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STOCHASTIC BEHAVIOUR OF PLANETARY ORBITS DURING THE ACCUMULATION PROCESS

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ABSTRACT

The evolution of orbital elements of a growing planet during the accumulation process is considered. The planetary orbit undergoes perturbations because of random encounters and collisions with bodies of its accretion zone and also because of gravitational interaction with an already formed massive planet ("Jupiter"). The mass and velocity distributions of the swarm bodies are assumed to be given time-dependent functions. The Fokker-Planck equation describing the behaviour of the distribution function of orbital elements of the growing planet is worked out and solved. The present mean values of the eccentricities and inclinations of orbits of the terrestrial planets can be explained in the case of their accumulation from a single swarm of bodies with mean mass $\sim 10^{-2}$ $M_{\oplus}$ and with mean eccentricities and inclinations ~ 0.2.

1. INTRODUCTION

Nearly circular and coplanar orbits of the planets are one of the main regularities of the Solar system. In Ziglina, Safronov (1976), Ziglina (1976), Pechernikova, Vitjazev (1980), Ziglina (1985) the parameters of the planetary orbits to be expected in the model of accumulation of the planets from a swarm of solid bodies have been evaluated. These parameters are the "initial conditions" for their further evolution to the present state. We investigate here the stochastic behaviour of the planetary orbit at the last stage of accumulation. From analytical evaluations (Safronov, 1972) and numerical simulations (Wetherill, 1978) it is known that the duration of the late stage of the terrestrial planets formation is a few tens of millions of years. On the other hand, the astrophical data restrict the growth time of the gaseous giant planets $\lesssim 10^7$ yr (Strom et al., 1989). In the present paper, besides collisions and encounters with the swarm bodies the secular perturbations of an already formed massive planet are included additionally.

2. THE MODEL

The following model is considered: a growing planet m undergoes random encounters and collisions (with merging) with the bodies of its feeding zone m' << m and interacts gravitationally with a planet m* >> m, where m* = const. For simplicity we suppose that the masses and velocities of the swarm bodies are distributed independently. We assume the Maxwell velocity distribution

$$f_o(\vec{v}') = \left(\frac{3}{2\pi j^2}\right)^{3/2} \exp(-3v'^2/2j^2), \quad (1)$$

where $\vec{v}'$ is velocity relative to the mean velocity of the bodies, which is approximately equal to the circular Keplerian velocity in the central plane, $v_c = \sqrt{GM_\odot/R}$, where G is the gravitational constant, $M_\odot$ is the Sun mass, R is the distance from the Sun to the projection of the considered point on the central plane (Fig. 1). The mean square of random velocity j^2 is usually written in the form

$$j^2 = \frac{Gm}{\theta\tilde{\tau}}, \quad (2)$$

where m and $\tilde{\tau}$ are the mass and radius of the growing planet, θ is the Safronov number. In this paper, we assume that θ is of the order of a few units ($\theta = 1 \div 5$). The mass distribution is described by some function n(m',t), where n(m',t)dm' is number of bodies in unit volume with masses in the range (m', m' + dm').

The encounters of the swarm bodies with the planet are taken into account according to the two-body problem when their mutual distance is < S, where S is usually taken equal to the half-thickness of the swarm. The frequencies of encounters are collisions are defined as in a "particle-in-a-box" scheme, i.e. as a product of number of bodies in unit volume, collision (encounter) cross-section and the relative velocity. It is also assumed that the planet is growing in an average manner, i.e., increase of planetary mass during the time Δt is equal to its mathematical expectation.

The orbit of the perturbing planet lies in the central plane and its eccentricity $e^* = \text{const}$. Our consideration involves only secular perturbations. Let us denote

$$u_1 = e \cos \tilde{\omega},\quad u_3 = \sin i \cos \Omega,$$

$$u_2 = e \sin \tilde{\omega},\quad u_4 = \sin i \sin \Omega,$$

where $\tilde{\omega}$ is the longitude of perihelion, Ω longitude of the ascending node, axis x (see Fig. 1) is directed to the perihelion of orbit of the planet m^*. Then to the first order in eccentricity and inclination the equations for secular variations read

$$du_1/dt = -x_1u_2, \qquad du_3/dt = x_1u_4,$$
$$du_2/dt = x_1u_1 - x_2e^*, \qquad du_4/dt = -x_1u_3, \qquad (3)$$

where $x_1 = \frac{nm^*a}{4M_\odot} B_1$, $x_2 = \frac{nm^*a}{4M_\odot} B_2$, $n = \frac{GM_\odot}{a^{3/2}}$ are the mean motions of m, a is the semi-major axis of its orbit.

