



A Note on the Vanishing Viscosity Limit in the Yudovich Class

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Abstract. We consider the inviscid limit for the two-dimensional Navier–Stokes equations in the class of integrable and bounded vorticity fields. It is expected that the difference between the Navier–Stokes and Euler velocity fields vanishes in L^2 with an order proportional to the square root of the viscosity constant ν . Here, we provide an order $(\nu/|\log \nu|)^{\frac{1}{2}} \exp(-Ct)$ bound, which slightly improves upon earlier results by Chemin.

1 Introduction

The convergence of solutions of the Navier–Stokes equations towards solutions of the Euler equations in the limit of vanishing viscosity has been an ongoing research topic for many years. Most of the progress has been made in the two-dimensional full space, in which both vortex stretching and boundary effects are absent. Configurations in which the vorticity field is non-smooth are of particular interest, as these include the important examples of vortex patches. In this paper, we study the inviscid limit for integrable and bounded vorticity fields.

To be more specific, we are interested in the rate of L^2 convergence of the Navier–Stokes velocity fields towards the Euler velocity fields. To the best of our knowledge, the best estimate available in the literature is due to Chemin [3], who provides an $O(\nu^{\frac{1}{2}} \exp(-Ct))$ bound on the velocity difference. In this paper, we slightly improve this result by a logarithm,

$$(1.1) \quad \|u^\nu(t) - u(t)\|_{L^2} = O\left(\left(\frac{\nu}{|\log \nu|}\right)^{\frac{1}{2}} \exp(-Ct)\right),$$

as $\nu \ll 1$. Moreover, for small times $t \ll 1/|\log \nu|$, we obtain

$$(1.2) \quad \|u^\nu(t) - u(t)\|_{L^2} = O(\sqrt{\nu}).$$

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Better convergence results are available in the literature under additional regularity assumptions. For instance, in [5, 6], Constantin and Wu obtained $O(\sqrt{\nu})$ convergence globally in time under additional gradient bounds on the velocity field. Such bounds are true, for instance, for vortex patches with smooth boundaries. For these particular solutions, however, better rates can be obtained. Indeed, Abidi and Danchin [1] established an $O(\nu^{\frac{3}{4}})$ estimate for vortex patches with smooth boundaries and showed the optimality of this convergence order. These results were further generalized and extended (to higher order Sobolev velocity fields) by Masmoudi [14]. We also mention the L^∞ bounds by Cozzi in the case of bounded but not necessarily decaying velocity fields [7, 8].

We will work on the Navier–Stokes and Euler equations in vorticity formulations. The (scalar) vorticity fields are computed as the rotations of the velocity vectors and are denoted by ω^ν in the case of the Navier–Stokes and ω in the case of the Euler equations. The Navier–Stokes equation in vorticity formulation is the advection-diffusion equation

$$(1.3) \quad \partial_t \omega^\nu + u^\nu \cdot \nabla \omega^\nu = \nu \Delta \omega^\nu,$$

which reduces to the Euler vorticity equation, a simple advection equation,

$$(1.4) \quad \partial_t \omega + u \cdot \nabla \omega = 0,$$

in the inviscid limit when $\nu \rightarrow 0$. The velocity vector fields can be reconstructed from the vorticities via the Biot–Savart law

$$u^\nu = K^* \omega^\nu, \quad u = K^* \omega, \quad \text{where } K(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2},$$

and x^\perp is the counterclockwise rotation by 90 degrees of a point x in the plane. We note for completeness that the velocity fields are divergence-free by construction, thus $\nabla \cdot u^\nu = \nabla \cdot u = 0$, which is the incompressibility assumption on the fluid.

