Can. J. Math., Vol. XXVIII, No. 3, 1976, pp. 523–532

ON THE GENUS OF STRONG TENSOR PRODUCTS OF GRAPHS

B. L. GARMAN, R. D. RINGEISEN AND A. T. WHITE

1. Introduction and definitions. The genus parameter for graphs has been studied extensively in recent years, with impetus given primarily by the Ringel-Youngs solution to the Heawood Map-coloring Problem [**15**]. This solution involved the determination of $\gamma(K_n)$, the genus of the complete graph K_n. It has been natural to consider also the genus of graphs closely related to K_n: the *complete n-partite graph* $G = K_{p_1, p_2, \ldots, p_n}$ has vertex set $V(G) = \bigcup_{i=1}^{n} V_i$ (a disjoint union of nonempty sets), with edge set $E(G) = \{v_i v_j | v_i \in V_i, v_j \in V_j, i \neq j\}$; $p_i = |V_i|, 1 \leqq i \leqq n$. If $p_i = m$ for each $i = 1, 2, \ldots, n$, then G is regular and we write $G = K_{n(m)}$. Thus $K_n = K_{n(1)}$. Several genus results have been established for these families; see [**12**]. The existing techniques for imbedding graphs (see, for example, [**6**; **7**; and **23**]) are most readily applied for graphs which can be factored as a (possibly iterated) product of some kind (such as Q_n and related graphs, see [**20**]) or for graphs which are Cayley graphs for some finite group (such as $K_{n(m)}$, see [**24**]). The situation is particularly nice where triangular imbeddings are produced, as they are necessarily minimal. In this paper we introduce a graphical product which iterates, under the proper conditions, to produce triangular imbeddings of many families of graphs, including some of the families $K_{n(m)}$.

The graphical product we introduce is related to both the tensor product and the cartesian product. Let graphs G_1 and G_2 have vertex sets $V(G_1)$, $V(G_2)$ and edge sets $E(G_1)$, $E(G_2)$ respectively. The *tensor product* $G_1 \otimes G_2$ has vertex set $V(G_1) \times V(G_2)$ and edge set $E(G_1 \otimes G_2) = \{(u_1, u_2)(v_1, v_2) | u_i v_i \in E(G_i), i = 1, 2\}$. The *cartesian product* $G_1 \times G_2$ has vertex set $V(G_1) \times V(G_2)$ and edge set $E(G_1 \times G_2) = \{(u_1, u_2)(v_1, v_2) | u_1 = v_1$ and $u_2 v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1 v_1 \in E(G_1)\}$. For example, the n-cube Q_n is defined by: $Q_1 = K_2$, and $Q_n = Q_{n-1} \times K_2$, $n \geqq 2$. For more information on the tensor and cartesian products, see [**10**; **17**; and **18**]. We now define the *strong tensor product* $G_1 \underline{\otimes} G_2$ to have vertex set $V(G_1) \times V(G_2)$ and edge set $E(G_1 \underline{\otimes} G_2) = E(G_1 \otimes G_2) \cup \{(u_1, u_2)(v_1, v_2) | u_1 = v_1$ and $u_2 v_2 \in E(G_2)\}$. Thus $G_1 \underline{\otimes} G_2 = G_1 \otimes G_2 + p_1 G_2$, where $p_1 = |V(G_1)|$, "+" denotes the "sum" operation of [**2**], and $p_1 G_2$ denotes p_1 disjoint copies of G_2, as in [**9**].

If G has components $G_1, G_2, \ldots, G_n$, we will write $G = \bigcup_{i=1}^{n} G_i$, consistent with [**9**]. If G_1 and G_2 are isomorphic, we will write $G_1 = G_2$. As usual, $\chi(G)$

Received May 2, 1975.

https://doi.org/10.4153/CJM-1976-052-9 Published online by Cambridge University Press

will denote the chromatic number of G. If G is imbedded in a closed orientable 2-manifold S_k, we let G^* denote the dual pseudograph for this imbedding. We say that the imbedding has *bichromatic dual* if $\chi(G^*) = 2$. The first Betti number of a connected graph G is given by $\beta(G) = q - p + 1$, where $q = |E(G)|$ and $p = |V(G)|$. The closed orientable 2-manifold of genus n will be denoted by S_n, $n = 0, 1, 2, \ldots$.