The values B_1 and B_2 can be expressed in terms of complete integrals of the first and second kind (Charlier, 1927). For the Earth-Jupiter system $x_1 = 3.6 . 10^{-5}$ yr^{-1}, $x_2 = 1.03 . 10^{-5}$ yr^{-1}.

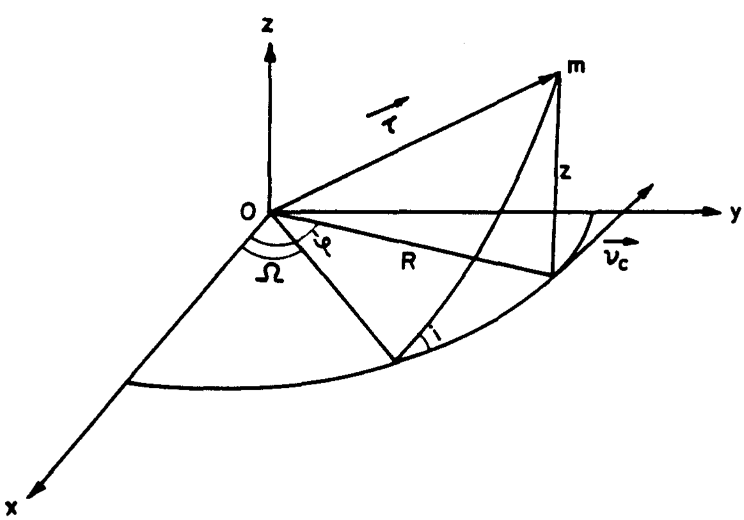

Figure 1: Geometry of the problem. The notations are in the text.

3. DETERMINATION OF THE DISTRIBUTION FUNCTION OF ORBITAL ELEMENTS FROM THE FOKKER-PLANCK EQUATION

The random process, in which the elements of orbit of the growing planet are formed, is defined by us completely, because the probabilities of encounters and collisions are determined for given values of orbital elements of the planet. So it is expected, at least in principle, that the distribution function of the planetary elements could be found depending on time. In result of encounters and collisions, the orbital elements undergo random walks occurring mainly by small portions. We can suppose in such situation that the behaviour of the distribution function is described by diffusion equation named the Fokker-Planck equation (Chandrasekhar,1943a)

$$\frac{\partial f}{\partial t} = -\sum_i \frac{\partial}{\partial u_i}\left(f \frac{\langle \Delta u_i \rangle}{\Delta t}\right) + \sum_i \frac{\partial^2}{\partial u_i^2}\left(f \frac{\langle (\Delta u_i)^2 \rangle}{2\Delta t}\right) + \sum_{i<j} \frac{\partial^2}{\partial u_i \partial u_j}\left(f \frac{\langle \Delta u_i \Delta u_j \rangle}{\Delta t}\right) \qquad (4)$$

where Δt is the time scale such that the increments of values u_i is small in spite of the large number of fluctuations. In the problem under consideration $\Delta t >> T$, where T is the period of rotation of the planet around the Sun. That is why we average over the orbit of the planet while finding the coefficients of the equation. Generally speaking, equation (4) is to be considered for the values $u_1, \ldots, u_4$, $u_5 = a$ simultaneously. To simplify the problem we neglect small terms and put in the coefficients a = constant. It is shown in Ziglina (1986) that for the Earth the characteristic change in semi-major axis is of order 0.1 AU. Note that in Ziglina (1985), where the perturbations of m^* were not considered, the Fokker-Planck equation was applied to the variables e, i.

3.1 Evaluation of Coefficients of the Fokker-Planck Equation

In order to calculate the coefficients of equation (4), at first we find the increments Δu_i $(i = 1,\ldots,4)$ due to one encounter (collision). Then we express the evaluated coefficients in terms of the increments due to many encounters and collisions during time Δt and average over the random variables are: m', the mass; $\vec{v}'$ the random velocity and v true anomaly of the planet. The encounter is characterised additionally by the encounter parameter D and by the angle between the plane of relative orbit of the body and the planet and the central plane χ. To account for the secular perturbations by the planet m*, we must add the right parts of the equations (3) to the values $<\Delta u_i>/\Delta t$ caused by the encounters and collisions.

