PART II

STABILITY AND COLLAPSE

THE STABILITY OF RELATIVISTIC SYSTEMS

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Abstract. The stability of relativistic systems is reviewed against the background of what is known in the corresponding contexts of the Newtonian theory. In particular, the importance of determining whether Dedekind-like points of bifurcation occur along given stationary axisymmetric sequences is emphasized: the occurrence of such points of bifurcation may signal the onset of secular instability induced by radiation-reaction. (At a Dedekind-like point of bifurcation, the system can be subject, quasistationarily, to a non-axisymmetric deformation with an $e^{2i\phi}$-dependence on the azimuthal angle ϕ.)

A formalism is described in terms of which the normal modes of axisymmetric oscillation of axisymmetric systems can be determined. Specialized to neutral modes of oscillation the formalism provides an alternative proof of Carter's theorem and clarifies the minimal requirements for its validity. A parallel formalism is described for ascertaining whether an axisymmetric system can be subject to a quasi-stationary non-axisymmetric deformation. The possibility of applying this latter formalism to determining whether a Dedekind-like point of bifurcation occurs along the Kerr sequence is considered.

1. Introduction

During recent years the stability of a variety of systems in general relativity has been considered by a number of investigators by differing methods. In this paper, an approach to a substantial class of them will be presented which will make them appear as special cases of an effectively single mathematical theory. Moreover, the approach will be motivated by an attempt to develop the problems in general relativity closely parallel to the corresponding developments in the Newtonian theory.

The systems whose stability we shall consider fall into two categories: those which are spherically symmetric and those which are axisymmetric (by virtue of rotation, for example). In the case of spherical systems, we shall be interested in both radial and non-radial oscillations and in the instabilities which derive from them; and in the case of axisymmetric systems we shall be interested in oscillations which preserve the axisymmetry and in those which do not. And in all cases, we shall be interested in criteria for the occurrence of neutral modes of oscillation since their occurrence often signals the onset of instabilities.

2. Known Results in the Newtonian Theory

We shall first enumerate the known results of stability analysis in the Newtonian theory.

With respect to spherical systems, it is known that stability with respect to radial oscillations depends on an average value of the adiabatic exponent, γ, which relates the Lagrangian changes, Δp in the pressure and $\Delta\varrho$ in the density, which a fluid element experiences during its motion:

$$\Delta p/p = \gamma\, \Delta\varrho/\varrho. \tag{1}$$

Thus, instability will occur if

$$\bar{\gamma} = \frac{\displaystyle\int_0^M \gamma p \, dM(r)}{\displaystyle\int_0^M p \, dM(r)} < \tfrac{4}{3} \qquad [dM(r) = 4\pi r^2 \varrho \, dr] \; ; \tag{2}$$

and the *e*-folding time of the instability (when it occurs) is of the same order as the period of pulsation (when the configuration is stable).

The *dynamical instability* to radial perturbations which occurs when $\bar{\gamma} < \tfrac{4}{3}$ is a *global instability* in the sense that its occurrence depends on the structure of the configuration in its entirety. It is a remarkable fact that there are no other global instabilities to which a spherical distribution of mass is subject: instabilities that may arise from non-radial perturbations are all *local* and originate (as Lebovitz first regorously showed) in the violations of the Schwarzschild criterion for convective stability.

When we consider rotating systems which are axially symmetric in the stationary state, the instabilities that may arise are of two kinds. First, the continuation into the rotating domain of the global instability that occurs in non-rotating systems for radial perturbations; and second, instabilities that derive from the centrifugal and coriolis forces that are operative in rotating systems, i.e. in instabilities that are peculiar to rotating systems.

It is known (cf. Lebovitz, 1970) that the former type of instabilities result from modes of axisymmetric oscillations; indeed, for slow rotation, the condition (2) is replaced by

$$\bar{\gamma} - \tfrac{4}{3} + \text{constant} \, \frac{\Omega^2}{\pi G \bar{\varrho}} < \tfrac{4}{3}, \tag{3}$$

where $\bar{\varrho}$ is the mean density of the configuration. More generally, an exact expression can be given for the fundamental frequency (σ) of axisymmetric oscillation of a slowly rotating configuration in terms of the frequency (σ_0) of radial oscillation of the non-rotating configuration and the ($l = 0$)-distortion of it by the rotation. Thus,

$$\sigma^2 = \sigma_0^2 + \Omega^2 \sigma_1^2 + O(\Omega^4), \tag{4}$$

where σ_1^2 depends only on the proper solution ξ_0 belonging to σ_0 and the spherically symmetric part of the distortion caused by the rotation.

With regard to the latter type of instabilities, i.e. those that are peculiar to rotating systems, they derive from non-axisymmetric perturbations; and these instabilities are of two kinds: *secular* and *dynamical*. We shall clarify their nature and their origin by considering the classical sequences of uniformly rotating homogeneous masses. (For more detailed information on what follows, see Chandrasekhar, 1969).

It is well known that a possible sequence of equilibrium figures for rotating homogeneous masses is represented by the Maclaurin sequence of oblate spheroids. When one examines the second-harmonic oscillations of the Maclaurin spheroid, in a frame of reference rotating with its angular velocity, one finds that for two of these modes, whose dependence on the azimuthal angle is given by $e^{2i\phi}$, the squares of the characteristic frequencies depend on the eccentricity e in the manner illustrated in Figure 1. It will be observed that one of the modes becomes neutral (i.e. $\sigma^2 = 0$) when $e = 0.813$ and that the two modes coincide when $e = 0.953$ and become complex conjugates of one another beyond this point. Accordingly, the Maclaurin spheroid becomes *dynamically instable* at the latter point (first isolated by Riemann). On the other hand, the origin of the neutral mode at $e = 0.813$ is that at this point a new equilibrium sequence of tri-axial ellipsoids – the ellipsoids of Jacobi – bifurcate. On this latter account it was conjectured by Lord Kelvin in 1883 that "if there be any viscosity, however slight, in the liquid, or if there be any imperfectly elastic solid, however small, floating on it or sunk within it, the equilibrium [beyond $e = 0.81$] cannot be secularly stable." Lord Kelvin's conjecture has been confirmed by an explicit investigation of the effect of a small amount of viscous dissipation on the two modes illustrated in Figure 1. It is found that viscous dissipation makes the mode, which becomes neutral at $e = 0.81$, unstable beyond this point with an e-folding time which depends inversely on the magnitude of the kinematic viscosity and which further decreases monotonically to zero at the point ($e = 0.953$) of onset of dynamical instability. (The e-folding time of the instability becomes proportional to $v^{1/2}$ in the immediate neighborhood of $e = 0.95$.)

However, it should not be concluded that *any* dissipative mechanism will make the Jacobi mode unstable beyond the point of bifurcation. If we ask, for example, what effect the dissipative forces derived from radiation-reaction of general-relativistic origin, has on the secular stability of the Maclaurin spheroid at $e = 0.813$, we find (Chandrasekhar, 1970) that it does *not* induce any stability in the Jacobi mode; instead, it induces instability in the *alternative* mode at the same eccentricity. In the first instance this may seem surprising; but the situation we encounter here clarifies some important issues.