Both evolution equations have to be equipped with an initial condition. In the sequel, we assume that the initial vorticities are identical, integrable and bounded; that is,

$$(1.5) \quad \omega^\nu(0) = \omega(0) = \omega_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2).$$

These assumptions are retained by the evolution, in the sense that for any time t ,

$$(1.6) \quad \|\omega^\nu(t)\|_{L^1} \leq \|\omega_0\|_{L^1}, \quad \|\omega^\nu(t)\|_{L^\infty} \leq \|\omega_0\|_{L^\infty},$$

$$(1.7) \quad \|\omega(t)\|_{L^1} = \|\omega_0\|_{L^1}, \quad \|\omega(t)\|_{L^\infty} = \|\omega_0\|_{L^\infty}.$$

We recall that in the class of integrable and bounded solutions, both the Navier–Stokes and Euler equations admit a unique global solution in the two-dimensional setting. Indeed, the well-posedness of the Navier–Stokes equations holds true under more general assumptions; see, e.g., the work of Ben-Artzi [2] for a proof in the L^1 setting. Roughly speaking, the results for (1.3) are a consequence of the parabolicity of the equation. In the case of the Euler equations, well-posedness for initial data in the class (1.5) was first obtained by Yudovich [11] and is essentially open for unbounded vorticities. Yudovich’s result was later recovered by Loeper [12], who

deduced uniqueness for the Euler equations from stability estimates in terms of the 2-Wasserstein distance. In either work, the central key for proving uniqueness for (1.4) is a log-Lipschitz estimate for the velocity field, which is valid in the Yudovich class $L^1 \cap L^\infty$, namely,

$$(1.8) \quad |u(x) - u(y)| \leq C|x - y| \left(1 + \log \left(1 + \frac{1}{|x - y|} \right) \right);$$

see, e.g., [13, Lemma 8.1]. In this estimate, the constant C depends on $\|\omega\|_{L^1}$ and $\|\omega\|_{L^\infty}$.

In our derivation of the bounds on the convergence order (1.1) and (1.2), we build up on Loeper’s approach. More precisely, we derive an estimate on certain 2-Wasserstein distances, which provides, on the one hand, a bound on the 1-Wasserstein distance between the viscous and the inviscid vorticity fields, and, on the other hand, the desired bound on the L^2 norm of the distance of the corresponding velocity fields. We first state and discuss the latter. A definition of Wasserstein distances will be given in the subsequent section.

Theorem 1.1 For $\nu \ll 1$ and $t \ll \frac{1}{|\log \nu|}$, it holds that

$$\|u^\nu(t) - u(t)\|_{L^2} \lesssim \sqrt{\nu t}.$$

Moreover, for any fixed $t > 0$, the following holds:

$$\|u^\nu(t) - u(t)\|_{L^2} = O \left(\left(\frac{\nu}{|\log \nu|} \right)^{\frac{1}{2} \exp(-Ct)} \right),$$

as $\nu \rightarrow 0$, where $C > 0$ is a constant dependent only on $\|\omega_0\|_{L^1}$ and $\|\omega_0\|_{L^\infty}$.

Here and in the following, we write $A \lesssim B$ if there exists a constant Λ independent of ν and t such that $A \leq \Lambda B$. Moreover, we write $A \sim B$ if both $A \lesssim B$ and $B \lesssim A$. Finally, $A \ll B$ means that $A \leq \Lambda B$ for some sufficiently large Λ .

As mentioned earlier, up to the logarithmic improvement, our second estimate has been established earlier by Chemin [3]. Our contribution here is essentially a new proof that is based on stability estimates. With regard to the scaling in ν , the first estimate is supposedly optimal, even globally in time.

The results can be restated as bounds on the homogeneous H^{-1} norm of the vorticity fields; for instance,

$$(1.9) \quad \|\omega^\nu(t) - \omega(t)\|_{\dot{H}^{-1}} = O \left(\left(\frac{\nu}{|\log \nu|} \right)^{\frac{1}{2} \exp(-Ct)} \right),$$

and thus complement a recent work by Constantin, Drivas, and Elgindi [4], in which the convergence in L^p is proved for vorticity fields in the Yudovich class

$$\lim_{\nu \rightarrow 0} \sup_t \|\omega(t) - \omega^\nu(t)\|_{L^p} = 0.$$

Note that there can be no rates of strong convergence without imposing additional regularity assumptions on the data, which can be easily seen on the linear level (cf. [16, Example 2]).

The results of our second theorem are similar in spirit to (1.9), in the sense that they provide estimates on a negative Sobolev norm.