In our calculations, we start with the following formulas expressing u_i in terms of the coordinates and random velocity components in the cylindrical coordinate system with the centre at the Sun, z axis perpendicular to the central plane and angle ψ calculated from the same axis as the node longitude.

$$u_1 = v_c^{-1} (v_R \sin\psi + 2v_\psi \cos\psi) + 0(e^2, ei^2),$$

$$u_2 = v_c^{-1} (-v_R \cos\psi + 2v_\psi \sin\psi) + 0(e^2, ei^2),$$

$$u_3 = v_c^{-1} v_z \cos\psi + zR^{-1} \sin\psi + 0(ei, i^3),$$

$$u_4 = v_c^{-1} v_z \sin\psi - zR^{-1} \cos\psi + 0(ei, i^3). \tag{5}$$

We admit that during an encounter the coordinates of the planet do not change and its velocity increment is as in the two-body problem. This approximation is valid when the relative velocity is not too small. Note that our calculations are carried out on assumption $v << j$, where v is random velocity of the planet, as far as $m >> m'$. The results obtained here agree with this assumption. Calculated in this way coefficients are as follows:

$$\frac{<\Delta u_i>}{\Delta t} = \begin{cases} (-\frac{2}{3}\frac{\dot{m}}{m} - \eta)u_i = -\beta u_i & i = 1,2 \\ -\frac{\beta}{2} u_i & i = 3,4 \end{cases}$$

$$\frac{<(\Delta u_i)^2>}{2\Delta t} = \begin{cases} \frac{5}{18}\frac{\bar{m}'\dot{m}}{m^2}\frac{j^2}{v_c^2} + \frac{5}{6} fG\bar{m}'m^{-1}\tilde{\tau}^{-1}v_c^{-2}\dot{m} = p & i = 1,2 \\ \frac{p}{5} & i = 3,4 \end{cases} \tag{6}$$

$$\frac{<\Delta u_i \Delta u_k>}{\Delta t} = \frac{<(\Delta u_i)^2>}{2\Delta t} \, 0(\frac{v^2}{j^2}) \approx 0 \qquad i \neq k$$

where $\dot{m} = \frac{dm}{dt} \approx 4(\frac{3}{2\pi})^{1/2} \pi\tilde{\tau}^2\rho\theta j,$ (7)

$$\eta = 2(6\pi)^{1/2} G^2 m\rho f j^{-3} , \tag{8}$$

$\overline{m'}$ is the mean mass of the bodies with the weight function $m'n(m',t)$, ρ is the spatial density of the bodies in the central plane,

$$j = 2 e_n \left(\frac{2S}{3\gamma\ \theta\tilde{\tau}} \right), \tag{9}$$

$\gamma = 1.781...$ is the Euler constant. $f = 10 \div 4$ for the terrestrial planets at $\theta = 1$.

In formulas (6), the terms caused by encounters are $\sim \theta f$ times larger than one caused by collisions.

It can be shown that in every point of the planetary orbit the encounters of the planet with the swarm bodies result in

$$\frac{\langle \Delta\vec{v} \rangle}{\Delta t} \approx -\eta\vec{v} \tag{10}$$

Here $\Delta t \ll T$. According to Chandrasekhar's (1943b) definition, η is the coefficient of dynamical friction. Correspondingly, the value $\eta^{-1} = \tau_{rel}$ is the relaxation time of the planetary velocty. From (7) and (8) it follows that the relaxation time of the planet is θf times less than the characteristic growth time of the planet $m/\dot{m}$, i.e. more than an order.

3.2 The Solution of the Fokker-Planck Equation

Now we substitute the coefficients (6) into the equation (4). Then the distribution function $f(u_1,...,u_4,t) = f_1(u_1, u_2,t).f_2(u_3,u_4,t)$, where f_1 and f_2 satisfy the equations

$$\frac{\partial f_1}{\partial t} = -\frac{\partial}{\partial u_1}[f_1(-\beta u_1 - x_1 u_2)] - \frac{\partial}{\partial u_2}[f_1(-\beta u_2 + x_1 u_1 - x_2 e^*)] + \frac{\partial^2(f_1 p)}{\partial u_1^2} + \frac{\partial^2(f_1 p)}{\partial u_2^2} , \tag{11}$$

$$\frac{\partial f_2}{\partial t} = -\frac{\partial}{\partial u_3}[f_2(-\frac{\beta}{2}u_3 + x_1 u_4)] - \frac{\partial}{\partial u_4}[f_2(-\frac{\beta}{2}u_4 - x_1 u_3)] + \frac{\partial^2(f_2\, p/5)}{\partial u_3^2} + \frac{\partial^2(f_2 p/5)}{\partial u_4^2}$$