If instead of analyzing the normal modes in the rotating frame, we had analyzed them in the inertial frame, we should have found that the mode, which becomes unstable by radiation-reaction at $e = 0.813$, is in fact neutral at this point. And the neutrality of *this* mode in the inertial frame corresponds to the fact that the neutral deformation is associated with the bifurcation at this point of a new tri-axial sequence – the sequence of Dedekind ellipsoids. These Dedekind ellipsoids, while they are congruent to the Jacobi ellipsoids, differ from them in that they are at rest in the inertial frame and owe their tri-axial figures to internal vortical motions. An important conclusion that would appear to follow from these facts is that in the framework of general relativity we can expect secular instability, derived from radiation-reaction, to arise by a Dedekind mode of deformation (which is quasi-stationary in the inertial frame) rather than by a Jacobi mode (which is quasi-stationary in a rotating frame). A

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further fact which requires to be emphasized in this context is that the notion of a neutral point is subject to ambiguity arising from the freedom we have in the choice of a coordinate frame in which we may wish to specify the characteristic frequencies belonging to the various normal modes. It is important to observe in this connection that while for uniformly rotating objects, the inertial frame and the frame rotating with the object are naturally distinguished, no rotating frame is naturally distinguished when the object is rotating non-uniformly. Accordingly, in the case of non-uniform
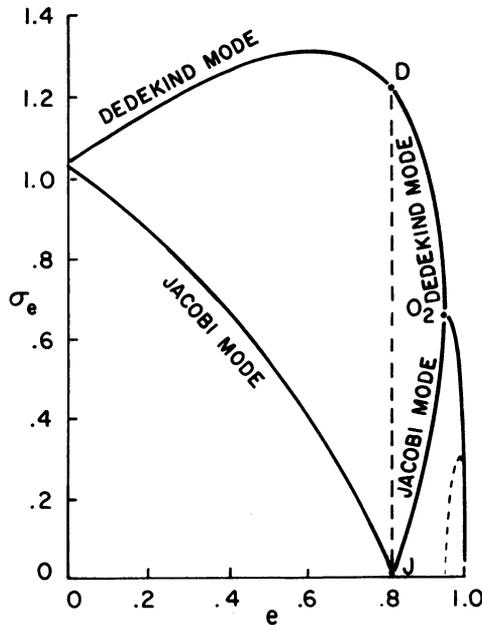

Fig. 1. The characteristic frequencies (in the unit $(\pi G\varrho)^{1/2}$) of the two even modes of second-harmonic oscillation of the Maclaurin spheroid. The Jacobi sequence bifurcates from the Maclaurin sequence by the mode that is neutral ($\sigma^2 = 0$) at $e = 0.813$; and the Dedekind sequence bifurcates by the alternative mode at D. At O_2 ($e = 0.9529$) the Maclaurin spheroid becomes dynamically unstable. The real and the imaginary parts of the frequency, beyond O_2, are shown by the full-line and the dashed curves, respectively. Viscous dissipation induces instability in the branch JO_2 of the Jacobi mode; and radiation-reaction induces instability in the branch DO_2 of the Dedekind mode.

rotation, the one secure concept is that of dynamical instability. The points at issue here are important particularly in view of differentially rotating compressible models that have recently been constructed by Ostriker and his associates (Tassoul and Ostriker, 1968; Ostriker and Tassoul, 1969; Ostriker and Bodenheimer, 1973; Durisen, 1973a, b) and their demonstration that these sequences have extraordinary similarity with the classical Maclaurin sequence. These investigations on differentially rotating systems would lead one to suppose that during the natural evolution of rotating systems, disc-like objects cannot come into being by virtue of secular or dynamical instability setting in long before the objects become anything like disc-like (cf. Chandrasekhar and Lebovitz, 1973; Chandrasekhar, 1974).

3. Known Results in General Relativity

The stability of spherically symmetric fluid masses in general relativity is one of the first problems that was fully and rigorously solved (Chandrasekhar, 1964; Fowler, 1964). And the theory parallels the Newtonian theory very closely. Thus, the criterion for instability is again an inequality for an average value of γ though it cannot in general be expressed as simply as in Equation (2). By actual numerical integrations (Bardeen *et al.*, 1966; Thorne and Meltzer, 1966) of the pulsation equation it has been shown that instability can occur under circumstances in which the Newtonian theory will predict stability. This *relativistic instability* is of particular importance when γ is close to, but greater than $\frac{4}{3}$, as it will be the case when radiation pressure is dominant (as in massive stars) or when the constituent particles that contribute to the pressure move with velocities close to the velocity of light (as in degenerate configurations near their limiting mass). Thus, if γ should be a constant through the star and exceeds $\frac{4}{3}$ by a small amount, then it follows from the theory that instability will occur when the radius of the star

$$R < \frac{2GM}{c^2} \frac{K}{\gamma - \frac{4}{3}}, \tag{5}$$

where K is a constant which, while it depends on the structure of the star, is generally of order unity. This last formula shows very clearly that effects arising from general relativity can induce instability even under circumstances when its effect on the structure of the equilibrium configuration are entirely negligible.

The corresponding theory of axisymmetric oscillations of slowly rotating stars has been recently worked out by Hartle and Thorne (1968) and by Chandrasekhar and Friedman (1972b). Again this theory parallels the Newtonian theory; and a formula identical in form with Equation (4) exists in the theory of general relativity as well.

The principal reason why the theory of radial oscillations of spherical systems and of axisymmetric oscillations in slowly rotating systems, in general relativity, closely parallel the Newtonian theory is that in both cases gravitational radiation plays no role: it is identically absent in the former case (by virtue of Birkhoff's theorem) and it is absent to the relevant order in the latter case. But this simplification does not obtain for non-radial oscillations of spherical systems and axisymmetric oscillations of rapidly rotating systems. We now turn our consideration to such systems.

4. A General Theory of Axisymmetric Oscillations

It is clear that the theory of radial oscillations of spherical systems can be included as a special case of a general theory of axisymmetric oscillations of axisymmetric systems. The theory of non-radial oscillations of spherical systems can also be included as a special case of the same general theory since the normal modes of such oscillations can be analyzed in spherical harmonics $Y_l^m(\theta, \phi)$; and while the characteristic frequencies belonging to the normal modes will depend on l they will be independent of m. In other words, the radial and the non-radial oscillations of spherical systems

as well as the axisymmetric oscillations of rotating axisymmetric systems can all be included in one general theory. We shall indicate later (in Section 6 below) how the same theory with changes only in notation can be adapted to isolate points of onset of instabilities by non-axisymmetric modes of oscillation.

The theory of non-radial oscillations of spherical systems with the emission of gravitational radiation has been developed in considerable detail by Thorne and his associates (Thorne and Campolattaro, 1967, 1970; Thorne, 1969a, b; Thorne and Price, 1969) and more recently by Ipser and Detweiler (1973). The theory we shall outline includes the theory of non-radial oscillations in an alternative version.

The account which follows is largely based on a series of papers by Chandrasekhar and Friedman that have been published during the past two years (Chandrasekhar and Friedman 1972a, b, 1973a, b, c).

