ON RELATIVELY INVARIANT MEASURES

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1. Introduction. In this note we will discuss the question of the measurability of the multiplier function of a relatively invariant measure on a group. That is, for a group G, σ-ring S, and a measure μ defined on the sets of S, we assume: E in S, x in G implies xE is in S and $\mu(xE) = \sigma(x)\mu(E)$ and study the measurability of the function $\sigma(x)$.

The problem was discussed by Halmos (**1**, p. 265), on locally compact groups and there the situation proved to be as nice as it could be, that is, if the measure is a non-trivial, relatively invariant Baire measure then the multiplier function is continuous. We prove two theorems for groups in which no topology is assumed. In the first theorem we assume a shearing condition and answer the question completely. The second theorem places a condition on the measure and weakens the shearing assumption. Its proof is complicated and occupies the major portion of this paper.

2. Definitions and Notation. We shall use the measure-theoretic notation and definitions of (**1**) with these modifications and additions. All measures which are considered are complete.

2.1. *A left-invariant ring*, R, is a ring of subsets of a group, G such that E in R implies xE is in R for all x in G.

2.2. When we say a function, f, is *S-measurable* we mean that for E in S and M a Borel set of the real line, $E \cap f^{-1}(M) \cap N(f)$ is in S. ($N(f) = \{x : f(x) \neq 0\}$.)

2.3. (G, S, μ) will be a measure space such that G is a group and S is a left-invariant σ-ring of subsets.

2.4. If E and xE are measurable and $\mu(xE) = \sigma(x)\mu(E)$ and if μ is not identically equal to zero and is σ-finite then μ is called *relatively invariant* and will be denoted by $(\sigma)\mu$. Note that the definition of $\sigma(x)$ implies that $0 < \sigma(x) < \infty$, all $x \in G$, $\sigma(xy) = \sigma(x)\sigma(y) = \sigma(yx)$, $\sigma(e) = 1$, $\sigma(x)\sigma(x^{-1}) = 1$.

2.5. By $H(S)$ we shall mean the hereditary σ-ring generated by S.

2.6. In $(G, H(S), (\sigma)\mu^*)$ we shall define an *outer measure integral* denoted by $\delta^*(E) = \int^*_E f(x)d\mu^*$, where f is an arbitrary non-negative function on G and

$$\delta^*(E) = \lim_{n\to\infty} \sum_{i=1}^{n2^n} (i-1)2^{-n}\mu^*(E_{ni} \cap E)$$

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where $E_{ni} = \{x: (i-1)2^{-n} \leqslant f(x) < i2^{-n}$ for $i = 1, \ldots, n2^n\}$. Note that $0 < \sigma(x) < \infty$ implies that if $\delta^* = \int^* \sigma(x^{-1})d\mu^*$, then $\delta^*(E) = 0$ if and only if $\mu(E) = 0$.

2.7. (G, S) will be said to satisfy the *shearing condition* if the transformation from $G \times G$ to $G \times G$ defined by $\theta(x, y) \to \theta(x, xy)$ is a measurability preserving transformation, (carries $S \times S$ onto $S \times S$).

2.8. By *weak shearing* we shall mean that if $f(x)$ is S-measurable then $g(x, y) = f(xy)$ is $S \times S$-measurable.

2.9. *By condition A* on a measure space we shall mean that the space is the union of a disjoint class $\mathscr{D}$ of measurable sets of finite measure with the property that every measurable set may be covered by countably many sets of $\mathscr{D}$ and a set of measure zero.

Remark. According to Halmos **(1**, p. 132**)** this implies that the Radon–Nikodym theorem is valid.

2.10. We say that (G, S, μ) is *countably coverable* if for every set E of positive measure and any other measurable set F, there exist x_i, $i = 1, 2, \ldots$, such that $F - \bigcup x_i E$ has measure zero.

Remark. Lebesgue measure is countably coverable.

2.11. By a *measure group* we shall mean a measurable space (G, S) such that G is a group and S is left-invariant and satisfies the shearing condition.