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At the initial moment $t = 0$; $u_1 = u_{10}, \ldots, u_4 = u_{40}$. So the initial conditions are: $f_1(u_1, u_2, 0) = (u_1 - u_{10})\delta(u_2 - u_{20})$, $f_2(u_3, u_4, 0) = \delta(u_3 - u_{30})\delta(u_4 = u_{40})$. The solutions can be easily found because the equations are linear. The complex eccentricity $z = u_1 + \sqrt{-1}u_2$ is a sum of the "forced" eccentricity $a + \sqrt{-1}f = \dfrac{x_1 x_2 e^* - \sqrt{-1}\,\beta x_2}{\beta^2 + x_1^2}$ and the "free" eccentricity, which moves along the circle with the frequency x_1 at time scales $\Delta t \ll \tau_{rel}$. At time $t \sim \tau_{rel}$ relation the initial values are "forgotten" and the distribution of the components of the free eccentricity tends to the Gaussian distribution symmetric relative 0 and with dispersion

$$c^2 = 4 \int_0^t P e^{-\int_y^t 2\beta\, dx}\, dy \qquad (12)$$

The value $\zeta = u_3 + \sqrt{-1}\,u_4$ circulates in retrograde direction with the same frequency. Since $\sin^2 i = u_3^2 + u_4^2$ the inclination is constant during time interval $\Delta t \ll \tau_{rel}$. At time $t \gg \tau_{rel}$, f_2 tends to the Gaussian distribution with dispersion

$$d^2 = \frac{4}{5} \int_0^t P\, e^{-\int_y^t \beta\, dx}\, dy \qquad (13)$$

3.3 The Distribution Function of e and i

The distributions $f_e(e,t)$ and $f_i(i,t)$ can be expressed in terms of the joint distributions $f_1(u_1, u_2, t)$ and $f_2(u_3, u_4, t)$ since $e = \sqrt{u_1^2 + u_2^2}$ and $i \approx \sin i = \sqrt{u_3^2 + u_4^2}$.

$$f_e(e,t) = \int_{-e}^{e} [f_1(u_1, \sqrt{e^2 - u_1^2}, t) + f_2(u_1, -\sqrt{e^2 - u_1^2}, t)] \times \frac{e\,d u_1}{\sqrt{e^2 - u_1^2}} \qquad (14)$$

The formula for $f_i(i,t)$ is analogous. The complete expressions for f_e and f_i are rather cumbersome, so we don't cite them here. The dependence of the distribution functions upon the initial values is damped quickly because of the action of dynamical friction. All distributions of eccentricity at $t \gg \tau_{rel}$ appear to be close to the distribution

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$$\tilde{f}_e(e,t) = \frac{2e}{c^2} \exp\left[- \frac{e^2 + a^2 + b^2}{c^2} \right] I_o\left(\frac{2e\sqrt{a^2+b^2}}{c^2} \right) \quad (15)$$

with the root mean square value

$$< e^2 >^{1/2} = \sqrt{a^2 + b^2 + c^2} \quad (16)$$

$I_o(z)$ is the Bessel function of imaginary argument of zero order. For inclination at $t >> \tau_{rel}$ all solutions tend to the Rayleigh distribution

$$\tilde{f}_i(i,t) = \frac{2i e^{-i^2/d^2}}{d^2} \quad (17)$$

The characteristic width of this curve $\sim d$. With probability 0.8 the inclination is enclosed in the interval $0.3d \leq i \leq 1.5d$. The distribution $\tilde{f}_e(e,t)$ is shown in Fig. 2a for some values of parameters.

