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ON A THEOREM OF GOLDSCHMIDT APPLIED TO GROUPS WITH A COPRIME AUTOMORPHISM

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1. Introduction. In a recent important paper of Goldschmidt [**3**], all finite simple groups were determined in which a non-trivial abelian 2-subgroup controls 2-fusion. Our purpose here is to present a straightforward application of this deep result to the following general question: If p is a prime and G is a finite group of order not divisible by p which admits an automorphism σ of order p^n, what conditions on the fixed point subgroup $C_G(\sigma)$ will ensure that G is solvable?

Probably the best known result in this direction is the theorem of Thompson [**1**, Theorem 10.2.1] that if $n = 1$ and σ is fixed-point-free on G (i.e. $C_G(\sigma) = 1$), then G is nilpotent. It is still an open conjecture that the existence of a fixed-point-free automorphism (of any order) is sufficient to imply solvability. Of course, in the case that $p = 2$, the theorem of Feit and Thompson on groups of odd order (which will be implicit in our argument) immediately disposes of the question of solvability without reference to the fixed point subgroup of σ.

By way of motivating the present discussion, we remark that in general, it seems that the non-abelian simple groups tend to admit rather a limited number of coprime automorphisms. Indeed, if G is a Chevalley group over a finite field, the automorphisms of order relatively prime to the order of G arise essentially from automorphisms of the field. In this case then, the group of fixed points of a coprime automorphism contains a subgroup isomorphic to the appropriate Chevalley group over the prime subfield and, in particular, has even order. It seems reasonable, therefore, to ask what can be said about the structure of a group admitting a coprime automorphism with a fixed point subgroup of odd order.

For a restricted class of finite groups, we have the following partial answer:

MAIN THEOREM. *Let G be a finite group which admits an automorphism σ of order a power of p, where p is a prime not dividing the order of G. Assume that G contains a unique σ-invariant Sylow 2-subgroup S and also, that σ has no non-trivial fixed points in S. If G does not involve the symmetric group of degree four, then G is solvable.*

To clarify the hypotheses somewhat, we remark that the mere existence of σ-invariant Sylow subgroups (for every prime) is guaranteed by the hypothesis of coprime order; the uniqueness of S is equivalent to the further assumption

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that $C_G(\sigma)$ normalizes S [**1**, Theorem 6.2.2]. The hypotheses are, of course, inductive.

As a consequence of this theorem, we have the

COROLLARY. *A finite group which admits a fixed-point-free automorphism of order a power of three is solvable.*

These results, it should be admitted, are completely subsumed by the recent spectacular success of Glauberman in classifying (also as a corollary of Goldschmidt's work) all simple groups in which the symmetric group of degree four is not involved. However, although we do make use of an earlier Glauberman result on "S_4-free" groups (Lemma 2.3), the argument presented here is independent of his most recent work.

All groups under consideration here are, of course, finite and our notation conforms to that of [**1**]. We shall assume familiarity with Sections 5.3 and 6.2 of [**1**].

2. The tools. If G is a group and A is a subset of the subgroup S of G, A is said to be *strongly closed in S with respect to G* if $S \cap A^g \subseteq A$ for every element g of G. As we indicated at the outset, the crucial tool in the argument presented here is the following theorem of Goldschmidt [**3**];

LEMMA 2.1. *Let G be a non-abelian finite simple group with a Sylow* 2*-subgroup S. Suppose S contains a non-trivial abelian subgroup A which is strongly closed in S with respect to G. Then G is isomorphic to one of*

(a) $L_2(2^n)$, $n \geqq 3$,
(b) $Sz(2^{2n+1})$, $n \geqq 1$,
(c) $U_3(2^n)$, $n \geqq 2$,
(d) $L_2(q)$, $q \equiv 3$ *or* $5 \pmod 8$ *or*
(e) *G is of "Janko-Ree type."*

For future reference, we note that each of the groups in Goldschmidt's list enjoys the property that it contains a unique conjugacy class of involutions.

In addition to the usual results on coprime operators, we shall need the following representation theoretic fact due to Shult and Gross:

LEMMA 2.2. *Let G be a finite group admitting an automorphism σ of order p^n, where p is an odd prime not dividing the order of G, and let H be the semi-direct product $G\langle\sigma\rangle$. Suppose K is a field of characteristic not dividing the order of H and A is a faithful $K[H]$-module with the property that σ has no non-trivial fixed points in A. Then $\sigma^{p^{n-1}}$ centralizes G in either of the following two situations:*

(a) *G has odd order* [**6**, *Corollary* 3.2] *or*
(b) *σ is fixed-point-free on G* [**4**, *Theorem* 2].