3. Measurability theorems.

THEOREM 1. *Let (G, S) be a measure group and let $(\sigma)\mu$ be a relatively invariant measure defined on S. Then σ is S-measurable.*

Proof. From the definition of shearing we have, for any subset E of $G \times G$, $(\theta(E))_x = xE_x$. (See **(1)**, p. 258.) Let $E = F \times F$, where F is in S. By Fubini's theorem we have that

$$\int \chi_{\theta(E)} d\mu(y) = \mu((\theta(E))_x) = \mu(xE_x) = \sigma(x)\mu(F)\chi_F$$

is a measurable function of x. Therefore, $\sigma(x)\mu(F)\chi_F$ is measurable but $\mu(F)$ is a constant and F is an arbitrary set in S; hence $\sigma(x)$ is S-measurable.

COROLLARY. *In a measure group the existence of one non-trivial measure $(\sigma)\mu$ implies that any other non-trivial $(\sigma')\mu'$ can be written as*

$$\mu'(E) = K \int_E \sigma'/\sigma \, d\mu.$$

Proof. The theorem implies that both σ and σ' are measurable. Let $\theta(E) = \int_E \sigma(x^{-1})d\mu$ and $\theta'(E) = \int_E \sigma'(x^{-1})d\mu'$. Both θ and θ' are invariant measures and (G, S, θ) and (G, S, θ') are measurable groups (see **(1)**, page 257). Therefore Theorem 60:B of **(1)** applies and shows that $K\theta = \theta'$. Let

$$f_n = \sum_{m=1}^{M} a_{nm} \chi_{E_{nm}}$$

be a sequence of simple functions monotonically converging to σ'. Then

$$\mu'(E) = \int_E \sigma'(x)d\theta' = \lim_{n\to\infty} \int_E f_n d\theta'$$

$$= \lim_{n\to\infty} K \sum_{m=1}^{M} a_{nm} \int_{E_{nm} \cap E} \sigma(x^{-1})d\mu = K \int_E (\sigma'/\sigma)\, d\mu.$$

For Theorem 2 we shall need the following lemmas:

LEMMA 1. *For arbitrary non-negative function f on G, δ^*, the outer measure integral of f in $(G, H, (S), \mu^*)$, is an outer measure on $H(S)$ and the σ-ring of μ^*-measurable sets is contained in the σ-ring of δ^*-measurable sets.*

Proof. The fact that δ^* is an outer measure follows immediately from the definition. Let E be μ^*-measurable. Then for arbitrary $A \in H(S)$ we have

$$\delta^*(A) = \lim_{n\to\infty} \sum_{i=1}^{n2^n} (i-1)2^{-n}\mu^*(A \cap E_{ni})$$

$$= \lim_{n\to\infty} \sum_{i-1}^{n2^n} (i-1)2^{-n}[\mu^*(A \cap E_{ni} \cap E) + \mu^*(A \cap E_{ni} \cap E')]$$

$$= \delta^*(A \cap E) + \delta^*(A \cap E').$$

This completes the proof of the lemma.

LEMMA 2. *If $\delta^*(E) = \int^*_E f d\mu^*$, then f is R-measurable, where R is the collection of δ^*-measurable sets.*

Proof. It is sufficient to show E_{Nj} satisfies the Carathéodory criterion for all $A \in H(S)$. For N and j fixed and $n > N$, we have either $E_{ni} \cap E_{Nj} = \phi$ or E_{ni}. Therefore, for arbitrary $A \in H(S)$ we have

$$\delta^*(A) = \lim_{n\to\infty} \sum_{i-1}^{n2-1} (i-1)2^{-n}(\mu^*(E_{Nj} \cap A \cap E_{ni}) + \mu^*(E'_{Nj} \cap A \cap E_{ni}))$$

$$= \delta^*(A \cap E_{Nj}) + \delta^*(A \cap E'_{Nj}).$$

LEMMA 3. *$\delta^*(E) = \int^*_E \sigma(x^{-1})d\mu^*$ is an invariant outer measure on $H(S)$ and the restriction of δ^* to S is an invariant measure on S.*