2. Basic properties of the strong tensor product. This product has interest in its own right, apart from genus considerations. In support of this contention, as well as for use in subsequent calculations, we offer the following elementary observations. Let G, G_1, G_2 have p, p_1, p_2 vertices and q, q_1, q_2 edges respectively. Let $d(v_i)$ be the degree of v_i in G_i, $i = 1, 2$.

PROPOSITION 1. *The degree of* (v_1, v_2) *in* $G_1 \underline{\otimes} G_2$ *is given by* $d(v_1, v_2) = d(v_2)(d(v_1) + 1)$.

Using Proposition 1, we obtain

PROPOSITION 2. *For* $G = G_1 \underline{\otimes} G_2$, $p = p_1p_2$ *and* $q = q_2(p_1 + 2q_1)$.

Proof. The equality for p is obvious. For q, we note that

$$2q = \sum_{v_i \in V(G_i)} d(v_1, v_2) = \sum_{v_i \in V(G_i)} d(v_2)(d(v_1) + 1)$$
$$= \sum_{v_2 \in V(G_2)} d(v_2) \sum_{v_1 \in V(G_1)} (d(v_1) + 1) = 2q_2(2q_1 + p_1)$$

so that $q = q_2(p_1 + 2q_1)$.

PROPOSITION 3. *For* $G \neq K_1$, $G \underline{\otimes} H$ *is connected if and only if* G *and* H *are both connected with* $H \neq K_1$. *Moreover, if* $G = \bigcup_{i=1}^m G_i$ *and* $H = \bigcup_{i=1}^n H_j$, *with no* $H_j = K_1$, *then*

$$G \underline{\otimes} H = \bigcup_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} G_i \underline{\otimes} H_j$$

(*where all unions are over components*).

PROPOSITION 4. $\chi(G_1 \underline{\otimes} G_2) = \chi(G_2)$.

As a special case of Proposition 4, we have

PROPOSITION 5. *If* G_2 *is bipartite, then* $G_1 \underline{\otimes} G_2$ *is bipartite.*

Moreover, we have

PROPOSITION 6. *If* G_1 *is bipartite, then* $G_1 \underline{\otimes} K_2 = G_1 \times K_2$.

Now define $G_1 = K_2$ and $G_n = G_{n-1} \underline{\otimes} K_2$ for $n \geqq 2$; using Proposition 6 and an easy induction argument, we have

PROPOSITION 7. $G_n = Q_n$.

https://doi.org/10.4153/CJM-1976-052-9 Published online by Cambridge University Press

Consistent with Proposition 4, we have the following:

PROPOSITION 8. $K_m \underline{\otimes} K_{p_1,p_2,\ldots,p_n} = K_{mp_1,mp_2,\ldots,mp_n}$.

Thus taking $G_1 = K_m$ in the strong tensor product, with G_2 complete n-partite, has the effect of multiplying the order of each partite set by m. In particular,

PROPOSITION 9. $K_m \underline{\otimes} K_{n(r)} = K_{n(mr)}$; *for* $r = 1$, *we have* $K_m \underline{\otimes} K_n = K_{n(m)}$.

Propositions 8 and 9 will prove useful in subsequent genus computations. Since $K_2 = K_{1,1}$, we can also apply Proposition 8 twice to see that $K_2 \underline{\otimes} (K_2 \underline{\otimes} K_2) = K_{4,4}$. But, by Proposition 7, $(K_2 \underline{\otimes} K_2) \underline{\otimes} K_2 = Q_3 \neq K_{4,4}$. Hence

PROPOSITION 10. *The strong tensor product operation is neither associative nor commutative.*

A special case of Proposition 3 gives $\bar{K}_p \underline{\otimes} G = pG$ (where $\bar{K}_p$ is the complement of K_p). Taking $p = 1$, we have $K_1 \underline{\otimes} G = G$. Thus

PROPOSITION 11. *The strong tensor product operation has* K_1 *as a left identity.*

Because $G_1 \underline{\otimes} G_2$ has p_1p_2 vertices, K_1 is also the only possible right identity. However, since $G \underline{\otimes} K_1 = \bar{K}_p$, where $p = V(G)$, we have

PROPOSITION 12. *The strong tensor product operation has no right identity.*

3. An imbedding technique. The strong tensor product is valuable not only for factoring graphs such as Q_n and $K_{n(m)}$; it is also amenable for the construction of triangular imbeddings of product graphs, given an appropriate triangular imbedding of one of the factors.