We start then with a form for the metric that is suitable to describe general time-dependent systems with the only restriction that the systems maintain axial symmetry, about a fixed axis, at all times. It can be shown that a form of the metric that is suitable for the purposes is

$$ds^2 = - e^{2\nu}(dt)^2 + e^{2\psi}(d\phi - q_{2,0}\,dx^2 - q_{3,0}\,dx^3 - \omega\,dt)^2 +$$
$$+ e^{2\mu_2}(dx^2)^2 + e^{2\mu_3}(dx^3)^2, \tag{6}$$

where ν, ψ, ω. q_2, q_3, μ_2, and μ_3 are functions of the time coordinate $t(=x^0)$ and the two space-like variables x^2 and x^3 but independent of the cyclic angle-variable $\phi(=x^1)$. As we have written, the metric depends on the seven functions enumerated; but it can be shown that the functions q_2, q_3, and ω can occur in the field equations only in the combinations

$$\omega_{,2} - q_{2,00}, \qquad \omega_{,3} - q_{3,00}, \quad \text{and} \quad q_{2,30} - q_{3,20}; \tag{7}$$

accordingly, we shall be concerned with only six independent quantities.

The metric (6) includes the form.

$$ds^2 = - e^{2\nu}(dt)^2 + e^{2\psi}(d\phi - \omega\,dt)^2 + e^{2\mu_2}(dx^2)^2 + e^{2\mu_3}(dx^3)^2, \tag{8}$$

which is generally chosen as appropriate for stationary axisymmetric systems. However, in the stationary case, when ν, ψ, ω, μ_2, and μ_3 are functions of x^2 and x^3 only, one has the additional gauge freedom to restrict the functions μ_2 and μ_3, by a coordinate condition, such as $\mu_2 = \mu_3 = \mu$ or $e^{\mu_3} = x^2 e^{\mu_2}$. In the non-stationary case, we do not have this freedom. Thus, if we should consider time-dependent departures from equilibrium, we must allow for a difference in the Eulerian changes in μ_2 and μ_3.

In writing out the field equations appropriate to the metric (6), we shall restrict ourselves to the case when the source of the gravitational field is a perfect fluid described by the energy-momentum tensor

$$T^{ij} = (\varepsilon + p)\,u^i u^j + p g^{ij}, \tag{9}$$

where ε and p denote the energy density and the pressure, respectively. We shall

further suppose that there exists an 'equation of state' which relates ε uniquely as a function of p and the baryon number N (per unit proper three-volume):

$$\varepsilon = \varepsilon(p, N). \tag{10}$$

The equations of the problem are then provided by the Einstein field-equations supplemented by the equation

$$(Nu^j\sqrt{-g})_{,j} = 0 \tag{11}$$

which ensures the conservation of the baryon number.

Since we have assumed that the system preserves its axisymmetry at all times, we shall have in addition to Equation (11), the further equation

$$u^j\left(\frac{\varepsilon+p}{N}u_1\right)_{,j} = 0 \tag{12}$$

which ensures the conservation of the angular momentum per baryon.

The problem to which we now address ourselves is the following. We are given a system that is axisymmetric, stationary and in a state of uniform rotation with an angular velocity Ω. We suppose that it is subjected to an infinitesimal perturbation and that in the non-stationary state which ensues, the axisymmetry of the system is maintained. What are the equations which govern the time evolution of such a perturbation?

We start with the full, non-linear, time-dependent equations that are appropriate to metric (6). The initial stationary system will satisfy these same equations: only the terms which explicitly involve the time derivatives will be absent. The equations which govern the perturbation can be obtained by linearizing the full time-dependent equations about the time-independent solution representing the initial stationary state.

In considering the changes in the various quantities caused by the perturbation, we shall distinguish between the Eulerian and the Lagrangian changes. These are respectively the changes that take place at a fixed location and the changes that accompany a fluid element as it moves. It is convenient to describe the perturbations in the various quantities that describe the fluid in terms of a Lagrangian displacement ξ which is the spatial displacement that an element of fluid experiences relative to its location in the unperturbed state. Since we have assumed that the perturbation does not affect the axisymmetry of the configuration, it is clear that the components ξ^α ($\alpha = 2, 3$) of ξ should suffice to describe the displacement. The Lagrangian displacement ξ^α is related to the velocity v^α ($= \mathrm{d}x^\alpha/\mathrm{d}t$) by

$$v^\alpha = \xi^\alpha_{,0}. \tag{13}$$

Among the equations which govern the departures from equilibrium of a system that is initially static or stationary we distinguish two classes: *initial-value equations* and *dynamical equations*. Initial-value equations are those that are of the first order in the time derivatives; and dynamical equations are those that are of second order in the time derivatives. Initial value equations can be directly integrated when the fluid

variables are expressed in terms of the Lagrangian displacement. In contrast, dynamical equations are the basic equations of motion that eventually lead to a characteristic-value problem which determines the normal modes of oscillation and the characteristic frequencies belonging to them.

For the problem we are presently considering, the initial value equations are (1) the equations expressing the conservation of baryon number

$$\frac{\Delta N}{N} = -\frac{1}{u^0 \sqrt{-g}} (\xi^\alpha u^0 \sqrt{-g})_{,\alpha} - \delta[\log(u^0 \sqrt{-g})] =$$

$$= -\frac{1}{u^0 \sqrt{-g}} (\xi^\alpha u^0 \sqrt{-g})_{,\alpha} - \frac{V \delta V}{1-V^2} - \delta(\psi + \mu_2 + \mu_3) ; \tag{14}$$

(2) the adiabatic condition expressing the conservation of entropy

$$\frac{\Delta p}{p} = \gamma \frac{\Delta N}{N} \quad \text{and} \quad \Delta\left(\frac{\varepsilon+p}{N}\right) = vp \frac{\Delta N}{N^2} ; \tag{15}$$

(3) the equation expressing the conservation of angular momentum

$$\frac{\delta V}{V(1-V^2)} = -\frac{\gamma p}{\varepsilon+p} \frac{\Delta N}{N} - \delta\psi - \xi^\alpha (\log u_1)_{,\alpha} ; \tag{16}$$

and finally (4) the (0, 2)- and the (0, 3)-components of the linearized field equations

$$(\delta\psi + \delta\mu)_{,2} - \nu_{,2}(\delta\psi + \delta\mu) + \psi_{,2}(\delta\psi - \delta\mu) =$$

$$= e^{2\mu_2}\left(8\pi \frac{\varepsilon+p}{1-V^2}\xi^2 - \frac{Q}{2\sqrt{-g}}\omega_{,3}\right) - \delta\tau_{,2} - (2\mu_3 + \psi - \nu)_{,2}\,\delta\tau \tag{17}$$

and

$$(\delta\psi + \delta\mu)_{,3} - \nu_{,3}(\delta\psi + \delta\mu) + \psi_{,3}(\delta\psi - \delta\mu) =$$

$$= e^{2\mu_3}\left(8\pi \frac{\varepsilon+p}{1-V^2}\xi^3 + \frac{Q}{2\sqrt{-g}}\omega_{,2}\right) + \delta\tau_{,3} + (2\mu_2 + \psi - \nu)_{,3}\,\delta\tau, \tag{18}$$

where

$$u^0 = \frac{e^{-\nu}}{\sqrt{(1-V^2)}}, \qquad u_1 = \frac{e^\psi V}{\sqrt{(1-V^2)}}, \qquad V = e^{\psi-\nu}(\Omega-\omega),$$

$$\delta\mu = \tfrac{1}{2}\delta(\mu_3 + \mu_2), \qquad \delta\tau = \tfrac{1}{2}\delta(\mu_3 - \mu_2), \tag{19}$$

and

$$Q = e^{3\psi + \nu - \mu_2 - \mu_3}(q_{2,3} - q_{3,2}). \tag{20}$$

The (1, 2)- and the (1, 3)-components of the linearized field-equations play a double