The proof of this lemma is long and will be given in § 4. We now have this

THEOREM 2. *Let $(G, S, (\sigma)\mu)$ satisfy condition A and be countably coverable and suppose that there exists a set $E \in S$ such that $0 < \delta^*(E) < \infty$, (with δ^* as in Lemma 3). Then there exists a σ-ring R containing S and a measure (σ) μ on R which is an extension of μ on S such that σ is R-measurable. If in addition S satisfies the weak shearing condition then $\sigma(x)$ is S-measurable.*

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Proof. By Lemma 3, δ^* restricted to S is a measure. Since $\delta^*(E) = 0$ if $\mu(E) = 0$, $\delta^* \ll \mu$ and condition A then implies the Radon–Nikodym theorem is valid. Let f be the $R-N$ derivative. Let $E \in S$ be such that $0 < \delta^*(E) < \infty$. Let A be any set of $\mathscr{D}$. There exist $\{x_i\}$, $i = 1, 2, \ldots, 1$ such that $\cup x_i E \supset A$. Therefore, on A, δ is σ-finite. Hence f can be chosen to be finite-valued on A, hence on G. On each subset F of A such that $\delta^*(F) < \infty$, we have,

$$\delta^*(F) = \int_F f(y)d\mu = \delta^*(xF) = \int_{xF} f(y)d\mu = \sigma(x^{-1}) \int_F f(x^{-1}y)\, d\mu.$$

Therefore, for each x

(1) $$f(y) = \sigma(x^{-1})f(x^{-1}y), \qquad [\mu] \text{ in } y \text{ for } y \text{ in } A.$$

Since the A are disjoint and a countable union of them cover any measurable set to within a set of measure zero the formula is valid for all x, $[y]$, when y is restricted to any measurable set.

(1) implies $\bar{\mu} = \int (f(x))^{-1} d\delta$ is a relatively invariant measure with δ as the multiplier function on the σ-ring of δ^* measurable sets R. Lemma 2 shows that σ is R-measurable. Therefore, we have only to show that $\bar{\mu}$ is an extension of μ. Using Theorem B, page 134 of **(1)**, we have

$$\int_E (f)^{-1} d\delta = \int_E (f)^{-1} f d\mu = \mu(E),$$

for every E in S. Therefore, $\bar{\mu}$ satisfies the theorem and this completes the proof of the first part of the theorem.

The weak shearing condition implies that $f(y)\sigma(x^{-1}) - f(xy) = g(x, y)$ is $R \times R$-measurable. On every set in $R \times R$, $g(x, y)$ is integrable and its integral will be zero by (1) and the Fubini theorem. Let A be any set in D with $\mu(A) > 0$. Then $\delta^*(A) > 0$ and A contains a set of points of positive μ-measure at which $0 < f(y) < \infty$. Let E be the subset of $A \times A$ for which $f(y)\sigma(x^{-1}) - f(xy) \neq 0$. Then $\bar{\mu} \times \bar{\mu}(E) = 0$. Therefore, for almost all y in E,

$$\frac{f(y)}{\sigma(x)} - f(xy) = 0 \qquad [\bar{\mu}].$$

If $A_y = \{x: f(y)\sigma^{-1}(x) - f(xy) = 0\}$, $\bar{\mu}(A_y) = 0$ for almost all y in A by the Fubini theorem. If $\bar{\mu}(A_y) = 0$, then $\delta^*(A_y) = 0$. Whence $\mu(A_y) = 0$, using 2.6. Thus there exists $y \in E$ with $0 < f(y) < \infty$ and such that $f(y)\sigma^{-1}(x) - f(xy) = 0$ for almost all x in $A[\mu]$. The measurability of $f(xy)$ then implies that $\sigma(x)$ is measurable in A, and the definition of A implies that $\sigma(x)$ is S-measurable.

4. Proof of Lemma 3. We shall prove a sequence of remarks which will lead to the lemma.

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REMARK 1. *$\mu^*(xE) = \sigma(x)\mu^*(E)$ for all E in $H(S)$.*

Proof. This statement is an immediate consequence of the definition of an outer measure and the relative invariance of μ.