3.4 The Root Mean Square Values of the Eccentricity and Inclination

The planet's mass is an increasing function of time t. Therefore in integrals (12) and (13), we can pass from the integration variable t to m. Neglecting in expressions (6) for β and P quantities resulting from collisions in comparison with ones resulting from encounters, we obtain

$$c^2 = \int_{m_o}^{m} \frac{10}{3} Gf\overline{m'}^{-1}\tau^{-1} v_c^{-2} \exp\left(-\int_x^m 2\theta f \tau^{-1} d\tau\right)dx,$$

$$d^2 = \int_{m_o}^{m} \frac{2}{3} Gf\overline{m'}^{-1}\tau^{-1} v_c^{-2} \exp\left(-\int_x^m \theta f \tau^{-1} d\tau\right)dx, \quad (18)$$

where m_o is the planet's mass at the initial moment. From these formulas for $t >> \tau_{rel}$, θ = const., $\overline{m'}/m$ = const. follows

$$c^2 = \frac{5\ Gm}{3\theta\tau v_c^2} \frac{\overline{m'}}{m} \left(1 + O(1/\theta f) \right),$$

$$d^2 = \frac{2\ Gm}{3\theta\tau v_c^2} \frac{\overline{m'}}{m} \left(1 + O(1/\theta f) \right) \quad (19)$$

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These relations are approximately valid also for variable ratio $\overline{m'}/m = I(m)$. The relative error is of order $\frac{dI}{dm} m/I\Theta f$.

For the Maxwell velocity distribution of the swarm bodies we find

$$< e'^2 > = \frac{5}{3}\frac{j^2}{v_c^2} \ , \ < i'^2 > = \frac{2}{3}\frac{j^2}{v_c^2} \tag{20}$$

Therefore, neglecting the values of the order $1/\Theta f$, we can rewrite the relations (19) in the form

$$c^2 = < e'^2 > \frac{\overline{m'}}{m} \ , \ d^2 = < i'^2 > \frac{m'}{m} \tag{21}$$

These relations correspond to equipartition of "free" random energy between the planet and the bodies with mean mass $\overline{m'}$. It can be shown that the values (21) are quasi-equilibrium meansings of $< e^2_{free} >$ and $< i^2 >$, where e_{free} is the modulus of free eccentricity.

4. CONCLUSION

In the present paper the distribution function of the orbital elements $u_1, \ldots, u_4$ and e, i have been found and their properties examined. In the course of the calculations, the expression for the coefficient of dynamical friction, which acts on the planet during the accumulation, is found. For concrete results, we need to define the functions $\overline{m'}(t)$, $\Theta(t)$ and $m(t)$. These values can be taken from the other works devoted to accumulation of the planets. According to Wetherill (1978,1980), Vitjazev and Pechernikova (1981) at the last stage of formation of the terrestrial planets $\Theta \sim 1$ and the mean mass of the swarm bodies $\overline{m'} \sim 10^{-2} M_\oplus$. The eccentricities and inclinations of the planets obtained in result of numerical simulations (Wetherill 1978,1980) are of order of the modern values. Our work agrees with this result. Actually, substituting the above mentioned quantities (and $m \sim M_\oplus$) in the formulas (21) we find $c = 0.034$, $d = 0.021$. At e^* equals to its maximum value per Jupiter (≈ 0.06), $e_{forced} = \sqrt{a^2 + b^2} = 0.017$, $e_{rms} = 0.038$. The increase of e_{rms} due to Jupiter perturbations is $\approx 10\%$. In the case $c > e_{forced}$ (Fig. 2b, curve 2) the value $\bar{e} = (e_{min} + e_{max})/2 = c$. According to Brower and Clemence (1961) the Earth's eccentricity varies between 0 and 0.067, the inclination varies between 0 and 0.051. So c and $\bar{e}$, d and τ turn out to be comparable. The rms eccentricity and inclination of the swarm bodies in the Earth zone at the last stage of accumulation at $\theta = 1 \div 3$ are

(f.20) $\langle e'^2 \rangle^{1/2} = 0.2 \div 0.34, \quad \langle i'^2 \rangle^{1/2} = 0.1 \div 0.2$

