POLYNOMIALS WITH REAL ROOTS

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In a recent issue of this Bulletin a problem equivalent to the following is proposed by Moser and Pounder [1]:

If ax^2+bx+c is a polynomial with real coefficients and real roots then $a+b+c \leq 9/4 \max(a,b,c)$.

The object of this note is to prove the following theorems which generalise this result.

THEOREM 1. Let α_n be the smallest constant such that for all polynomials

(1) $p(x) = a_0+a_1x+\ldots+a_nx^n$

of degree n, with real coefficients and only real roots:

(2) $a_0+a_1+\ldots+a_n \leq \alpha_n \max a_k$.

Then

(3) $$\alpha_n = \frac{(n+1)^n}{\binom{n}{s}(n-s)^{n-s}(s+1)^s} \sim \sqrt{\frac{\pi n}{2}} \quad \text{where} \quad s = [\tfrac{n}{2}] .$$

THEOREM 2. Let β_n be the largest constant such that for all polynomials (1)

(4) $\min a_k \leq \beta_n \max a_k$.

Then

(5) $$\beta_n = \binom{n}{s}^{-1} \sim 2^{-n}\sqrt{\pi n} \quad \text{where} \quad s = [\tfrac{n}{2}] .$$

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It is interesting to note how the condition that p have real roots results in a dispersion of the values of the coefficients. In the case where no restriction is made on the reality of the roots, the constants corresponding to α_n and β_n are n+1 and 1 respectively.

The proofs of the theorems will proceed from a series of lemmas. We first confine attention to theorem 1 and note at once that if $\Sigma a_i \leq 0$ then condition (2) is satisfied for the given value of α_n. Therefore we consider the class $\mathcal{P}$ of polynomials (1) with real coefficients and real roots such that $\Sigma a_i > 0$. For each $p \varepsilon \mathcal{P}$ we define

$$M(p) = (\max a_k)^{-1} \sum_{i=0}^{n} a_i$$

and then $\alpha_n = \sup M(p)$ taken over all $p \varepsilon \mathcal{P}$.

Lemma 1. If $p \varepsilon \mathcal{P}$ has a positive root u then $M(p) \leq M(q)$ where $q(x) = \varepsilon\, xp(x)/(x-u)$ and $\varepsilon = \operatorname{sgn}(1-u)$ is inserted to ensure that $q \varepsilon \mathcal{P}$.

Proof. We first note that $u \neq 1$ because of the definition of $\mathcal{P}$. Let $p(x) = (x-u)(b_0+b_1x+\ldots+b_{n-1}x^{n-1}) = a_0+a_1x+\ldots+a_nx^n$. Then $q(x) = \varepsilon\, b_0x+\varepsilon\, b_1x^2+\ldots+\varepsilon\, b_{n-1}x^n$. Hence $\max a_k = \max(b_{k-1} - ub_k) \geq (\max \varepsilon\, b_k)|1-u|$ and so $(\max a_k)\, \Sigma\, \varepsilon\, b_i \geq (\max \varepsilon\, b_k)|1-u|\, \Sigma\, \varepsilon\, b_i = (\max \varepsilon\, b_k)\, \Sigma\, a_i$ and the lemma is proved.

Since $D_x^{n-k-1} D_y^{k-1} (a_0y^n+a_1y^{n-1}x+\ldots+a_nx^n)$ has its roots real whenever (1) has, we have the following well known result (see [2] theorem 51).

Lemma 2. If $p(x) = a_0+a_1x+\ldots+a_nx^n$ has all its roots real then $h_ka_k^2 \geq a_{k-1}a_{k+1}$ $(k = 1, 2, \ldots, n-1)$ where $h_k = \dfrac{k(n-k)}{(k+1)(n-k+1)} < 1$. There is equality for each k if and only if all the roots of p are equal.

Lemma 3. If $p \in \mathcal{P}$ and all the roots of p are negative then $M(p) \leq M(q)$ where $q \in \mathcal{P}$ is a certain polynomial whose roots are all equal.

Proof. Since the roots are all negative the coefficients a_k of p are all positive. From lemma 2 it then follows inductively that

$$(6) \quad a_{k \pm r} \leq h_{k\pm 1}^{r-1} h_{k\pm 2}^{r-2} \cdots h_{k\pm(r-1)} \frac{a_{k\pm 1}^{r}}{a_k^{r-1}} = H(k, \pm r) a_{k\pm 1}^{r} / a_k^{r-1},$$

say.