role in the theory: they provide initial-value equations of $\delta\omega$ while their integrability condition leads to a dynamical equation for Q. Thus

$$
\begin{aligned}
\delta\omega_{,2} - q_{2,00} = {} & 16\pi(\varepsilon+p)\, u^0 u_1 e^{-2\psi+2\nu+2\mu_2}\xi^2 - \\
& - \omega_{,2}(3\delta\psi-\delta\nu+2\delta\tau) - e^{-3\psi+\nu+\mu_2+\mu_3}Q_{,3}
\end{aligned}
\tag{21}
$$

and

$$
\begin{aligned}
\delta\omega_{,3} - q_{3,00} = {} & 16\pi(\varepsilon+p)\, u^0 u_1 e^{-2\psi+2\nu+2\mu_3}\xi^3 - \\
& - \omega_{,3}(3\delta\psi-\delta\nu-2\delta\tau) + e^{-3\psi+\nu+\mu_2+\mu_3}Q_{,2}\,;
\end{aligned}
\tag{22}
$$

and the elimination of $\delta\omega$ from these equations gives

$$
\begin{aligned}
(e^{-3\psi+\nu+\mu_3-\mu_2}Q_{,2})_{,2} + {} & (e^{-3\psi+\nu+\mu_2-\mu_3}Q_{,3})_{,3} = e^{-3\psi-\nu+\mu_2+\mu_3}Q_{,00} - \\
& - [\omega_{,2}(3\delta\psi-\delta\nu+2\delta\tau)]_{,3} + [\omega_{,3}(3\delta\psi-\delta\nu-2\delta\tau)]_{,2} + \\
& + 16\pi\{[(\varepsilon+p)\,u^0 u_1 e^{-2\psi+2\nu+2\mu_2}\xi^2]_{,3} - \\
& - [(\varepsilon+p)\,u^0 u_1 e^{-2\psi+2\nu+2\mu_3}\xi^3]_{,2}\},
\end{aligned}
\tag{23}
$$

which, as we have stated, is a dynamical equation for Q.

Turning to the remaining dynamical equations, we shall separate the time and the space variables and seek solutions whose dependence on time is given by

$$
e^{i\sigma t},
\tag{24}
$$

where σ is a characteristic-value parameter to be determined. This time-dependence will appear as a factor in all the equations. We shall suppose that this common factor has been removed and that all quantities (such as ξ^α, ΔN, etc.) which appear in the equations from now on represent the space-dependent amplitudes of the respective quantities; thus it will be assumed, for example, that the chosen Lagrangian displacement is of the form

$$
\xi^\alpha(x^2, x^3)\, e^{i\sigma t}.
\tag{25}
$$

The dynamical equation which follows from the linearization of the equation

$$
u^j u_{i;j} = -\frac{1}{\varepsilon+p}\,(\delta^j{}_i + u^j u_i)\, p_{,j},
\tag{26}
$$

(which in turn is a consequence of the identity $T^{ij}{}_{;j}=0$) is the pulsation equation:

$$
\begin{aligned}
-\sigma^2(\varepsilon+p)(u^0)^2\, e^{2\mu_\alpha}\xi^\alpha = {} & -u^0\left(\frac{\gamma p}{u^0}\frac{\Delta N}{N}\right)_{,\alpha} + \frac{\Delta N}{N}\,p_{,\alpha} + \\
& + (\varepsilon+p)\left(\frac{\Delta u^0}{u^0}\right)_{,\alpha} - \\
& - (\varepsilon+p)\,u^0 u_1(\Delta\Omega_{,\alpha} - q_{\alpha,00}).
\end{aligned}
\tag{27}
$$

Besides this pulsation equation we must include, at most, two of the linearized field equations; 'at most two' since Equations (17), (18), 21), and (22) already account for four equations and there can be no more than six linearly independent field equations.

As the remaining dynamical equations we may take, for example, the $(1, 1)$- and the $[(2, 2) + (3, 3)]$-components of the field equations. We shall not write these equations out explicitly.

Our problem is to solve Equation (27) consistently with the initial-value Equations (14)–(18), (21), and (22), the remaining dynamical equations, and the appropriate boundary conditions. The boundary conditions are that ξ^α vanishes at the origin and remains bounded and continuous over its domain; that Δp vanishes on the boundary and that all the remaining field variables (such as $\delta\psi$, $\delta\nu$, etc.,) vanish sufficiently rapidly at infinity and satisfy the necessary conditions on the horizon if we are dealing with vacuum solutions exterior to a black hole.

The problem to which we are thus led is a characteristic value problem for σ^2.

5. The Variational Principle: Applications to Vacuum Metrics

The characteristic-value problem to which we were led in Section 4 can be formulated in terms of a variational principle. We shall clarify its nature by restricting ourselves to the case of vacuum metrics external to an event horizon and which are asymptotically flat at infinity, i.e. effectively to the Kerr and the Schwarzschild metrics.

First by making use of the equations governing the problem (it may be noted parenthetically that the initial-value equations and the dynamical equations must be treated differently) we derive the following formulae for σ^2.

$$
\sigma^2 \int\int \left[\!\left[2e^{-2\nu}\sqrt{-g}\left\{(\delta\tau)^2 + (\delta\psi)^2 - [\delta(\psi+\mu)]^2\right\} + \right.\right.
$$

$$
\left.\left. + \tfrac{1}{2}e^{-3\psi-\nu+\mu_2+\mu_3}Q^2 \right]\!\right] dx^2\, dx^3 =
$$

$$
= \int\int \left[\!\left[X\, e^{3\psi-\nu}[4(\delta\psi)^2 + (\delta\tau)^2] + 4Y\, e^{3\psi-\nu}\,\delta\psi\delta\tau - \right.\right.
$$

$$
- 4U(\delta\tau)^2 + 2e^{\beta}\left[e^{\mu_3-\mu_2}(\delta\psi_{,2})^2 + e^{\mu_2-\mu_3}(\delta\psi_{,3})^2 \right] -
$$

$$
- 4e^{\beta}\left\{ e^{\mu_3-\mu_2}[\beta_{,2}\delta\mu_{3,2} - (\psi-\nu)_{,2}\,\delta\psi_{,2}] - \right.
$$

$$
- e^{\mu_2-\mu_3}[\beta_{,3}\delta\mu_{2,3} - (\psi-\nu)_{,3}\,\delta\psi_{,3}]\right\}\,\delta\tau -
$$

$$
- 2[2(Q_{,2}\omega_{,3} - Q_{,3}\omega_{,2})\,\delta\psi - (Q_{,2}\omega_{,3} + Q_{,3}\omega_{,2})\,\delta\tau] +
$$

$$
+ \tfrac{1}{2}e^{-3\psi+\nu}\left[e^{\mu_3-\mu_2}(Q_{,2})^2 + e^{\mu_2-\mu_3}(Q_{,3})^2 \right]\!\right]\! dx^2\, dx^3 +
$$

$$
+ \int \left[\!\left[e^{\mu_3-\mu_2+\beta}\left[-\delta(\psi-\nu)_{,2}\,\delta\psi - 2(\psi-\nu_{,2}\,\delta\psi\delta\tau + \right.\right.\right.
$$

$$
+ \delta(\psi+\nu)_{,2}\,\delta\mu_3 + 2(\psi+\nu)_{,2}\,\delta\tau\delta\mu_3] +
$$