As remarked in the Introduction, the restriction that p be odd (which is, in fact, a necessary hypothesis) is, for our purposes, no restriction at all.

We say the group G is S_4-*free* if the symmetric group of degree four is not involved in G. The final non-standard device we shall make use of is a difficult theorem due to Glauberman on the structure of such groups [**2**, Corollary 10].

LEMMA 2.3. *Let S be a Sylow 2-subgroup of the finite group G and suppose $C_G(O_2(G)) \subseteq O_2(G)$. If G is S_4-free, then $G = \langle N_G(J(S)), C_G(Z(S))\rangle$, where $Z(S)$ is the centre of S and $J(S)$ is a certain characteristic subgroup of S containing $Z(S)$.*

3. Fusion in S_4-free groups.

The object of this section is to exploit the preceding result of Glauberman to attain some control over the fusion of 2-elements in S_4-free groups.

LEMMA 3.1. *Let S be a Sylow 2-subgroup of G and suppose T is a normal subgroup of S. If G is S_4-free, then $N_G(T) \subseteq C_G(T)\langle N_G(J(S)), C_G(Z(S))\rangle$.*

Proof. Let $N = N_G(T)$ and $U = TC_S(T)$, so $C_S(U) \subseteq U$. By the Dedekind lemma, $U = S \cap TC_G(T)$ so since $S \subseteq N$ and $TC_G(T)$ is normal in N, we conclude that U is a Sylow 2-subgroup of $TC_G(T)$. Then by the Frattini argument, $N = C_G(T)N_N(U)$, so it suffices to show that $N_G(U) \subseteq \langle N_G(J(S)), C_G(Z(S))\rangle$.

Let $M = N_G(U)$. S is contained in M so $C_S(U) = Z(U)$ is a Sylow 2-subgroup of $C_G(U)$. By Burnside's normal complement theorem, $C_G(U) = Z(U) \times K$, where K is a subgroup of odd order. If X is any subgroup of M, denote by $\bar{X}$ the image of X in M/K. Then $\bar{U}$ is normal in $\bar{M}$ and $C_{\bar{M}}(\bar{U}) = \overline{C_G(U)} \subseteq \bar{U}$, so from Lemma 2.3, it follows that $\bar{M} = \langle N_{\bar{M}}(J(\bar{S})), C_{\bar{M}}(Z(\bar{S}))\rangle$. A Frattini argument implies that $N_{\bar{M}}(J(\bar{S})) = \overline{N_M(J(S))}$ and $C_{\bar{M}}(Z(\bar{S})) = \overline{C_M(Z(S))}$, so $M = \langle N_M(J(S)), C_M(Z(S)), K\rangle$. Finally, $K \subseteq C_G(U) \subseteq C_G(Z(S))$ since $Z(S) \subseteq U$, whence $M = \langle N_M(J(S)), C_M(Z(S))\rangle$ and the proof is complete.

Definition. If S is a Sylow subgroup of G and T is a subgroup of S, let $P_1(T) = N_S(T)$, $N_1(T) = N_G(T)$, and define recursively $P_{i+1}(T) = N_S(J(P_i(T)))$ and $N_{i+1}(T) = N_G(J(P_i(T)))$. Then T is said to be *well placed in S (with respect to J)* if $P_i(T)$ is a Sylow subgroup of $N_i(T)$ for all $i \geqq 1$.

LEMMA 3.2. *Let S be a Sylow 2-subgroup of G and T be a well-placed subgroup of S with respect to J. If G is S_4-free, then $N_G(T) \subseteq C_G(T)\langle N_G(J(S)), C_G(Z(S))\rangle$.*

Proof. Suppose the lemma is false and assume a counterexample T is chosen with $T_1 = N_S(T)$ of maximal possible order.

Since T is well-placed, T_1 is a Sylow 2-subgroup of $N_G(T)$. Applying Lemma 3.1 to $N_G(T)$, we conclude that $N_G(T) \subseteq C_G(T)\langle N_G(J(T_1)), C_G(Z(T_1))\rangle$. Also, since $Z(S) \subseteq N_S(T) = T_1$, we have $Z(S) \subseteq Z(T_1) \subseteq J(T_1)$, so certainly $C_G(Z(T_1)) \subseteq C_G(Z(S))$. Finally, $T_1 \neq S$ by Lemma 3.1, so $T_1 \subset N_S(T_1) \subseteq N_S(J(T_1))$. Since $J(T_1)$ is well placed in S, our particular choice of T forces $N_G(J(T_1)) \subseteq \langle N_G(J(S)), C_G(Z(S))\rangle$ and we have a contradiction.

Now we state the main result of this section.

