ON SUBTOURNAMENTS OF A TOURNAMENT

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Beineke and Harary [1] recently showed that the maximum number of strong tournaments with k nodes that can be contained in a tournament with n nodes is

$$\binom{n}{k} - \left[\frac{1}{2}(n+1)\right] \cdot \binom{[\frac{1}{2}n]}{k-1} - \left[\frac{1}{2}n\right] \binom{[\frac{1}{2}(n-1)]}{k-1},$$

if $3 \leq k \leq n$. The object of this note is to obtain some additional results of this type. We will use essentially the same terminology as was used in [1], so we will not repeat the standard definitions here.

L. Moser (see [5], p. 305) proved that a strong tournament T_n with n nodes contains a cycle of length k, for $k = 3, 4, \ldots, n$. (His argument is a refinement of the argument Camion [2] used to prove that a strong tournament contains a complete cycle.) We will need the following slightly stronger result which can be proved in essentially the same way.

THEOREM 1. Each node of a strong tournament T_n is contained in a cycle of length k, for $k = 3, 4, \ldots, n$.

For any integers n and k such that $3 \leq k \leq n$, let $s(n, k)$ denote the minimum number of strong tournaments T_k that can be contained in a strong tournament T_n. (If the tournament T_n is not strong then it need not contain any strong tournaments T_k.)

THEOREM 2. $s(n, k) = n - k + 1$.

Proof. We will first show that $s(n, k) \geq n - k + 1$. This inequality certainly holds when $n = k$, by Theorem 1. If $n > k \geq 3$, then it follows from Theorem 1 that any strong

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tournament T_n contains a strong tournament T_{n-1}. Now T_{n-1} contains at least $s(n-1, k)$ strong subtournaments T_k, by definition, and the node not in T_{n-1} is contained in at least one cycle of length k. The nodes of this cycle determine a strong tournament T_k that is not contained in T_{n-1}. Consequently,

$$s(n, k) \geq s(n - 1, k) + 1.$$

The earlier inequality now follows by induction on n, for each fixed value of k.

To show that $s(n,k) \leq n - k + 1$, consider the tournament T'_n in which $p_i \to p_j$ if and only if $i = j - 1$ or $i \geq j + 2$, for $i, j = 1, 2, \ldots, n$ and $i \neq j$. (The tournament T'_5 is shown in Figure 1.) It is not difficult to see that this tournament contains precisely $n - k + 1$ strong subtournaments T_k, for $k = 3, 4, \ldots, n$. This completes the proof of the theorem.

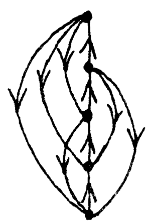
COROLLARY 2.1. The minimum number of cycles of length k a strong tournament T_n can contain is $n - k + 1$.

This follows from Theorems 1 and 2 and the fact that each strong subtournament T_k of T'_n contains exactly one cycle of length k. The case $k = 3$ of this corollary is given in [5, p. 306].

The problem of determining the maximum number of cycles of length k a strong tournament T_n can contain seems very difficult in general. The case $k = 3$ was settled by Kendall and Smith [6] and Szele [7]; the case $k = 4$ was settled by Colombo [3] and Beineke and Harary [1].

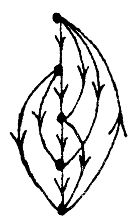
COROLLARY 2.2. The minimum number of cycles a strong tournament T_n can contain is $\binom{n-1}{2}$.

This follows from Corollary 2.1 upon summing from $k = 3$ to $k = n$.



T_5'

Figure 1



T_5''

Figure 2

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Let $u(n, k)$ denote the maximum number of transitive tournaments T_k that can be contained in a strong tournament T_n. (If T_n is not strong, then the problem is trivial.)

THEOREM 3. If $3 \leq k \leq n$, then $u(n, k) = \binom{n}{k} - \binom{n-2}{k-2}$.