$$
+ Q\omega_{,3}\delta(2\psi-\tau) - \tfrac{1}{2}e^{-3\psi+\nu+\mu_3-\mu_2}QQ_{,2}\right]\!\right]_{[x^2]} dx^3 +
$$

$$
+ \int \left[\!\left[e^{\mu_2-\mu_3+\beta}\left[-\delta(\psi-\nu)_{,3}\,\delta\psi + 2(\psi-\nu)_{,3}\,\delta\psi\delta\tau + \right.\right.\right.
$$

$$
+ \delta(\psi+\nu)_{,3}\,\delta\mu_2 - 2(\psi+\nu)_{,3}\,\delta\tau\delta\mu_2] -
$$

$$
- Q\omega_{,2}\delta(2\psi+\tau) - \tfrac{1}{2}e^{-3\psi+\nu+\mu_2-\mu_3}QQ_{,3}\right]\!\right]_{[x^3]} dx^2. \qquad (28)
$$

where

$$\beta = \psi + \nu,$$
$$X = e^{\mu_3 - \mu_2}(\omega_{,2})^2 + e^{\mu_2 - \mu_3}(\omega_{,3})^2,$$
$$Y = e^{\mu_3 - \mu_2}(\omega_{,2})^2 - e^{\mu_2 - \mu_3}(\omega_{,3})^2$$

and

$$U = e^{\beta}\left[e^{\mu_3 - \mu_2}(\beta_{,2}\mu_{3,2} + \psi_{,2}\nu_{,2}) + e^{\mu_2 - \mu_3}(\beta_{,3}\mu_{2,3} + \psi_{,3}\nu_{,3})\right]. \tag{29}$$

Also, in equation (28) the symbol

$$[\![\cdots]\!]_{[x^\alpha]}$$

in the integrands of the surface integrals has the following meaning. For a fixed $x^\beta (\beta \neq \alpha)$ let the appropriate limits of x^α be $x^\alpha(1)$ and $x^\alpha(2) \geqslant x^\alpha(1)$; the symbol stands for the difference in the values of the quantity enclosed by the double brackets at $x^\alpha(2)$ and $x^\alpha(1)$.

Equation (28) provides the basis for a variational determination of σ^2 in the following sense.

First, we assume for Q and $\delta\tau$ certain 'trial functions' which are arbitrary in the first instance except for the requirement that they satisfy the same boundary conditions as are demanded of the true proper solutions. Then, we evaluate $\delta\psi$ and $\delta\mu$ in terms of the assumed trial functions for Q and $\delta\tau$ with the aid of the initial-value equations (cf. Equations (17) and (18))

$$e^{\nu}\left[e^{-\nu}(\delta\psi + \delta\mu)\right]_{,\alpha} + \psi_{,\alpha}(\delta\psi - \delta\mu) = \mathfrak{F}_\alpha \quad (\alpha = 2, 3), \tag{30}$$

where

$$\mathfrak{F}_\alpha = (-1)^\beta \left[\frac{e^{2\mu_\alpha}Q\omega_{,\beta}}{2\sqrt{-g}} + \delta\tau_{,\alpha} + (2\mu_\beta + \psi - \nu)_{,\alpha}\,\delta\tau\right] \quad (\alpha \neq \beta). \tag{31}$$

Equation (30) represents simple quasi-linear differential equations for $\delta\psi$ and $\delta\mu$ and can be solved by standard methods. Thus

$$(\delta\psi + \delta\mu)_{\text{along }\psi = \text{constant}} = e^{\nu} \int_{\psi = \text{constant}} e^{-\nu}\mathfrak{F}_\alpha\,dx^\alpha, \tag{32}$$

where the integral on the right-hand side is taken along a contour $\psi = \text{constant}$; with $\delta\psi + \delta\mu$ determined by equation (32), $\delta\psi - \delta\mu$ follows directly from Equation (30).

With the assumed trial functions for Q and $\delta\tau$ and the deduced values for $\delta\psi$ and $\delta\mu$, we can formally evaluate σ^2 given by Equation (28). We can similarly evaluate $\sigma^2 + \delta\sigma^2$ which follows from using the trial functions $Q + \delta Q$ and $\delta\tau + \delta^2\tau$ which differ from Q and $\delta\tau$ by some (arbitrarily specified) increments δQ and $\delta^2\tau$. If we now demand that $\delta\sigma^2$ vanishes *identically*, i.e. for *all* δQ and $\delta^2\tau$ (restricted only by the boundary conditions that must be satisfied), then it can be shown that the originally

chosen Q and $\delta\tau$ must satisfy the correct dynamical equations of the problem and that σ^2 given by Equation (28) is a true characteristic value.

It is clear from the foregoing remarks that Equation (28) can be used to evaluate the radiation damping of odd-parity, magnetic-type modes of axisymmetric oscillation of the Kerr black hole – 'odd-parity, magnetic-type modes' since the source of the radiation, as is evident from Equation (23), resides in the angular momentum of the field (as manifested by the appearance on the right-hand side of the equation, the function ω which represents the dragging of the inertial frame).

5.1. CARTER'S THEOREM

Equation (28) has an important application to the problem of whether we can deform quasi-stationarily an asymptotically flat axisymmetric vacuum metric external to an event horizon, i.e. to Carter's theorem (1971).

It can be shown that a necessary and sufficient condition for the existence of such a quasi-stationary deformation is obtained by setting both σ^2 and $\delta\tau$ equal to zero in Equation (28). That σ^2 should be set equal to zero for quasi-stationary deformations is clear. That we can also set $\delta\tau = 0$ is a consequence of the fact that during a quasi-stationary deformation, we can continue to maintain the coordinate condition that we are entitled to impose under stationary conditions. Accordingly in the gauge $\mu_2 = \mu_3 = \mu$, Equation (28) gives (see Equation (42) below)

$$\int\int \left[e^\beta\, \delta\psi_{,\alpha}\, \delta\psi_{,\alpha} + 2X\, e^{3\psi-\nu}(\delta\psi)^2 - 2(Q_{,2}\omega_{,3} - Q_{,3}\omega_{,2})\, \delta\psi + \right.$$
$$\left. + \tfrac{1}{4}e^{-3\psi+\nu}\, Q_{,\alpha}Q_{,\alpha} \right]\, \mathrm{d}x^2\, \mathrm{d}x^3$$
$$= \int\int \left[e^\beta\, \delta\psi_{,\alpha}\, \delta\psi_{,\alpha} + 2X\, e^{3\psi-\nu}(\delta\psi)^2 - \tfrac{1}{4}e^{-3\psi+\nu}\, Q_{,\alpha}Q_{,\alpha} \right]\, \mathrm{d}x^2\, \mathrm{d}x^3 = 0.$$

(33)

The surface integrals in Equation (28) do not survive under the present conditions: at infinity by the requirements of asymptotic flatness and on the horizon and on the axis by the requirements

$$e^\beta = 0 \quad \text{on the horizon and on the rotation axes;} \tag{34}$$

besides, it must be supposed that all the perturbations vanish on the horizon in such a way that (see below)

$$e^\beta\, \delta v\, \delta v_{,\alpha} = e^\beta\, \delta\psi\, \delta\psi_{,\alpha} = e^\beta\, QQ_{,\alpha} = Q\, \delta\psi = 0. \tag{35}$$

By a sequence of elementary transformations, the integrands in Equation (32) can be brought to positive-definite forms; and Carter's theorem (that quasi-stationary, axisymmetric deformations, of asymptotically flat axisymmetric vacuum metrics external to an event horizon, are impossible) follows.