Theorem 1.2 For $\nu \ll 1$ and $t \ll \frac{1}{|\log \nu|}$, we have that

$$W_1(\omega^\nu(t), \omega(t)) \lesssim \sqrt{\nu t}.$$

Moreover, for any fixed $t > 0$, we have

$$W_1(\omega^\nu(t), \omega(t)) = O\left(\left(\frac{\nu}{|\log \nu|}\right)^{\frac{1}{2} \exp(-Ct)}\right)$$

as $\nu \rightarrow 0$, where $C > 0$ is a constant dependent only on $\|\omega_0\|_{L^1}$ and $\|\omega_0\|_{L^\infty}$.

Notice that, by the Kantorovich–Rubinstein theorem, the 1-Wasserstein distance W_1 is dual to the homogeneous Lipschitz norm $\|\cdot\|_{\dot{W}^{1,\infty}}$; cf. (2.3) below. Moreover, as Wasserstein distances metrize weak convergence; cf. [17, Theorem 7.12], estimates on the Wasserstein distance translate into estimates on the convergence order. That is, our second theorem shows that the vorticity fields of the viscous fluid, ω^ν , converge towards the vorticity field of the inviscid fluid, ω , weakly with order $O((\nu/|\log \nu|)^{\frac{1}{2} \exp(-Ct)})$. Again, the author believes that $O(\sqrt{\nu})$ convergence is optimal. In this regard, the situation is very similar to the inviscid limit problem for linear advection–diffusion equations in the DiPerna–Lions setting considered earlier by the author by using new stability estimates for the continuity equation; see [15, 16]. (In a certain sense, the estimates in [15] are the linear analogues of Loeper’s estimates for the 2D Euler equations [12].)

The remainder of the paper is organized as follows. In the following section, we recall the definition of the Wasserstein distances and collect a number of properties that will be useful in our proofs. The proofs of Theorems 1.1 and 1.2 will be provided in the last section.

2 Some Tools from the Theory of Optimal Transportation

In this section, we collect definitions and properties of Wasserstein distances that will be used in the sequel. For a general comprehensive introduction into the topic of optimal transportation, we refer the reader to Villani’s popular monograph [17].

Given two nonnegative integrable functions f and g of the same total mass,

$$(2.1) \quad \int_{\mathbb{R}^2} f \, dx = \int_{\mathbb{R}^2} g \, dx,$$

we define the set of transport plans $\Pi(f, g)$ as the set of joint measures π on the product space $\mathbb{R}^2 \times \mathbb{R}^2$ having f and g as marginals; that is,

$$\int_{\mathbb{R}^2 \times \mathbb{R}^2} (\varphi(x) + \psi(y)) \, d\pi(x, y) = \int_{\mathbb{R}^2} \varphi(x) f(x) \, dx + \int_{\mathbb{R}^2} \psi(y) g(y) \, dy$$

for any continuous functions φ and ψ . The p -Wasserstein distance $W_p(f, g)$ between f and g is then defined by the formula

$$W_p(f, g) = \left(\inf_{\pi \in \Pi(f, g)} \int_{\mathbb{R}^2 \times \mathbb{R}^2} |x - y|^p \, d\pi(x, y) \right)^{\frac{1}{p}}.$$

In this paper, we will only consider the cases $p = 1$ and $p = 2$. Both are ordered in the sense that

$$(2.2) \quad W_1(f, g) \leq \|f\|_{L^1}^{\frac{1}{2}} W_2(f, g)$$

by Jensen’s inequality, where we used the fact that $\pi[\mathbb{R}^2 \times \mathbb{R}^2] = \|f\|_{L^1}$.

As a consequence of the Kantorovich–Rubinstein duality theorem,

$$(2.3) \quad W_1(f, g) = \sup \left\{ \int_{\mathbb{R}^2} (f - g)\zeta \, dx : \|\nabla \zeta\|_{L^\infty} \leq 1 \right\}$$

(cf. [17, Theorem 1.14]), the 1-Wasserstein distance is a transshipment cost that only sees the difference of the marginals, and thus, W_1 can be naturally extended as a measure on the space of not necessarily nonnegative configurations with same spatial average (2.1). This is particularly convenient in our application to solutions to the Navier–Stokes equations, as these conserve the spatial average but not the L^1 norm.