THEOREM 1. *Let* G_2 *have a triangular imbedding in a closed orientable* 2-*manifold* S_h, *with bichromatic dual. Let* G_1 *be connected and bichromatic, with maximum degree at most two. Then* $G_1 \otimes G_2$ *has a triangular imbedding in a closed orientable* 2-*manifold* S_k, *with bichromatic dual.*

Proof. We regard $G_1 \underline{\otimes} G_2$ as $p_1G_2 + G_1 \otimes G_2$. Let $M_i = S_h$, $i = 1, 2, \ldots, p_1$, be imbedded in R^3, with M_i exterior to M_j for $i \neq j$. Begin with identical imbeddings of G_2 on $M_1, M_3, M_5, \ldots$, each triangular and with bichromatic dual and (say) clockwise orientation. Now form the same imbeddings, but with counterclockwise orientation, of G_2 on $M_2, M_4, M_6, \ldots$; thus these imbeddings are "mirror images" of those previously formed. Let regions $R_1, R_3, \ldots, R_{r_2-1}$ be colored black and regions $R_2, R_4, \ldots, R_{r_2}$ (where r_2 is the number of regions for the imbedding of G_2 on S_h) be colored white on each of $M_1, M_3, M_5, \ldots$, in consonance with $\chi(G_2^*) = 2$. (Since the dual is bichromatic, the numbers of regions of the two colors agree—each is $q_2/3$, since each edge of G_2 appears in exactly one (triangular) region of each color.)

Let R_i' be the mirror image of R_i ($i = 1, 2, \ldots, r_2$) on $M_2, M_4, M_6, \ldots$. Color regions $R_1', R_3', \ldots, R_{r_2-1}'$ with white and regions $R_2', R_4', \ldots, R_{r_2}'$ with black on each of $M_2, M_4, M_6, \ldots$. We must add the tensor product edges to form $G_1 \underline{\otimes} G_2$ from $p_1 G_2$. This will be accomplished by attaching cylinders among the surfaces M_i so as to form S_k, and then triangulating the cylinders with these edges. We will see that the resulting triangular imbedding of $G_1 \underline{\otimes} G_2$ also has bichromatic dual.

Consider R_1 in M_1 and R_1' in M_2. Excise two open disks, D_1 and D_1', from the interiors of R_1 and R_1' respectively, as indicated in Figure 1. Let simple closed boundary curves C_1 and C_1' bound D_1 and D_1' respectively. Let T_1 be a topological cylinder, with simple closed boundary curves B_1 and B_1'. Identify B_1 with C_1 and B_1' with C_1'. The edges xy', xz', yz', yx', zx', zy' can now be imbedded along T_1, as shown in Figure 1. Note that six triangular regions are formed along T_1 and that these can be 2-colored consistently with the 2-colorings of M_1 and M_2. Now repeat this process, joining region R_i in M_1 with R_i' in M_2 by cylinder T_i, $i = 3, 5, \ldots, r_2 - 1$, adding the six required tensor product edges along each cylinder. At this stage we have added precisely $6(r_2/2) = 6(q_2/3) = 2q_2$ edges and $K_2 \underline{\otimes} G_2$ is triangularly imbedded on $M_1 \cup M_2$ (as altered by cylinder attachment), with bichromatic dual. (Since each edge of G_2 appears in exactly one (triangular) region colored black in M_1, these $2q_2$ edges are exactly those needed at this stage.)

