (1973), 145–220.

6. R. Nevanlinna, *Le Théorème de Picard-Borel et la Théorie des Fonctions Méromorphes*, (Gauthier-Villars, Paris, 1929).

7. A. Vitter, *The lemma of the logarithmic derivative in several complex variables*, Duke Math. J. *44* (1977), 89–104.

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