Concerning W_2 , we will use the fact that the Wasserstein distance dominates the \dot{H}^{-1} norm,

$$(2.4) \quad \|f - g\|_{\dot{H}^{-1}} \leq \max\{\|f\|_{L^\infty}, \|g\|_{L^\infty}\}^{\frac{1}{2}} W_2(f, g)$$

(cf. [12, Theorem 2.9]).

3 Proofs

Due to the fact that the vorticity fields are not necessarily nonnegative and our argument is essentially based on W_2 -distances that are defined for nonnegative quantities only, we need the following construction.

For a general initial vorticity distribution ω_0 , we consider separately the evolution of the positive and negative parts given by the linear equations

$$(3.1) \quad \partial_t \omega^\pm + u^v \cdot \nabla \omega^\pm = \nu \Delta \omega^\pm, \quad \omega^\pm(0) = \omega_0^\pm,$$

$$(3.2) \quad \partial_t \omega_\pm + u \cdot \nabla \omega_\pm = 0, \quad \omega_\pm(0) = \omega_0^\pm,$$

where the superscript plus and minus signs indicate the positive and negative parts; *i.e.*, $\omega_0^+ = \max\{0, \omega_0\}$ and $\omega_0^- = \max\{0, -\omega_0\}$, while the subscript plus and minus signs just mark the solutions. Notice that both equations are well-posed (and so is the hypoelliptic equation in (3.3), which is a hybrid version of (3.1) and (3.2)) as a consequence of the DiPerna–Lions theory [9], because ∇u and ∇u^v both belong to the class $L^\infty(\mathbb{R}_+; L^p(\mathbb{R}^2))$ for any $p \in [1, \infty)$ by the assumptions on the initial vorticity (1.5), the a priori estimates in (1.6) and (1.7), and Calderón–Zygmund theory.

By the maximum principles for the respective equations, the solutions are nonnegative. Moreover, as both equations are conservative thanks to the incompressibility condition $\nabla \cdot u^v = \nabla \cdot u = 0$, the total masses are preserved: $\|\omega^\pm(t)\|_{L^1} = \|\omega_\pm(t)\|_{L^1} = \|\omega_0^\pm\|_{L^1}$. Finally, by uniqueness and linearity, it holds that $\omega^v = \omega^v_+ - \omega^v_-$ and

$\omega = \omega_+ - \omega_-$. We then have by the triangle inequality, (2.3), and (2.2) that

$$\begin{aligned} W_1(\omega^v, \omega) &\leq W_1(\omega^v_+, \omega_+) + W_1(\omega^v_-, \omega_-) \\ &\leq \|\omega_0^+\|_{L^1}^{\frac{1}{2}} W_2(\omega^v_+, \omega_+) + \|\omega_0^-\|_{L^1}^{\frac{1}{2}} W_2(\omega^v_-, \omega_-) \\ &\leq \|\omega_0\|_{L^1}^{\frac{1}{2}} (W_2(\omega^v_+, \omega_+) + W_2(\omega^v_-, \omega_-)). \end{aligned}$$

An analogous estimate holds true for the \dot{H}^{-1} norm via (2.4), namely,

$$\|\omega^v - \omega\|_{\dot{H}^{-1}} \leq \|\omega_0\|_{L^\infty}^{\frac{1}{2}} (W_2(\omega^v_+, \omega_+) + W_2(\omega^v_-, \omega_-)).$$

Theorems 1.1 and 1.2 are thus consequences of the following result.

Theorem 3.1 For $v \ll 1$ and $t \ll \frac{1}{|\log v|}$, it holds that

$$(W_2(\omega^v_+(t), \omega_+(t)) + W_2(\omega^v_-(t), \omega_-(t))) \lesssim \sqrt{vt}.$$

Moreover, for any fixed $t > 0$, we have

$$(W_2(\omega^v_+(t), \omega_+(t)) + W_2(\omega^v_-(t), \omega_-(t))) = O\left(\left(\frac{v}{|\log v|}\right)^{\frac{1}{2}} \exp(-Ct)\right)$$

as $v \rightarrow 0$, where $C > 0$ is a constant dependent only on $\|\omega_0\|_{L^1}$ and $\|\omega_0\|_{L^\infty}$.