THEOREM 3.3. *Let S be a Sylow 2-subgroup of G and assume that G is S_4-free. Then two subsets of S are conjugate in G if and only if they are conjugate in* $\langle N_G(J(S)), C_G(Z(S))\rangle$.

Proof. This is an immediate consequence of Lemma 3.2 and the well known results of Alperin and Gorenstein on the localization of fusion [**1**, Theorem 8.4.5 and Remark, p. 288].

4. The main theorem. To make use of Goldschmidt's result, we clearly must construct, under the hypotheses of the Main Theorem, a strongly closed abelian 2-subgroup. We shall employ for this purpose the following simple consequence of Lemma 2.2:

LEMMA 4.1. *Let G be a solvable group admitting an automorphism σ of order p^n, where p is an odd prime not dividing the order of G. Assume that G contains a unique σ-invariant Sylow 2-subgroup S and also, that σ has no non-trivial fixed points in S. Then $[S, \sigma^{p^{n-1}}]$ is normal in G.*

Proof. Suppose false and assume G is a counterexample of minimal order. If $O_{2'}(G) \neq 1$, then $O_q(G) \neq 1$ for some prime $q \neq 2$ so, applying the inductive hypothesis to $G/O_q(G)$, we conclude that $[S, \sigma^{p^{n-1}}]O_q(G)$ is normal in G. Now the Frattini quotient A of $O_q(G)$ is a module for $S\langle\sigma\rangle$ over $GF(q)$ and since, by hypothesis, $C_G(\sigma)$ normalizes S, $C_A(\sigma)$ centralizes S so $A/C_A(\sigma)$ is also a module for $S\langle\sigma\rangle$. It follows from Lemma 2.2 (b) that $[S, \sigma^{p^{n-1}}]$ is contained in the kernel of this representation, so $[S, \sigma^{p^{n-1}}]$ centralizes A and hence, $O_q(G)$. Therefore, $[S, \sigma^{p^{n-1}}]$ is normal in G, a contradiction.

Thus, $O_{2'}(G) = 1$ so $C_G(O_2(G)) \subseteq O_2(G)$. If Q is a σ-invariant Sylow subgroup of G of odd order, $Q\langle\sigma\rangle$ is then faithfully represented on the Frattini quotient of $O_2(G)$. Since $C_S(\sigma) = 1$, we conclude from Lemma 2.2 (a) that $\sigma^{p^{n-1}}$ centralizes Q. Therefore, $G = C_G(\sigma^{p^{n-1}})S$ so $[S, \sigma^{p^{n-1}}] = [G, \sigma^{p^{n-1}}]$ which is normal in G. This contradiction completes the proof.

Now we bring these results to bear on the

Proof of the main theorem. Suppose the theorem is false and assume G is a non-solvable S_4-free group of minimal order subject to admitting an automorphism σ with the stated properties.

If N is a characteristic subgroup of G, σ induces automorphisms of order a power of p on both N and G/N, so G must be characteristically simple. Thus, $G = G_1 \times \ldots \times G_m$, where m is a power of p, the G_i's are isomorphic simple groups, and σ permutes the G_i's transitively. Since each G_i is normal in G, $S \cap G_i$ is a Sylow 2-subgroup of G_i so $S = (S \cap G_1) \times \ldots \times (S \cap G_m)$.

$N_G(J(S))$ and $C_G(Z(S))$ are proper σ-invariant subgroups of G and hence, are solvable. By Lemma 4.1, it follows that if σ has order p^n, then $\langle N_G(J(S)),$

$C_G(Z(S))\rangle \subseteq N_G([S, \sigma^{p^{n-1}}])$. Theorem 3.3 then yields that $Z([S, \sigma^{p^{n-1}}])$ is strongly closed in S with respect to G.

If $Z([S, \sigma^{p^{n-1}}]) = 1$, then $S \subseteq C_G(\sigma^{p^{n-1}})$ so we may choose $k \geqq 1$ to be minimal subject to $S \subseteq C_G(\sigma^{p^k})$. Let $\tau = \sigma^{p^k}$ and suppose x and y are two elements of S such that $g^{-1}xg = y$ for some g in G. Then

$$g^{-1}xg = y = y^\tau = (g^{-1}xg)^\tau = (g^\tau)^{-1}xg^\tau$$

so $g^\tau g^{-1}$ centralizes x. That is, τ stabilizes the coset $C_G(x)g$. But the orbit equation then implies that τ fixes some element of this coset so $cg \in C_G(\tau)$ for some $c \in C_G(x)$. Since $(cg)^{-1}x(cg) = g^{-1}xg = y$, we have shown that two elements of S are conjugate in G if and only if they are conjugate in $C_G(\tau)$. However, $C_G(\tau)$ is solvable if $k \leqq n - 1$ so Lemma 4.1 implies that $C_G(\tau) \subseteq N_G([S, \sigma^{p^{k-1}}])$. From this and the preceding paragraph, we conclude that for some $k \leqq n$, $Z([S, \sigma^{p^{k-1}}])$ is non-trivial and strongly closed in S with respect to G.