There is a direct and a simple way in which we can derive the basic Equation (33) and which clarifies some important aspects of the theorem.

By linearizing the field equations appropriate to the stationary metric (8) about a given solution that corresponds to an axisymmetric black hole, we readily obtain, in the gauge $\mu_2 = \mu_3 = \mu$, the equations

$$(e^\beta \, \delta\beta)_{,\alpha\alpha} = 0, \tag{36}$$

$$(e^{-3\psi+\nu} \, Q_{,\alpha})_{,\alpha} = [\omega_{,3}(3\delta\psi - \delta\nu)]_{,2} - [\omega_{,2}(3\delta\psi - \delta\nu)]_{,3}, \tag{37}$$

and

$$(e^\beta \, \delta\psi_{,\alpha})_{,\alpha} = 2e^{3\psi-\nu} \, X \, \delta\psi - (Q_{,2}\omega_{,3} - Q_{,3}\omega_{,2}). \tag{38}$$

From Equation (36) it readily follows that

$$\delta\beta = \delta\psi + \delta\nu = 0; \tag{39}$$

and Equation (37) becomes

$$(e^{-3\psi+\nu} \, Q_{,\alpha})_{,\alpha} = 4(\omega_{,3} \, \delta\psi_{,2} - \omega_{,2} \, \delta\psi_{,3}). \tag{40}$$

Now multiplying Equations (38) and (40) by $\delta\psi$ and Q, respectively, and integrating over all three-space external to the horizon, we obtain after integrations by parts

$$\int\int [e^\beta \, \delta\psi_{,\alpha} \, \delta\psi_{,\alpha} + 2e^{3\psi-\nu} \, X(\delta\psi)^2 - (Q_{,2}\omega_{,3} - Q_{,3}\omega_{,2}) \, \delta\psi] \, dx^2 \, dx^3 = 0 \tag{41}$$

and

$$\int\int e^{-3\psi+\nu} \, Q_{,\alpha}Q_{,\alpha} \, dx^2 \, dx^3 = 4 \int (Q_{,2}\omega_{,3} - Q_{,3}\omega_{,2}) \, \delta\psi \, dx^2 \, dx^3. \tag{42}$$

The integrated parts vanish in both cases: at infinity by the requirement of asymptotic flatness and on the horizon and on the axes by virtue of the boundary conditions (34) and (35). Equations (41) and (42) are clearly equivalent to the identity from which Carter's theorem follows.

It is important to observe that the proof of Carter's theorem at no stage requires that the perturbations satisfy any continuity requirements on the horizon: all that is needed is that the squares and products of the perturbations vanish on the horizon and also that the derivatives with respect to x^2 and x^3 of such squares and products remain bounded so that when multiplied by e^β they vanish.

5.2. SPHERICALLY SYMMETRIC BLACK HOLES

In the case of spherical symmetry and in the absence of rotation ($\omega = 0$), Equation (28) gives

$$\sigma^2 \int\int [\![e^{-2\nu} \sqrt{-g} \, \{(\delta\tau)^2 + (\delta\psi)^2 - [\delta(\psi + \mu)]^2 \}]\!] \, dx^2 \, dx^3 =$$

$$= \int\int [\![e^\beta [e^{\mu_3 - \mu_2}(\delta\psi_{,2})^2 + e^{\mu_2 - \mu_3}(\delta\psi_{,3})^2] - 2U(\delta\tau)^2 -$$

$$-2e^{\beta}\{e^{\mu_3-\mu_2}[\beta_{,2}\,\delta\mu_{3,2}-(\psi-\nu)_{,2}\,\delta\psi_{,2}]-$$
$$-e^{\mu_2-\mu_3}[\beta_{,3}\,\delta\mu_{2,3}-(\psi-\nu)_{,3}\,\delta\psi_{,3}]\}\,\delta\tau]\!]\,dx^2\,dx^3+\text{surface integrals},$$

(43)

where for the sake of brevity we have not written out the surface integrals explicitly. It should be noted that the term in Q^2 on the left-hand side of Equation (28) cancels directly with the terms in $(Q_{,2})^2$ and $(Q_{,3})^2$ on the right-hand side by virtue of the equation satisfied by Q.

By using spherical polar coordinates and expanding the various functions in Legendre polynomials, we can separate Equation (43) to obtain the characteristic frequencies belonging to the different harmonics; and this resulting equation can be used to determine the damping of the different non-radial modes of oscillation of the Schwarzschild black hole.

For quasi-stationary deformations, Equation (43) gives, in the gauge $\mu_2 = \mu_3$,

$$\iint e^{\beta}\,\delta\psi_{,\alpha}\,\delta\psi_{,\alpha}\,dx^2\,dx^3 = 0;$$

(44)

and the impossibility of neutral deformations follows. Since the restriction to axi-symmetric modes involves no loss of generality when dealing with spherically sym-metric systems, Equation (44) excludes general non-axisymmetric deformations as well. (For an alternative demonstration of this result, see Vishveshwara, 1970).

6. On a Criterion for the Occurrence of a Dedekind-Like Point of Bifurcation Along an Axisymmetric Sequence

As we have remarked earlier, radiation-reaction can induce secular instability along a sequence of axisymmetric configurations at a Dedekind-like point of bifurcation. In the framework of general relativity, we can obtain a criterion for the occurrence of such a point of bifurcation by considering the field equations valid for stationary non-axisymmetric systems and linearizing them about a stationary axisymmetric so-lution for deformations whose dependence on the azimuthal angle is $e^{2i\phi}$.

It is readily seen that for describing stationary non-axisymmetric systems, a suit-able form for the metric is (cf. Equation (6))

$$ds^2 = -e^{2\mathfrak{n}}(dt - \mathfrak{w}\,d\phi - \mathfrak{q}_{2,1}\,dx^2 - \mathfrak{q}_{3,1}\,dx^3)^2 + e^{2\mathfrak{p}}(d\phi)^2 +$$
$$+ e^{2\mu_2}(dx^2)^2 + e^{2\mu_3}(dx^3)^2,\qquad (45)$$

where $\mathfrak{n}$, $\mathfrak{p}$, $\mathfrak{w}$, $\mathfrak{q}_2$, $\mathfrak{q}_3$, μ_2, and μ_3 are functions of the three space-variables $\phi\,(=x^1)$, x^2, and x^3 only; and as in the case of the time-dependent axisymmetric metric (6), the three functions $\mathfrak{w}$, $\mathfrak{q}_2$, and $\mathfrak{q}_3$ can occur in the field equations only in the com-binations

$$\mathfrak{w}_{,2} - \mathfrak{q}_{2,11}, \mathfrak{w}_{,3} - \mathfrak{q}_{3,11}, \quad \text{and} \quad \mathfrak{q}_{2,31} - \mathfrak{q}_{3,21},\qquad (46)$$

so that we have again only six independent functions to consider.