We now turn to the proof of Theorem 3.1, in which we roughly follow and extend Loeper’s stability estimate for the Euler equations [12]. Loeper’s proof is based on the Lagrangian formulation of the advection equation (1.4). Its viscous version leads to the stochastic differential equation

$$dX_t(x) = u^v(t, X_t(x)) dt + \sqrt{2vd}W_t, \quad X_t(x) = x,$$

and the Lagrangian representation of the vorticity, $\omega^v(t) = \mathbb{E}[\omega_0 \circ X_t^{-1}]$. Instead of working with the stochastic flow, we propose a deterministic (or Eulerian) derivation of the stability-type estimate in Theorem 3.1 via the coupling method; see, e.g., [10]. For that purpose, we choose functions $\eta_0^\pm \in \Pi(\omega_0^\pm, \omega_0^\pm)$ and consider the linear hypoelliptic advection-diffusion equations

$$(3.3) \quad \partial_t \eta_\pm + u(x) \cdot \nabla_x \eta_\pm + u^v(y) \cdot \nabla_y \eta_\pm = v \Delta_y \eta_\pm$$

on the product space $\mathbb{R}^2 \times \mathbb{R}^2$, with initial conditions $\eta_\pm(0) = \eta_0^\pm$. By construction, the marginals of η_\pm coincide with the unique solutions of equations (3.1) and (3.2),

$$\int_{\mathbb{R}^2} \eta_\pm(t, x, y) dx = \omega^v_\pm(t, y), \quad \int_{\mathbb{R}^2} \eta_\pm(t, x, y) dy = \omega_\pm(t, x),$$

and thus, by definition of the Wasserstein distance,

$$(3.4) \quad W_2^2(\omega^v_\pm(t), \omega_\pm(t)) \leq \iint_{\mathbb{R}^2 \times \mathbb{R}^2} |x - y|^2 \eta_\pm(t, x, y) dx dy =: Q_\pm(t).$$

We also set

$$Q(t) = Q_+(t) + Q_-(t).$$

By standard approximation procedures, we can assume that η_0^\pm is smooth and compactly supported so that $Q_\pm(0)$ is finite, and that $Q_\pm(t)$ is smooth. In fact, since

the Wasserstein distance between the Navier–Stokes and Euler vorticities is initially vanishing by (1.5), we can choose η_0 such that $Q(0)$ is arbitrarily small, say

$$(3.5) \quad Q(0)\left(1 + \log\left(1 + \frac{1}{Q(0)}\right)\right) < \nu.$$

Our first goal is the following differential inequality.

Lemma 3.2 *We have that*

$$(3.6) \quad \frac{dQ}{dt} \lesssim Q\left(1 + \log\left(1 + \frac{1}{Q}\right)\right) + \nu.$$

Proof In the following computation, we neglect the time dependences of the involved functions. We first derive simultaneously estimates on the rates of change of Q_+ and Q_- . Differentiation, (multiple) integration by parts, and the fact that the velocity fields are both divergence-free yield

$$\begin{aligned} \frac{dQ_{\pm}}{dt} &= 2 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} (x - y) \cdot (u(x) - u^v(y)) \eta_{\pm}(x, y) \, dx \, dy \\ &\quad + 2\nu \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \eta_{\pm}(x, y) \, dx \, dy. \end{aligned}$$

Because (3.3) can be put in conservation form (because u and u^v are divergence-free), and in view of the marginal conditions for η_0^{\pm} , we notice that

$$\iint_{\mathbb{R}^2 \times \mathbb{R}^2} \eta_{\pm}(x, y) \, dx \, dy = \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \eta_0^{\pm}(x, y) \, dx \, dy = \|\omega_0^{\pm}\|_{L^1}.$$

We thus have and write

$$\begin{aligned} \frac{dQ_{\pm}}{dt} &= 2 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} (x - y) \cdot (u(x) - u(y)) \eta_{\pm}(x, y) \, dx \, dy \\ &\quad + 2 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} (x - y) \cdot (u(y) - u^v(y)) \eta_{\pm}(x, y) \, dx \, dy + 2\nu \|\omega_0^{\pm}\|_{L^1} \\ &=: I_{\pm}^1 + I_{\pm}^2 + 2\nu \|\omega_0^{\pm}\|_{L^1}. \end{aligned}$$