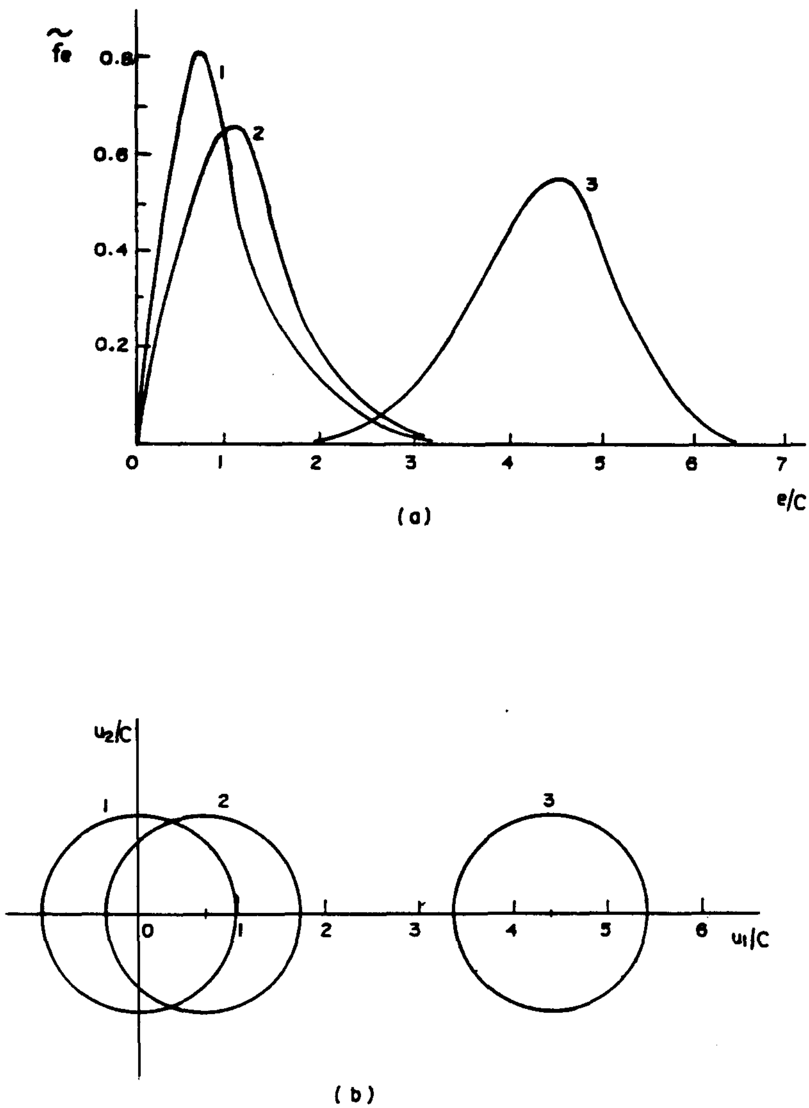

Figure 2: (a) Distribution of eccentricity $\tilde{f}_e$ for three values of the forced eccentricity modulus: 1-e_{forced} =0, 2-e_{forced} = 0.7c, 3-e_{forced}=4.4e. (b) Complex eccentricity $z = u_1 + \sqrt{-1}u_2$ in these three cases at e_{free} = c, b = 0. The eccentricity e = |z|.

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At time scale of a few millions of years, the secular variations of elements of planetary orbits are described by the averaged equations. These variations are quasi-periodic. From recent studies it is known (Lekar, 1989) that at larger time scales (for the terrestrial planets $\sim 10^8$ yr) the behaviour of the planetary orbits is stochastic. However, that does not mean that the orbits had undergone drastic changes (Milani,1989). We suppose that the orbits did not change significantly during the life of the Solar System. Then by analogy with the above example we can assume that the mean values of eccentricities and inclinations of the planets are stipulated mainly by the encounters with the bodies during the accumulation process, i.e. that $\bar{e} \sim c$, $\bar{i} \sim d$, where $\tau = (i_{min} + i_{max})/2$. For the terrestiral planets the ratio of the mean eccentricity to the mean inclination varies from 1.25 (for Venus) to 1.42 (for Mars). These quantities quite well agree with the ratio c/d = 1.6 obtained here. Therefore, from our point of view, the fact that the eccentricities and inclinations of the planetary orbits are of the same order is a consequence of the planetary accumulation from the bodies with comparable eccentricities and inclinations of orbits. It is easy to note that the mean (and also the maximum) eccentricities and inclinations of orbits of the terrestrial planets are in rough random energy equipartition. This can be a result of a considerable overlapping of the feeding zones of these planets, which led to approximately the same mean masses of the bodies ($\sim 10^{-2} M_{\oplus}$) and mean eccentricities and inclinations of their orbits ($\langle e'^2 \rangle^{1/2} \sim \langle i'^2 \rangle^{1/2} \sim 0.2$).

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