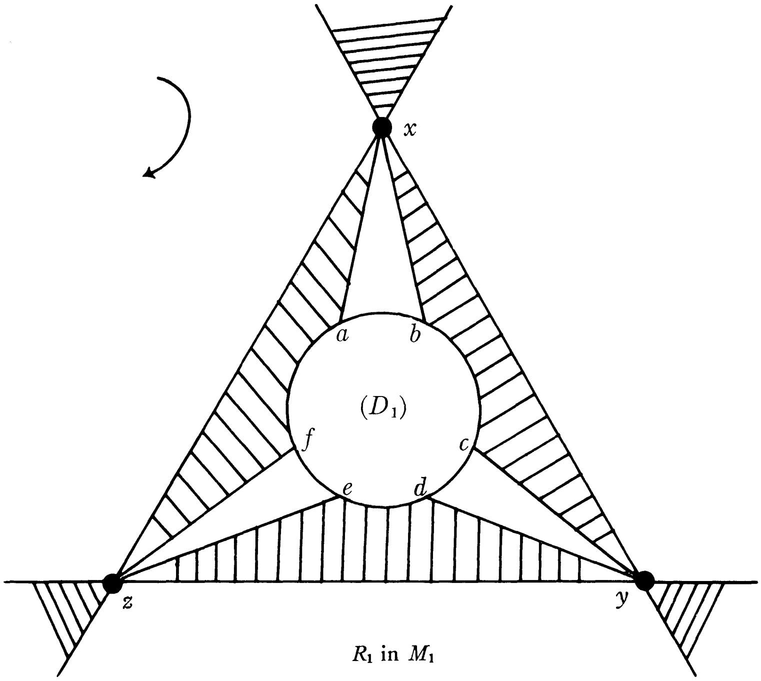
We now repeat this process, joining M_i to M_{i+1}, $i = 2, 3, \ldots, p_1 - 1$, by attaching cylinders between mirror-image regions of opposite color. (Regions colored black in M_i are joined to their counterparts colored white in M_{i+1}.) At each stage we have a triangular imbedding, with bichromatic dual. We now note that the hypotheses on G_1 imply that G_1 is either a path or an even cycle. If G_1 is a path, we are done. If G_1 is an even cycle, then M_{p_1} is joined also to M_1, in the same fashion; a triangular imbedding of $G_1 \underline{\otimes} G_2$, with bichromatic dual, again results.

COROLLARY 1. *Under the conditions of Theorem* 1, *where* $h = \gamma(G_2)$, $\gamma(G_1 \underline{\otimes} G_2) = k = p_1 h + q_1((q_2/3) - 1) + \delta$, *where* $\delta = 0$ *if* G_1 *is a path and* $\delta = 1$ *if* G_1 *is an even cycle.*

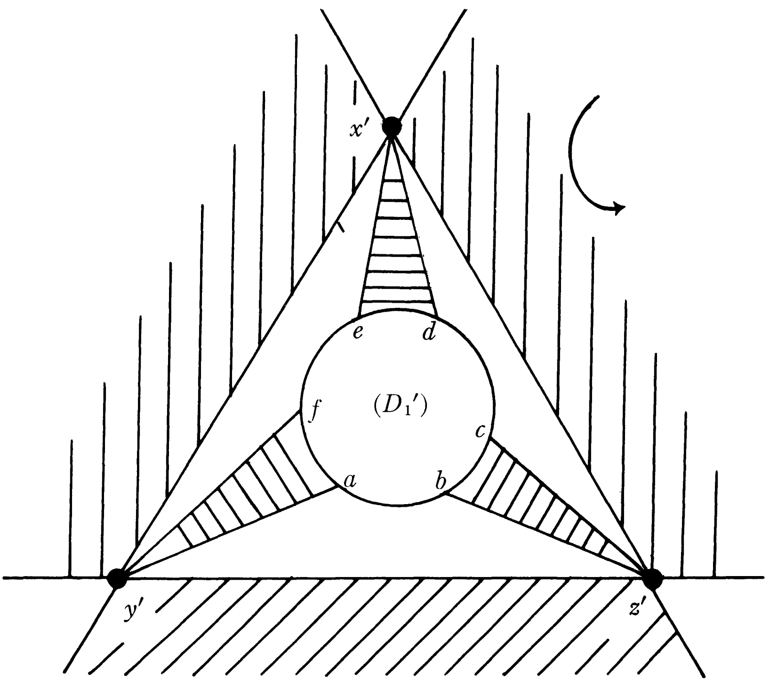
Proof. Use Proposition 2 and the well-known fact that if G is triangularly imbedded in S_n, then $n = 1 - p/2 + q/6$. Alternately, we observe that we commenced our construction with p_1 disjoint copies of S_n, that we constructed q_1 joins—each contributing $q^2/3 - 1$ to the genus of S_k, and that $\beta(G_1) = \delta$. (See [**21**].)