In order to estimate the first integral term, we use the log-Lipschitz estimate (1.8) for the velocity field,

$$|I_{\pm}^1| \lesssim Q_{\pm} + \iint_{\mathbb{R}^2 \times \mathbb{R}^2} |x - y|^2 \log\left(1 + \frac{1}{|x - y|}\right) \eta_{\pm}(x, y) \, dx \, dy.$$

Because $s \mapsto s \log\left(1 + \frac{1}{s}\right)$ is concave, we furthermore have with Jensen’s inequality

$$\begin{aligned} &\iint_{\mathbb{R}^2 \times \mathbb{R}^2} |x - y|^2 \log\left(1 + \frac{1}{|x - y|}\right) \eta_{\pm}(x, y) \, dx \, dy \\ &\lesssim \left(\iint_{\mathbb{R}^2 \times \mathbb{R}^2} |x - y|^2 \eta_{\pm}(x, y) \, dx \, dy \right) \\ &\quad \times \log\left(1 + \frac{\iint_{\mathbb{R}^2 \times \mathbb{R}^2} |x - y| \eta_{\pm}(x, y) \, dx \, dy}{\iint_{\mathbb{R}^2 \times \mathbb{R}^2} |x - y|^2 \eta_{\pm}(x, y) \, dx \, dy}\right) \end{aligned}$$

$$\begin{aligned} &\leq Q_{\pm} \log \left(1 + \left(\frac{\|\omega_0^{\pm}\|_{L^1}}{Q_{\pm}} \right)^{\frac{1}{2}} \right) \\ &\lesssim Q_{\pm} \left(1 + \log \left(1 + \frac{1}{Q_{\pm}} \right) \right). \end{aligned}$$

It thus follows that

$$I_{\pm}^1 \lesssim Q_{\pm} \left(1 + \log \left(1 + \frac{1}{Q_{\pm}} \right) \right).$$

For the second integral term, we use the Cauchy–Schwarz inequality and the marginal condition for η_{\pm} ,

$$\begin{aligned} |I_{\pm}^2| &\leq \left(\iint_{\mathbb{R}^2 \times \mathbb{R}^2} |x - y|^2 \eta_{\pm}(x, y) \, dx \, dy \right)^{\frac{1}{2}} \\ &\quad \times \left(\iint_{\mathbb{R}^2 \times \mathbb{R}^2} |u(y) - u^v(y)|^2 \eta_{\pm}(x, y) \, dx \, dy \right)^{\frac{1}{2}} \\ &= Q_{\pm}^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |u(y) - u^v(y)|^2 \omega_{\pm}^v(y) \, dy \right)^{\frac{1}{2}} \\ &\leq Q_{\pm}^{\frac{1}{2}} \|\omega_{\pm}^v\|_{L^{\infty}}^{\frac{1}{2}} \|u^v - u\|_{L^2}. \end{aligned}$$

It remains to notice that $\|\omega_{\pm}^v\|_{L^{\infty}} \leq \|\omega_0\|_{L^{\infty}}$. Moreover, the L^2 norm of the velocity difference is the \dot{H}^{-1} norm of the vorticity difference, which is bounded by the 2-Wasserstein distance; cf. (2.4). Hence, via the triangle inequality,

$$|I_{\pm}^2| \lesssim Q_{\pm}^{\frac{1}{2}} (W_2(\omega_+, \omega^v_+) + W_2(\omega_-, \omega^v_-)) \leq Q_{\pm}^{\frac{1}{2}} Q^{\frac{1}{2}}.$$

Combining the previous estimates yields

$$\frac{dQ}{dt} \lesssim Q_+^{\frac{1}{2}} Q^{\frac{1}{2}} + Q_-^{\frac{1}{2}} Q^{\frac{1}{2}} + Q_+ \left(1 + \log \left(1 + \frac{1}{Q_+} \right) \right) + Q_- \left(1 + \log \left(1 + \frac{1}{Q_-} \right) \right) + \nu,$$

which implies the statement of the lemma, because $s \mapsto s \left(1 + \log \left(1 + \frac{1}{s} \right) \right)$ is an increasing function on \mathbb{R}_+ . ■

It remains to integrate the differential inequality (3.6).