If $Z([S, \sigma^{p^{k-1}}]) \cap G_1 = 1$, then since $[S, \sigma^{p^{k-1}}] \cap G_1$ is normal in $[S, \sigma^{p^{k-1}}]$ it follows that $[S, \sigma^{p^{k-1}}] \cap G_1 = 1$. Then $S \cap G_1 \subseteq C_G(\sigma^{p^{k-1}})$ and since σ permutes the G_i's transitively, we have $S \subseteq C_G(\sigma^{p^{k-1}})$, contradicting the assumption on k.

The upshot of this is that $Z([S, \sigma^{p^{k-1}}]) \cap G_1$ is a non-trivial abelian subgroup of G_1, strongly closed in $S \cap G_1$ with respect to G_1, so by Lemma 2.1, the G_i's are isomorphic to certain known simple groups. But we remarked in Section 2 that each of the groups in Goldschmidt's list has a unique class of involutions. In other words, if x is an involution in G_1, then G_1 contains $i = |G_1 : C_{G_1}(z)|$ involutions, so G has $(i + 1)^m - 1$ involutions. But m is a power of p so $(i + 1)^m \equiv i + 1 \pmod{p}$. Since i is not divisible by p, σ must fix some involution of G, which contradicts the hypothesis that $C_G(\sigma)$ has odd order. This completes the proof of the Main Theorem.

5. Some remarks. Without the assumption of S_4-freeness in our theorem, we lose control of 2-fusion and dealing with the question of solvability appears much more difficult. However, with more stringent assumptions on the fixed point subgroups, something may be said. For example, a glance at the present proof reveals that if we assume $C_G(\sigma^{p^{n-1}})$ is solvable and contains S, we already have sufficient control of the fusion in S to complete the proof of solvability. At the other extreme, if we replace the S_4-free hypothesis by the assumption that $C_G(\sigma^{p^{n-1}})$ has odd order, then by Lemma 4.1, every σ-invariant solvable subgroup of G is 2-closed and, as in the proof of [**5**], it follows from a fusion result of Glauberman that G is 2-closed. In particular, the assumption in the Main Theorem that G is S_4-free may be dispensed with if σ has prime order, in which case the result may be regarded as a generalization of the theorem of Thompson mentioned in the Introduction.

The hypothesis that σ be fixed-point-free on S cannot be omitted. For example, if $G = L_2(3^q)$ where $q = p^n$, p an odd prime, then G is S_4-free and the

Frobenius automorphism of $GF(3^q)$ induces a coprime automorphism on G of order q whose fixed point subgroup is isomorphic to $L_2(3)$ and hence, is the normalizer of a Sylow 2-subgroup of G. However, if p is not a Fermat prime, then Lemma 2.2 holds without either of the restrictions (see [**6**]) and, at least in the S_4-free situation, it seems that with a minor modification of the above argument, the solvability of G does follow if the hypothesis that σ be fixed-point-free on S is replaced by the assumption that for every prime q, $C_S(\sigma)$ normalizes a σ-invariant Sylow q-subgroup of G. In particular, if p is not Fermat and G is S_4-free with a unique σ-invariant Sylow q-subgroup for every prime q, then G is solvable.

Finally, in light of Glauberman's characterization of the S_4-free simple groups as precisely the groups in Goldschmidt's list (Lemma 2.1), it is apparent that our assumption that G have a unique σ-invariant Sylow 2-subgroup is superfluous. However, it is not clear to the author how the present argument might be modified to avoid this hypothesis.

References

1. D. Gorenstein, *Finite groups* (Harper and Row, New York, 1968).

2. G. Glauberman, *Weakly closed elements of Sylow subgroups*, Math. Z. *107* (1968), 1–20.

3. D. Goldschmidt, *2-fusion in finite groups*, Ann. Math. *99* (1974), 70–117.

4. F. Gross, *Solvable groups admitting a fixed-point-free automorphism of prime power order*, Proc. Amer. Math. Soc. *17* (1966), 1440–1446.

5. M. R. Pettet, *A note on finite groups having a fixed-point-free automorphism*, Proc. Amer. Math. Soc. *52* (1975), 79–80.

6. E. Shult, *On groups admitting fixed point free abelian operator groups*, Illinois J. Math. *9* (1965), 701–720.

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