We remark that the construction of Theorem 1 fails if any one of the following occurs:

(i) G_1 has a vertex of degree 3 or more.
(ii) G_1 is not bichromatic.
(iii) The imbedding of G_2 is not triangular.
(iv) The imbedding of G_2 does not have bichromatic dual.



R_1 in M_1



R_1' in M_2

FIGURE 1

https://doi.org/10.4153/CJM-1976-052-9 Published online by Cambridge University Press

Yet the conjunction of the four conditions (in their affirmative forms) occurs sufficiently often, as we will see in Section 5, so as not to detract from the power of the theorem.

4. The main theorem. Let $H_0 = G_2$, and $H_n = G_1 \otimes H_{n-1}$, for $n \geqq 1$. By Proposition 3 and a result of Battle, Harary, Kodama, and Youngs [**1**] that the genus parameter is additive over the components of a graph, we take G_1 and G_2 to be connected, without loss of generality.

THEOREM 2. *Let G_2 have a triangular imbedding in a closed orientable 2-manifold, with bichromatic dual. Let G_1 be bichromatic with maximum degree at most two. Then*

$$\gamma(H_n) = 1 + \frac{q_2}{6}(p_1 + 2q_1)^n - \frac{p_2}{2}(p_1)^n, \quad n = 0, 1, 2, \ldots .$$

Proof. We apply Theorem 1 n times, to obtain a triangular imbedding of H_n. Let H_n have p_n vertices and q_n edges; then $\gamma(H_n) = 1 - p_n/2 + q_n/6$. But easy induction arguments show that $p_n = p_1^n p_2$ and (using Proposition 2) $q_n = q_2(p_1 + 2q_1)^n$.

5. Applications to new genus results. In applying Theorem 2 (or the case $n = 1$ of Theorem 2, which is Theorem 1), we must choose G_1 as the path P_m or an even cycle C_{2m}. As the special case $G_1 = P_2 = K_2$ allows us to apply Propositions 8 and 9, this choice will often be productive. There are many possible choices for G_2. Each such choice will lead to a family of genus formulas, via Theorem 2. (We will not state specific formulas explicitly, unless we believe them to be of particular interest.)

It is well-known (see, for example, [**25**]) that a connected planar graph G has bichromatic dual if and only if it is eulerian. Moreover, such a graph G triangulates the sphere if and only if it is maximal planar. Thus:

THEOREM 3. *A graph G triangulates the sphere, with bichromatic dual, if and only if it is maximal planar and eulerian.*

Using the fact that a maximal planar graph G has $q = 3p - 6$, we apply Theorem 1 to find a family of strong tensor products with genus asymptotic to the order of the second factor.

THEOREM 4. *If G is maximal planar eulerian, then $\gamma(K_2 \otimes G) = p - 3$.*

Unfortunately, the characterization of Theorem 3 does not extend to surfaces of positive genus. The graph G of Figure 2 triangulates the torus and is eulerian, yet its dual has chromatic number at least three, as indicated by the 7-cycle running vertically through the middle of the rectangle depicting the torus. It is easy to verify, however, that if G on $S_k (k \geqq 0)$ has $\chi(G^*) = 2$, then G must be eulerian.