Lemma 3.3 For $\nu \ll 1$ and $t \ll \frac{1}{|\log \nu|}$, we have that

$$Q(t) \lesssim Q(0) + \nu t.$$

Moreover, for any fixed $t > 0$,

$$Q(t) = O \left(\left(Q(0) + \frac{\nu}{|\log \nu|} \right)^{\exp(-Ct)} \right)$$

as $\nu \rightarrow 0$, where $C > 0$ is a constant dependent only on $\|\omega_0\|_{L^1}$ and $\|\omega_0\|_{L^{\infty}}$.

Proof The argument is very elementary. We provide it for the convenience of the reader.

We remark that by replacing $Q(t)$ by $Q(t) + \nu t$, which satisfies the same differential inequality (3.6) because $s \mapsto s \left(1 + \log \left(1 + \frac{1}{s}\right)\right)$ is an increasing function, we may without loss of generality assume that

$$(3.7) \quad Q(t) \geq \nu t.$$

As a consequence of (3.5), there exists a time t_1 up to which $Q(t)$ is small in the sense that

$$Q\left(1 + \log\left(1 + \frac{1}{Q}\right)\right) \leq \nu,$$

and thus, (3.6) reduces to

$$\frac{dQ}{dt} \lesssim \nu,$$

which yields that

$$Q(t) \lesssim Q(0) + \nu t \quad \text{for } t \in [0, t_1].$$

This already proves the first statement. We only have to give an estimate on t_1 . For this, we notice that we can furthermore assume that $Q(0) \leq \nu t_1$, so that $Q(t_1) \sim \nu t_1$ thanks to (3.7). Then the final time t_1 is given by the estimate

$$(3.8) \quad \nu t_1 \left(1 + \log\left(1 + \frac{1}{\nu t_1}\right)\right) \sim \nu.$$

For $\nu \ll 1$, this relation and, more precisely, the behavior of the function on the left-hand side enforce $\nu t_1 \ll 1$. In particular, (3.8) can be restated as

$$\frac{1}{t_1} \sim \log \frac{1}{\nu t_1},$$

which entails that $t_1 \ll 1$. Therefore, $1/t_1$ is much larger than its logarithm, and, thus, by the product rule for the logarithm, we must have

$$\frac{1}{t_1} \sim \log \frac{1}{\nu}.$$

We now turn to the second estimate of Lemma 3.3. Since we are interested in an asymptotic statement for fixed times, we can always assume that $Q < 1$. Moreover, it is enough to consider the case

$$Q\left(1 + \log\left(1 + \frac{1}{Q}\right)\right) \geq \nu$$

for $t \geq t_1$, because otherwise Q would have to decrease in time. In this situation, (3.6) simplifies to

$$\frac{dQ}{dt} \lesssim Q\left(1 + \log \frac{1}{Q}\right).$$

This differential inequality can be rewritten as $C \log Q + \frac{d}{dt} \log Q \leq C$, for some $C > 0$, and thus $\frac{d}{dt} (e^{Ct} \log Q) \leq C e^{Ct}$. A short computation reveals that

$$Q(t) \lesssim Q(t_1)^{\exp(C(t_1-t))} \lesssim \left(\frac{\nu}{\log \frac{1}{\nu}} + Q(0) \right)^{\exp(-Ct)},$$

where we have used the above estimate on $Q(t_1)$ and the estimate for t_1 . ■

Proof of Theorem 3.1. From (3.4) and Lemma 3.3, we deduce that

$$W_2^2(\omega^v_+(t), \omega_+(t)) + W_2^2(\omega^v_-(t), \omega_-(t)) \lesssim Q(0) + \nu t$$

for times $t \ll \frac{1}{|\log \nu|}$. It remains to notice that we can set $Q(0) = 0$ by optimizing over $\eta_0^\pm \in \Pi(\omega_0^\pm, \omega_0^\pm)$. This concludes the proof of the first statement. The second one follows analogously. ■

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