The metric (46) includes stationary axisymmetric metrics, now written in the form (cf. Equation (8))

$$ds^2 = -e^{2\mathfrak{n}}(dt - \mathfrak{w}\, d\phi)^2 + e^{2\mathfrak{p}}(d\phi)^2 + e^{2\mu_2}(dx^2)^2 + e^{2\mu_3}(dx^3)^2, \tag{47}$$

where the functions $\mathfrak{n}$, $\mathfrak{p}$, $\mathfrak{w}$, μ_2, and μ_3 are now functions of x^2 and x^3 only. However, as in the case of the metric (8), we have the gauge-freedom to impose a coordinate condition on μ_2 and μ_3.

From a comparison of the metrics (6) and (45), it is clear that we should be able to pass from equations which are valid for time-dependent axisymmetric systems to equations which are valid for time-independent non-axisymmetric systems by some simple rules of transcription. Thus, in place of Equation (12), expressing the conservation of angular momentum per baryon, we now have the conservation equation

$$u^j \left(\frac{\varepsilon + p}{N} u_0 \right)_{,j} = 0. \tag{48}$$

(It may be noted here that in the Newtonian limit Equation (48) reduces to the Bernoulli integral – a fact one might not have suspected.)

Quite generally, it can be shown that by the replacements

$$u^0 \to -iu^1, \quad u^1 \to iu^0, \quad u_0 \to iu_1, \quad u_1 \to -iu_0,$$

$$\psi \to \mathfrak{n}, \quad \nu \to \mathfrak{p}, \quad \omega \to -\mathfrak{w}, \quad \frac{\partial}{\partial t} \to \Omega \frac{\partial}{\partial \phi},$$

$$V = e^{\psi - \nu}(\Omega - \omega) \to -\frac{1}{V} = -e^{\mathfrak{n} - \mathfrak{p}}\left(\frac{1}{\Omega} - \mathfrak{w} \right),$$

and

$$Q = e^{3\psi - \nu - \mu_2 - \mu_3}(q_{2,3} - q_{3,2}) \to \mathfrak{Q} = e^{3\mathfrak{n} - \mathfrak{p} - \mu_2 - \mu_3}(\mathfrak{q}_{2,3} - \mathfrak{q}_{3,2}), \tag{49}$$

we can transcribe the equations which are valid for time-dependent axisymmetric systems into equations which are valid for time-independent non-axisymmetric systems.

By subjecting a stationary axisymmetric system (described in conformity with the metric (27)) to non-axisymmetric deformations with a ϕ-dependence

$$e^{im\phi}$$

(where m is an integer greater than or equal to 1), we can write down, with the aid of the rules of transcription (49), the equations that must be satisfied if the system considered can be so deformed quasi-stationarily. And these equations will lead to a characteristic-value problem for m^2 even as the parallel analysis of axisymmetric perturbations with a time-dependence $e^{i\sigma t}$ led to a characteristic-value problem for σ^2. However, in the present context the solution to the characteristic-value problem will be physically meaningful only if the problem allows an integral characteristic value for m. Nevertheless, the characteristic-value problem is *itself* meaningful regard-

less of whether m happens to be an integer or not. And considered solely as a characteristic-value problem for m^2, we can construct a variational base for evaluating m^2 even as we constructed a variational base in Section 5 for evaluating σ^2. The required formula for m^2, for vacuum metrics, for example, can be written down be letting $\sigma^2 \to -m^2$ and making the replacements (49) in Equation (28). We thus obtain

$$
m^2 \iint \llbracket 2e^{-2\mathfrak{p}} \sqrt{-g} \{[\delta(\mathfrak{n}+\mu)]^2 - (\delta\tau)^2 - (\delta\mathfrak{n})^2\} - \tfrac{1}{2} e^{-3\mathfrak{n}-\mathfrak{p}+\mu_2+\mu_3} \mathfrak{Q}^2 \rrbracket \, dx^2 \, dx^3 =
$$

$$
= \iint \llbracket e^{3\mathfrak{n}-\mathfrak{p}} \mathfrak{X} [4(\delta\mathfrak{n})^2 + (\delta\tau)^2] + 4e^{3\mathfrak{n}-\mathfrak{p}} \mathfrak{Y} \, \delta\mathfrak{n} \, \delta\tau - 4\mathfrak{U}(\delta\tau)^2 +
$$

$$
+ 2e^{\beta} [e^{\mu_3-\mu_2}(\delta\mathfrak{n}_{,2})^2 + e^{\mu_2-\mu_3}(\delta\mathfrak{n}_{,3})^2] -
$$

$$
- 4e^{\beta} \{e^{\mu_3-\mu_2}[\beta_{,2}\,\delta\mu_{3,2} - (\mathfrak{n}-\mathfrak{p})_{,2}\,\delta\mathfrak{n}_{,2}] -
$$

$$
- e^{\mu_2-\mu_3}[\beta_{,3}\,\delta\mu_{2,3} - (\mathfrak{n}-\mathfrak{p})_{,3}\,\delta\mathfrak{n}_{,3}]\} \, \delta\tau +
$$

$$
+ 2[2(\mathfrak{Q}_{,2}\mathfrak{w}_{,3} - \mathfrak{Q}_{,3}\mathfrak{w}_{,2}) \, \delta\mathfrak{n} - (\mathfrak{Q}_{,2}\mathfrak{w}_{,3} + \mathfrak{Q}_{,3}\mathfrak{w}_{,2}) \, \delta\tau] +
$$

$$
+ \tfrac{1}{2} e^{-3\mathfrak{n}+\mathfrak{p}} [e^{\mu_3-\mu_2}(\mathfrak{Q}_{,2})^2 + e^{\mu_2-\mu_3}(\mathfrak{Q}_{,3})^2] \rrbracket \, dx^2 \, dx^3 + \text{surface integrals}
$$

$$
\tag{50}
$$

where

$$
\mathfrak{X} = e^{\mu_3-\mu_2}(\mathfrak{w}_{,2})^2 + e^{\mu_2-\mu_3}(\mathfrak{w}_{,3})^2 \, ,
$$

$$
\mathfrak{Y} = e^{\mu_3-\mu_2}(\mathfrak{w}_{,2})^2 - e^{\mu_2-\mu_3}(\mathfrak{w}_{,3})^2 \, ,
$$

and

$$
\mathfrak{U} = e^{\beta} [e^{\mu_3-\mu_2}(\beta_{,2}\mu_{3,2} + \mathfrak{n}_{,2}\mathfrak{p}_{,2}) + e^{\mu_2-\mu_3}(\beta_{,3}\mu_{2,3} + \mathfrak{n}_{,3}\mathfrak{p}_{,3})] \, . \tag{51}
$$

In Equation (50), the functions that are to be varied are $\mathfrak{Q}$ and $\delta\tau$ while $\delta\mathfrak{n}$ and $\delta\mu$ are to be evaluated in terms of them with the aid of the initial-value equations (cf. Equations (30) and (31))

$$
e^{\mathfrak{p}} [e^{-\mathfrak{p}}(\delta\mathfrak{n} + \delta\mu)]_{,\alpha} + \mathfrak{n}_{,\alpha}(\delta\mathfrak{n} - \delta\mu) = \mathfrak{F}_\alpha, \quad (\alpha = 2, 3) \tag{52}
$$

where

$$
\mathfrak{F}_\alpha = (-1)^\alpha \, e^{2\mu_\alpha} \frac{\mathfrak{Q}\mathfrak{w}_{,\beta}}{2\sqrt{-g}} + (-1)^\beta [\delta\tau_{,\alpha} + (2\mu_\beta + \mathfrak{n} - \mathfrak{p})_{,\alpha}] \quad (\alpha \neq \beta). \tag{53}
$$

The variational expression for m^2, including the terms in the fluid variables, has been written down by Chandrasekhar and Friedman (1973c); it can be used to determine whether Dedekind-like points of bifurcation occur along given equilibrium sequences of axisymmetric configurations. If such points of bifurcation occur, then we may anticipate the onset of secular instability by radiation-reaction at these points.