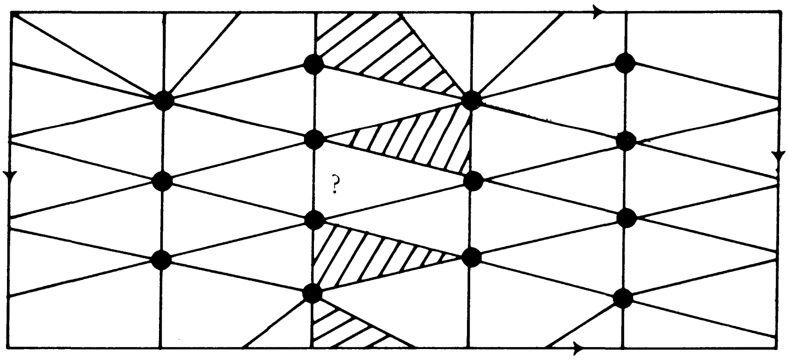


FIGURE 2

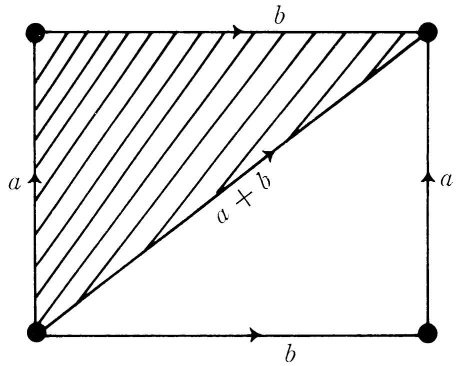


FIGURE 3

We next produce an infinite collection of triangular imbeddings on the torus, each with bichromatic dual. We use the *voltage graph* construction introduced by Gross in [**6**], to imbed a variety of toroidal *Cayley graphs* (see [**24**]) as follows. Consider the voltage graph of Figure 3, imbedded in the torus. (Here the "graph" is actually a loop graph H, but this is allowed by the theory.) Let Γ be any abelian group with generating set $\Delta = \{a, b, a + b\}$, where $\Delta \cap \Delta^{-1} = \phi(\Delta^{-1} = \{\delta^{-1} | \delta \in \Delta\})$. Then the "Kirchoff Voltage Law" holds around each region boundary, and the theory of voltage graphs guarantees not only that the Cayley graph $G_\Delta(\Gamma)$ covering H is triangularly imbedded in S_1 (among the surfaces S_k, only S_1 can cover S_1; see [**3**]), but also that the dual of $G_\Delta(\Gamma)$ in S_1 is bichromatic—since the dual of H in S_1 is bichromatic.

For example, if we take $a = 1$ and $b = 2$ in $\Gamma = Z_n$ $(n \geqq 7)$, we get an infinite collection of Cayley graphs $G_\Delta(Z_n)$, each triangulating the torus with bichromatic dual. The first two cases are of special interest: $G_\Delta(Z_7) = K_7$, and $G_\Delta(Z_8) = K_{4(2)}$. Or, we can take $a = (1, 0)$ and $b = (0, 1)$ for

https://doi.org/10.4153/CJM-1976-052-9 Published online by Cambridge University Press

$\Gamma = Z_m + Z_n$ $(m, n \geqq 3)$. The simplest case here gives $G_\Delta(Z_3 + Z_3) = K_{3(3)}$.

Letting $G_1 = K_2$ and $G_2 = K_7$ in Theorem 2, we obtain the following formula for the genus of a class of 7-partite graphs.

THEOREM 5. $\gamma(K_{7(2^k)}) = 1 + 7 \cdot 2^{k-1}(2^k - 1)$, $k \geqq 0$.

Proof. The case $k = 0$ is $\gamma(K_7) = 1$, since $K_{7(1)} = K_7$. For $k > 0$, apply Theorem 2 to compute $\gamma(H_k)$; now apply Proposition 8 repeatedly, to see that $H_k = K_{7(2^k)}$.

It can in fact be shown that *every* minimal imbedding of K_7 triangulates S_1 with bichromatic dual, although we do not do so here. We do establish, however, a similar result for $K_{3(m)}$. Recall that, for each natural number m, every minimal imbedding of $K_{3(m)}$ is a triangulation (see [**16**] or [**19**]). The case $m = 3$ below has been verified independently by Figure 3.

THEOREM 6. *Let* $G = K_{3(m)}$ *be minimally imbedded; then* $\chi(G^*) = 2$.

Proof. Let $V(G) = V_1 \cup V_2 \cup V_3$ be the partite set partition. Then every region is triangular and has an oriented clockwise boundary of exactly one of the two following forms:

(a) v_1, v_2, v_3, v_1

(b) v_1, v_3, v_2, v_1,

where $v_i \in V_i$, $i = 1, 2, 3$. Color each region of form (a) black, while coloring each of form (b) white. This provides a 2-coloring of the regions, as we can see by considering an arbitrary edge of G, say v_1v_2. This edge bounds two regions, colored as indicated in Figure 4.

In [**22**] it has been shown that K_{p_1,p_2,p_3} has an orientable triangular imbedding if and only if $p_1 = p_2 = p_3$. Thus among complete tripartite graphs, it is exactly those which are regular which can serve as the graph G_2 in Theorem 2. Moreover, these can be taken with arbitrarity large genus.