In the following let E be any set in $H(S)$ such that $\mu^*(E_{ni} \cap E) < \infty$ for all n and $i \neq 1$.

From Remark 1 we have

$$\delta^*(E) = \lim_{n\to\infty} \sum_{i=1}^{n2^n} (i-1)/2^n \mu^*(E_{ni} \cap E) = \lim_{n\to\infty} \sum_{i=1}^{n2^n} (i-1)/2^n \sigma(y). \mu^*(yE_{ni} \cap yE).$$

Let $A(N, i) = \{j: (i-1)/2^n\sigma(y) < (j-1)/2^N < j/2^N < i/2^n\sigma(y)\}$ for $i = 1, \ldots, n2^n$. Note that j in $A(N, i)$ implies $E_{Nj} \subset yE_{ni}$ and that

$$\bigcup_{j \in A(N,i)} E_{Nj} \subset \bigcup_{j \in A(M,i)} E_{Mj}$$

if $N \leqslant M$.

In Remarks 2 and 3 we shall be concerned with a particular i and fixed n and y; hence we shall suppress the i in the notation $A(N)$.

REMARK 2.

$$yE_{ni} = \lim_{N\to\infty} \bigcup_{j \in A(N)} E_{Nj} \cup I$$

where $I = \{x: \sigma(x) = 2^n\sigma(y)/(i-1)\}$.

Proof. From the definition of A we see that the right side is a subset of the left side. Let z be a member of the left side. Then

$$\sigma(z^{-1}) = (i-1)(2^n\sigma(y))^{-1}$$

or

$$(i-1)(2^n\sigma(y))^{-1} < \sigma(z^{-1}) < i(2^n\sigma(y))^{-1}.$$

The first case implies z is in I. For the second case there exists an M and $j \in A(M)$ such that

$$(i-1)/2^n\sigma(y) \leqslant (j-1)2^{-M} < \sigma(z^{-1}) < j/2^M < i/2^n\sigma(y).$$

Therefore z is in the union over $A(M)$ and $\bigcup_{j \in A(N)} E_{Nj}$ is an increasing sequence of sets; hence the remark follows.

REMARK 3. *Let $a > 0$ be arbitrary; then, for any $E \in H(S)$ such that $\mu^*(E_{ni} \cap E) < \infty$, there exists an M such that*

$$\mu^*(yE_{ni} \cap yE) \leqslant \sum_{j \in A(M)} \mu^*(E_{Mj} \cap yE) + \mu^*(I \cap yE) + a2^{-n}n^{-3}.$$

Proof. From Remark 2 we have

$$\mu^*(yE_{ni} \cap yE) \leqslant \mu^*(\lim_{N\to\infty} \cup_{j\epsilon A(N)} E_{Nj} \cap yE) + \mu^*(I \cap yE)$$
$$= \lim_{N\to\infty} \mu^*(\cup_{j\epsilon A(N)} E_{Nj} \cap yE) + \mu^*(I \cap yE)$$
$$\leqslant \lim_{N\to\infty} \sum_{j\epsilon A(N)} \mu^*(E_{Nj} \cap yE) + \mu^*(I \cap yE).$$

Since the left side is finite, there exists an M such that the remark holds.

We can do this for each i obtaining an M_i. If we are given y and fix n such that $\sigma(y)n > 1$ and if we let $N_0 = \max\{M_i, \log_2\sigma(y) + n\}$ then Remark 3 holds uniformly in i for all $N \geqslant N_0$. In addition, since $1/2^n\sigma(y) > 1/2^N$, there exists one distinct j_i for each i such that

$$E_{Nj_i} \supset I.$$

We then can prove

REMARK 4.

$$\sum_{i=1}^{n2^n} (i-1)(2^n\sigma(y))^{-1}\mu^*(yE_{ni} \cap yE)$$
$$\leqslant \sum_{j=1}^{N2^N} (j-1)2^{-N}\mu^*(E_{Nj} \cap yE) + \sum_{i=1}^{n2^n} 2^{-N}\mu^*(E_{Nj_i} \cap yE) + a$$

for all $N \geqslant N_0$.