Returning to Equation (50), we shall consider its application to the Kerr metric with a view to determining whether along the Kerr sequence a Dedekind-like point of bifurcation occurs.

By writing the Kerr metric in the Boyer-Lundquist coordinates in the form (47), we find that

$$e^{2\eta} = \frac{\Sigma}{\varrho^2}, \qquad e^{2\mathfrak{p}} = \frac{\Delta \varrho^2 \sin^2 \theta}{\Sigma}, \qquad e^{2\beta} = \Delta \sin^2 \theta,$$

$$e^{2\mu_2} = \frac{\varrho^2}{\Delta}, \qquad e^{2\mu_3} = \varrho^2, \quad \text{and} \quad \mathfrak{w} = -\frac{2Mar \sin^2 \theta}{\Sigma}, \tag{54}$$

where

$$\Delta = r^2 - 2Mr + a^2, \quad \Sigma = r^2 - 2Mr + a^2 \cos^2 \theta, \quad \text{and} \quad \varrho^2 = r^2 + a^2 \cos^2 \theta. \tag{55}$$

It should be noted that

$$\Delta = 0 \text{ on the horizon and } \Sigma = 0 \text{ on the stationary limit.} \tag{56}$$

Inserting the foregoing expressions in Equation (50), we obtain

$$
m^2 \int\int \left[\frac{2\Sigma}{\Delta \sin \theta} \left[(\delta\eta + \delta\mu)^2 - (\delta\tau)^2 - (\delta\eta)^2 \right] - \frac{\varrho^4}{2\Sigma\Delta \sin \theta} \, \mathfrak{Q}^2 \right] dx^2 \, dx^3 =
$$

$$
= \int\int \left\{ \frac{4}{\Sigma^2} \left[\frac{Ma \sin^2 \theta}{\varrho^2} (r^2 - a^2 \cos^2 \theta)(2\delta\eta + \delta\tau) - \tfrac{1}{2}\varrho^2 \mathfrak{Q}_{,3} \right]^2 + \right.
$$

$$
+ \frac{16\Delta}{\Sigma^2} \left[\frac{Mar \sin \theta \cos \theta}{\varrho^2} (2\delta\eta - \delta\tau) - \tfrac{1}{4}\varrho^2 \mathfrak{Q}_{,2} \right]^2 -
$$

$$
- \frac{\varrho^4 \sin \theta}{2\Sigma^2} \left[\Delta(\mathfrak{Q}_{,2})^2 + (\mathfrak{Q}_{,3})^2 \right] +
$$

$$
+ 4\Delta \sin \theta \left[\frac{M}{\Sigma\varrho^2} (r^2 - a^2 \cos^2 \theta) \delta\tau + \delta\eta_{,2} \right]^2 +
$$

$$
+ 16 \sin \theta \left[\frac{Ma^2 r \sin \theta \cos \theta}{\Sigma\varrho^2} \delta\tau + \tfrac{1}{2}\delta\eta_{,3} \right]^2 -
$$

$$
- 2 \sin \theta \left[\Delta(\delta\eta_{,2})^2 + (\delta\eta_{,3})^2 \right] - 4 \sin \theta \frac{(r - M)^2 - a^2 \cos^2 \theta}{\Sigma} (\delta\tau)^2 -
$$

$$
\left. - 4 \sin \theta \left[(r - M)(\delta\mu_{,2} + \delta\eta_{,2}) - \cot \theta(\delta\mu_{,3} + \delta\eta_{,3}) \right] \delta\tau \right\} dx^2 \, dx^3 +
$$

$$
+ \text{surface integrals.} \tag{57}
$$

We have not explicitly written out the expressions for the surface integrals in Equation (57) since the extent of their survival depends on the boundary conditions which obtain on the horizon and at infinity. (There is, however, no difficulty in showing that we have no contributions from infinity if the requirements of asymptotic flatness are met.)

We observe that as written out, the integrands on both sides of Equation (57) appear to diverge on the stationary limit at $\Sigma = 0$ *and* on the horizon at $\Delta = 0$. However, by examining in detail the equations governing the perturbations, we find that on the stationary limit, the equations allow solutions with the behaviours

$$\delta\mathfrak{n} = n\Sigma^2, \quad \delta\mathfrak{p} = p\Sigma^2, \quad \delta\tau = t\Sigma^2, \quad \delta\mu = m\Sigma^2, \quad \text{and} \quad \mathfrak{Q} = \frac{\phi}{\varrho^2}\Sigma^3,$$

(58)

where

$$n = \tfrac{1}{4}\phi\left(\frac{r-M}{a\cos\theta} - \frac{a\cos\theta}{r-M}\right) = -p,$$

$$m = -\tfrac{1}{4}\phi\left(\frac{r-M}{a\cos\theta} - \frac{a\cos\theta}{r-M}\right),$$

$$t = -\tfrac{1}{12}\phi\left(\frac{r-M}{a\cos\theta} + \frac{a\cos\theta}{r-M}\right),$$

(59)

and ϕ is an arbitrary function on $\Sigma = 0$. Similarly, we find that on the horizon, the equations allow solutions with the behaviours

$$\mathfrak{Q} = -\frac{4CMar\cos\theta}{\varrho^4(M^2-a^2)^{1/2}}\Delta^{3/2},$$

$$\delta\tau = C\Delta^{1/2}, \quad \delta\mu = -C\Delta^{1/2},$$

$$\delta\mathfrak{p} = C\left(\frac{m^2a^2}{M^2-a^2} - 1\right)\Delta^{1/2},$$

and

$$\delta\mathfrak{n} = \frac{CM(r^2 - a^2\cos^2\theta)}{\varrho^2a^2(M^2-a^2)^{1/2}\sin^2\theta}\Delta^{3/2},$$

(60)

where C is a constant.

With the foregoing behaviours on $\Sigma = 0$ and $\Delta = 0$, the integrands in Equation (50) are bounded and the integrals are well defined. But it may be argued that the behaviours on the horizon given by Equations (60) are 'unacceptable' since the derivatives of these functions are singular here. It is, however, worth noting that the behaviours we find satisfy the minimal requirements which we found in Section 5.1 were necessary for the validity of Carter's theorem. And apart from the acceptability or otherwise of the behaviours given in Equations (60), the question remains whether a value for m (not necessarily integral) exists for which the perturbation equations allow solutions with the behaviours (58) and (60) on the stationary limit and on the horizon (respectively) and still satisfy the requirements of asymptotic flatness at infinity. If such solutions do exist, the next question concerns whether along the Kerr

sequence there is a point where *m* has the value 2. And even if such a point exists, the question whether the occurrence of such a point signifies the secular instability of the Kerr metric at that point will still remain.

It is realized that the foregoing remarks are at variance with the conclusions that have been reached by Press and Teukolsky (1973), Wald (1973), and Stewart (1973). But it is not clear to the writer whether the regularity requirements imposed by these authors on the horizon are not too severe. In any event it would appear that one should be able to arrive at the correct result by the present analysis equally well.

Acknowledgement

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