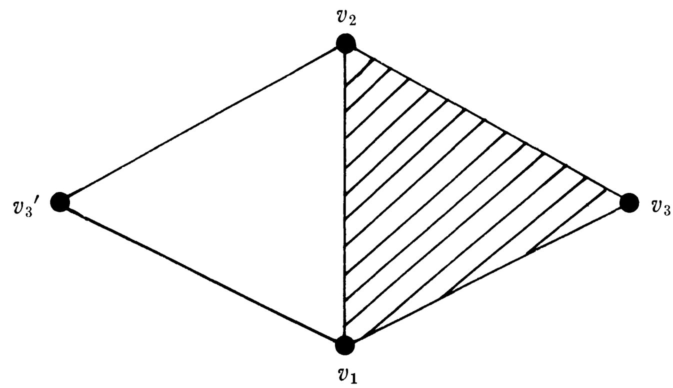


FIGURE 4

https://doi.org/10.4153/CJM-1976-052-9 Published online by Cambridge University Press

We next consider the family $K_{4(n)}$. Ringel [**13**] conjectured in 1969 that $\gamma(K_{4(n)}) = (n-1)^2$ for all n. In 1973, Garman [**4**] affirmed this conjecture, for $n \equiv 2 (\mathrm{mod}\ 4)$. In 1974 Jungerman [**11**] affirmed the conjecture, for all $n \neq 3$. Both solutions employ *current graphs* (see [**7**] or [**8**]), dual to voltage graphs. The current graphs of Garman [**5**] are bichromatic, and the theory of current graphs guarantees that the imbeddings of $K_{4(n)}$ ($n \equiv 2 \ (\mathrm{mod}\ 4)$) they produce have bichromatic dual. We can now show:

THEOREM 7. *$K_{4(n)}$ has a triangular imbedding with bichromatic dual if and only if n is even.*

Proof. If n is odd, $K_{4(n)}$ is regular of odd degree $3n$ and hence *no* dual can be 2-colored. If $n \equiv 2 \ (\mathrm{mod}\ 4)$, we are done by the remarks preceding the theorem. If $n \equiv 0 \ (\mathrm{mod}\ 4)$, we write $n = 2^s \cdot m = 2^{s-1} \cdot 2m$, where $s \geqq 2$ and m is odd; thus $2m \equiv 2 \ (\mathrm{mod}\ 4)$. If $m = 1$, we take $G_2 = K_{4(2)}$ imbedded in S_1 as given by Figure 3 and $G_1 = K_2$ in Theorem 2 to get a triangular imbedding for $K_{4(2^s)}$; Theorem 1 guarantees that the dual is bichromatic. If $m \geqq 3$, we take $G_2 = K_{4(2m)}$ imbedded in the manner described by Garman and $G_1 = K_2$; again Theorems 1 and 2 give the desired result.

Finally, we consider $G_2 = K_{12r+3} = K_{(12r+3)(1)}$ and its triangular imbeddings as given by Ringel in [**14**]; these all have bichromatic dual. Taking $G_1 = K_2$ in Theorem 2, we have the following:

THEOREM 8.

$$\gamma(K_{n(m)}) = \frac{(mn-3)(mn-4)}{12} - \frac{mn(m-1)}{12}, \quad \textit{for } n \equiv 3 \ (\mathrm{mod}\ 12)$$
$$\textit{and } m = 2^k, \quad k \geqq 0.$$

For other values of m and n, the equality of Theorem 8 will hold if and only if $K_{n(m)}$ has a triangular imbedding in some closed orientable 2-manifold. It is well-known that K_n has an orientable triangular imbedding if and only if $n \equiv 0, 3, 4, 7 \ (\mathrm{mod}\ 12)$. For $n \equiv 0, 4 \ (\mathrm{mod}\ 12)$, K_n is not eulerian and hence the duals are not bichromatic.

The case $n \equiv 3$ is analyzed above. For $n = 12r + 7$, $r \geqq 1$, the existing current graphs are *not* bichromatic, so that the construction of Theorem 8 is not yet known to apply.