Suppose that $a_k = \max a_i$ and that

$$(7) \quad a_{k-1} = y a_k, \; a_{k+1} = z a_k \quad (0 < y, z \leq 1),$$

then, using (6), we obtain

$$(8) \quad a_0 + a_1 + \ldots + a_n \leq a_k \{ 1 + \sum_{r=1}^{k} H(k, -r) y^r + \sum_{r=1}^{n-k} H(k, r) z^r \}$$

The inequality (8) remains true if y and z are increased and, in particular, if they are changed to y', z' where $y'z' = h_k < 1$ and $y \leq y' \leq 1$, $z \leq z' \leq 1$. These new values for y and z, the given value of a_k, the relations (7) and equality through the relations (6) define the coefficients of a polynomial q with equal roots (lemma 2), and the sum of the coefficients of q will be the right hand side of (8). Since $H(k, \pm r) \leq 1$, the largest coefficient of q will still be a_k and so (8) shows that $M(p) \leq M(q)$.

The last lemma has reduced our search for a polynomial $p \in \mathcal{P}$ for which $M(p)$ is maximal to the case where p has equal roots and this final case is disposed of in the final lemma.

Lemma 4. Let $c(a, k) = \binom{n}{k} a^k (1-a)^{n-k}$ be the $(k+1)$st coefficient of $q(x) = (ax + (1-a))^n$ with $0 \leq a \leq 1$. Then

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$$\min_a \max_k c(a,k) = c(a_0, k_0)$$

where $a_0 = \frac{k_0+1}{n+1}$ and $k_0 = [\frac{n}{2}]$.

Proof. From symmetry we may suppose $a \geq 1/2$ and hence $k \leq n/2$.

For given a, $c(a,k)/c(a,k-1) = \frac{n-k+1}{k}\frac{a}{1-a} \gtrless 1$ according to whether $a \gtrless \frac{k}{n+1}$. Therefore $c(a,k)$ is maximal when k satisfies $\frac{k}{n+1} \leq a \leq \frac{k+1}{n+1}$.

For given k and $\frac{k}{n+1} \leq a \leq \frac{k+1}{n+1}$, $c(a,k)$ is minimal when $a^k(1-a)^{n-k}$ is minimal and this occurs at one of the boundary points $a = \frac{k}{n+1}$ or $\frac{k+1}{n+1}$. Using the fact that $(\frac{m+1}{m})^m$ is a monotonically increasing function it follows that $c(a,k)$ is minimal (for $k \leq n/2$) at $a = \frac{k+1}{n+1}$.

Finally $c(\frac{k+1}{n+1}, k)/c(\frac{k}{n+1}, k-1) = (\frac{k+1}{k})^k/(\frac{n-k+1}{n-k})^{n-k} \gtrless 1$ according to whether $k \gtrless n-k$. Therefore for $k \leq n/2$, the maximum of $c(\frac{k+1}{n+1}, k)$ occurs at $k = [\frac{n}{2}]$ and the lemma is proved.

Proof of theorem 1. From lemmas 3 and 4 it follows at once that $c(a_0, k_0)^{-1} = \sup M(p)$ taken over all $p \in \mathscr{P}$ all of whose roots are negative. However, from lemma 1 it follows that if $p \in \mathscr{P}$ has a positive or zero root then $M(p) \leq \alpha_{n-1}$. Since our values (3) of α_n increase monotonically with n we can conclude, by induction on n, that $\alpha_n = c(a_0, k_0)^{-1}$ as given. The asymptotic estimate comes from an application of Stirling's formula.

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Proof of theorem 2. If p has any non-positive coefficient, (4) is satisfied for the given value of β_n. Therefore we need consider only the case where all the coefficients of p are positive. Let $a_k = \max a_i$. Then, as in the proof of lemma 3, the ratio $\frac{\min a_i}{\max a_i}$ is maximal when equality holds in all the relations (6) and $h_k a_k^2 = a_{k+1} a_{k-1}$. By lemma 2 this occurs only when p has all its roots equal. Suppose that $p(x) = (x+u)^n$ $(u > 0)$, where from symmetry we may suppose that $u \geq 1$. Since in this case $\min a_i = 1$ and $\max a_i \geq \binom{n}{s} u^s \geq \binom{n}{s}$ where $s = [\frac{n}{2}]$, we have

$$\frac{\min a_i}{\max a_i} \leq \binom{n}{s}^{-1}$$

where the limit is attained for $p(x) = (x+1)^n$. This proves theorem 2.

REFERENCES

1. L. Moser and J. R. Pounder, Problem 53, Canadian Mathematical Bulletin, vol. 5 (1962) 70.

2. Hardy, Littlewood and Polya, Inequalities, Cambridge University Press (1952).

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