Proof. We shall call the left side of the inequality K_n. Then, from Remark 3 and the definition of $A(N, i)$, we have

$$K_n \leqslant \sum_{i=1}^{n2^n} (i-1)/2^n\sigma(y)\left[\sum_{j\epsilon A(N,i)} \mu^*(E_{Nj} \cap yE) + \mu^*(I \cap yE)\right]$$
$$+ \sum_{i=1}^{n2^n} a(i-1)/n^3 2^{2n}\sigma(y)$$
$$\leqslant \sum_{i=1}^{n2^n} \sum_{j\epsilon A(N,i)} [(j-1)2^{-N}]\mu^*(E_{Nj} \cap yE)$$
$$+ \sum_{i=1}^{n2^n} (i-1)(2^n\sigma(y))^{-1}\mu^*(yE \cap I) + a.$$

Since there exists one j_i for each i, we have for all $N \geqslant N_0$

$$K_n \leqslant \sum_{j=1}^{N2^N} (j-1)2^{-N}\mu^*(E_{Nj} \cap yE)$$
$$+ a + \sum_{i=1}^{n2^n} [(i-1)(2^n\sigma(y))^{-1} - (j_j - 1)2^{-N}]\mu^*(E_{Nj_i} \cap yE).$$

Since $(i-1)(2^n\sigma(y))^{-1} - (j_i - 1)2^{-N} \leqslant 2^{-N}$, the remark follows.

REMARK 5. *The lemma is true if* $\mu^*(E) < \infty$.

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Proof. If $\mu^*(E) = 0$ we are finished. Therefore we shall assume that $0 < \mu^*(E) < \infty$. Then from Remark 4 and the monotonicity of the outer measure, we have

$$K_n \leqslant a + \delta^*(yE) + (n2^n/2^N)\mu^*(yE).$$

Letting $N \to \infty$ we have $K_n \leqslant a + \delta^*(yE)$. This is true for all n from some point on; therefore, $\delta^*(E) \leqslant a + \delta^*(yE)$. Since a is arbitrary we have $\delta^*(E) \leqslant \delta^*(yE)$. Applying this inequality to the set yE and y^{-1} we conclude that $\delta^*(E) \geqslant \delta^*(yE)$ and the remark follows.

REMARK 6. *If $\mu^*(E) = \infty$, then the lemma is true if there exists a K such that $\mu^*(yE_{ni} \cap yE) < 2^nK$ for all n and $i \neq 1$.*

Proof. Since $1/2^N < 1/2^n\sigma(y)$, we have

$$E_{Nji} \subset yE_{ni} \cup yE_{n,\,i-1}.$$

Then from Remark 4 we have

$$K_n \leqslant \delta^*(yE) + \sum_{i=1}^{n2^n} 2^{-N}[\mu^*(yE_{ni} \cap yE) + \mu^*(yE_{n,\,i-1} \cap yE)] + a$$
$$\leqslant \delta^*(yE) + 2\,2^nK\,n2^n\,2^{-N} + a.$$

The remark now follows as in Remark 5.

REMARK 7. *If $\mu^*(E) = \infty$ and there does not exist a K as in remark* 6, *then the lemma is true.*

Proof. Let K' be given. Then there exists an n and $i_0 \neq 1$ such that

$$\mu^*(yE_{n\,i_0} \cap yE) = \sigma(y)\mu^*(E_{n\,i_0} \cap E) > 2^nK'\sigma(y).$$

This implies

$$\delta^*(E) \geqslant \sum_{i=1}^{n2^n} (i-1)2^{-n}\mu^*(E_{ni} \cap E) > K'.$$

Hence $\delta^*(E) = \infty$. The result follows as in Remark 5.

This completes the proof of the Lemma. The case which was excluded just before Remark 2, that is, E such that $\mu^*(E_{ni} \cap E) = \infty$ for some n and $i \neq 1$, is clearly covered in Remark 7.

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