We give here a summary of the current knowledge about this problem; $K_{n(m)}$ has an orientable triangular imbedding for

(1) $n = 2$: no values of m (bipartite graphs have no odd cycles)
(2) $n = 3$ and all m ([**16**] or [**19**])
(3) $n = 4$ if and only if $m \neq 3$ ([**4**] and [**11**])
(4) $n = 7$ and $m = 2^k$ ($k \geqq 0$) (Theorem 5)
(5) $n \equiv 3 \ (\mathrm{mod}\ 12)$, $m = 2^k$ ($k \geqq 0$) (Theorem 8)
(6) $m = 1$ if and only if $n \equiv 0, 3, 4, 7 \ (\mathrm{mod}\ 12)$ ([**15**])
(7) $m = 2$ and $n = 6, 7$; $n \equiv 4 \ (\mathrm{mod}\ 6)$, or $n \equiv 3 \ (\mathrm{mod}\ 12)$ ([**24**, Theorem 5 above, **7**, Theorem 8 above, respectively]).

https://doi.org/10.4153/CJM-1976-052-9 Published online by Cambridge University Press

References

1. J. Battle, F. Harary, Y. Kodama, and J. W. T. Youngs, *Additivity of the genus of a graph*, Bull. Amer. Math. Soc. *68* (1962), 565–568.
2. M. Behzad and G. Chartrand, *An introduction to the theory of graphs* (Allyn and Bacon, Boston, 1971).
3. E. S. Cairns, *Introductory topology* (Ronald Press, New York, 1968).
4. B. Garman, *On the genus of* $K_{n,n,n,n}$, Notices Amer. Math. Soc. *20* (1973), A-564.
5. ——— *Imbedding Cayley graphs*, Specialist thesis, Western Michigan University, 1974.
6. J. L. Gross, *Voltage graphs*, Discrete Math. *9* (1974), 239–246.
7. J. L. Gross and S. R. Alpert, *Branched coverings of current graph imbeddings*, Bull. Amer. Math. Soc. *79* (1973), 942–945.
8. W. Gustin, *Orientable embedding of Cayley graphs*, Bull. Amer. Math. Soc. *69* (1963), 272–275.
9. F. Harary, *Graph theory* (Addison-Wesley, Reading, Mass., 1969).
10. F. Harary and G. Wilcox, *Boolean operations on graphs*, Math. Scand. *20* (1967), 41–51.
11. M. Jungerman, *The genus of the symmetric quadripartite graph*, J. Combinatorial Theory *B19* (1975), 181–187.
12. H. V. Kronk, R. D. Ringeisen, and A. T. White, *On 2-cell imbeddings of complete n-partite graphs*, to appear, Colloq. Math.
13. G. Ringel, *Genus of graphs*, Combinatorial Structures and their Applications, R. Guy, H. Hanani, N. Sauer, and J. Schonheim, editors (Gordon and Breach, New York, 1970), 361–366.
14. ——— *Map color theorem* (Springer-Verlag, Berlin, 1974).
15. G. Ringel and J. W. T. Youngs, *Solution of the Heawood map-coloring problem*, Proc. Nat. Acad. Sci. U.S.A. *60* (1968), 438–445.
16. ——— *Das Geschlecht des symmetrische vollstandige dreifarbaren Graphen*, Comment. Math. Helv. *45* (1970), 152–158.
17. G. Sabidussi, *Graph multiplication*, Math. Z. *72* (1960), 446–457.
18. P. M. Weichsel, *The Kronecker product of graphs*, Proc. Amer. Math. Soc. *13* (1963), 47–52.
19. A. T. White, *The genus of the complete tripartite graph* $K_{mn,n,n}$, J. Combinatorial Theory *7* (1969), 283–285.
20. ——— *The genus of repeated cartesian products of bipartite graphs*, Trans. Amer. Math. Soc. *151* (1970), 393–404.
21. ——— *On the genus of the composition of two graphs*, Pacific J. Math. *41* (1972), 275–279.
22. ——— *The genus of cartesian products of graphs*, Ph.D. Thesis, Michigan State University, 1969.
23. ——— *Graphs, groups, and surfaces* (North-Holland, Amsterdam, 1973).
24. ——— *Orientable imbeddings of Cayley graphs*, Duke Math. J. *41* (1974), 353–371.
25. R. Wilson, *Introduction to graph theory* (Academic Press, New York and London, 1972).

Western Michigan University,
Kalamazoo, Michigan 49008;
Purdue University,
Fort Wayne, Indiana 46805