Proof. When $k = 3$ the result follows from Corollary 2.1 since every subtournament T_3 is either a cycle or it is transitive. We now show that $u(n, k) \leq \binom{n}{k} - \binom{n-2}{k-2}$ for any fixed value of $k \geq 4$. The inequality certainly holds when $n = k$. If $n > k \geq 4$, then it follows from Theorem 1 that any strong tournament T_n contains a strong subtournament T_{n-1}. If p is the node not in T_{n-1}, then there are at most $u(n-1, k-1)$ transitive subtournaments T_k of T_n that contain p and at most $u(n-1, k)$ that do not. Therefore,

$$u(n, k) \leq u(n-1, k-1) + u(n-1, k).$$

The required inequality now follows by induction on n and k.

To show that $u(n, k) \geq \binom{n}{k} - \binom{n-2}{k-2}$, it suffices to consider the tournament T''_n in which $p_1 \to p_n$ but otherwise $p_j \to p_i$ if $j > i$. (The tournament T''_5 is shown in Figure 2.) This tournament has exactly $\binom{n}{k} - \binom{n-2}{k-2}$ transitive subtournaments T_k, if $3 \leq k \leq n$, for every subtournament T_k is transitive unless it contains both p_1 and p_n. This completes the proof of the theorem.

If we count the trivial tournaments with only one or two nodes as transitive then the following result holds.

COROLLARY 3.1. The maximum number of transitive tournaments that can be contained in a strong tournament T_n is 3.2^{n-2}, if $n \geq 2$.

Let $t(n, k)$ denote the minimum number of transitive tournaments T_k that can be contained in a tournament T_n. Erdős and Moser [4] showed that $t(n, k) = 0$ if $k > [2 \log_2 n] + 1$

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and conjectured that $t(n, k) = 0$ if $k > [\log_2 n] + 1$. They also showed that every tournament with 2^{k-1} nodes contains at least one transitive tournament T_k. This yields the inequality

$$t(n, k) \geq \binom{n}{2^{k-1}} \Big/ \binom{n-k}{2^{k-1}-k} = n_{(k)} / 2^{k-1}_{(k)} \geq \left(\frac{n}{2^{k-1}}\right)^k ,$$

if $n \geq 2^{k-1}$. The following result gives a sharper bound in general.

THEOREM 4. Let

$$\tau(n, k) = \begin{cases} n \cdot \dfrac{(n-1)}{2} \cdot \dfrac{(n-3)}{4} \cdots \dfrac{(n-2^{k-1}+1)}{2^{k-1}} & \text{if } n > 2^{k-1} - 1, \\ 0 & \text{if } n \leq 2^{k-1} - 1. \end{cases}$$

Then

$$t(n, k) \geq \tau(n, k).$$

Proof. When $k = 1$ the result is certainly true if we count the tournament T_1 as transitive. If $k \geq 2$, then clearly

$$t(n, k) \geq \sum_{i=1}^{n} t(s_i, k-1) ,$$

where $(s_1, s_2, \ldots, s_n)$ denotes the score vector of the tournament T_n. Let us suppose that $t(s_i, k-1) \geq \tau(s_i, k-1)$; since $\tau(n, k)$ is a convex function of n for fixed values of k we may apply Jensen's inequality and conclude that

$$t(n, k) \geq \sum_{i=1}^{n} \tau(s_i, k-1) \geq n\,\tau(\tfrac{1}{2}(n-1), k-1) = \tau(n, k).$$

The theorem now follows by induction on k.

We remark in closing that it can be shown that the distribution of the number of transitive subtournaments T_k in a random tournament T_n is asymptotically normal with mean

$$\mu' = (n)_k \, 2^{-\binom{k}{2}},$$

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and variance

$$\sigma^2 = (k!\, 2^{-\binom{k}{2}})^2 \sum_{r=3}^{k} \binom{n}{k}\binom{k}{r}\binom{n-k}{k-r}\left(\frac{1}{r!}\, 2^{\binom{r}{2}} - 1\right)$$

for each fixed value of k greater than two.

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