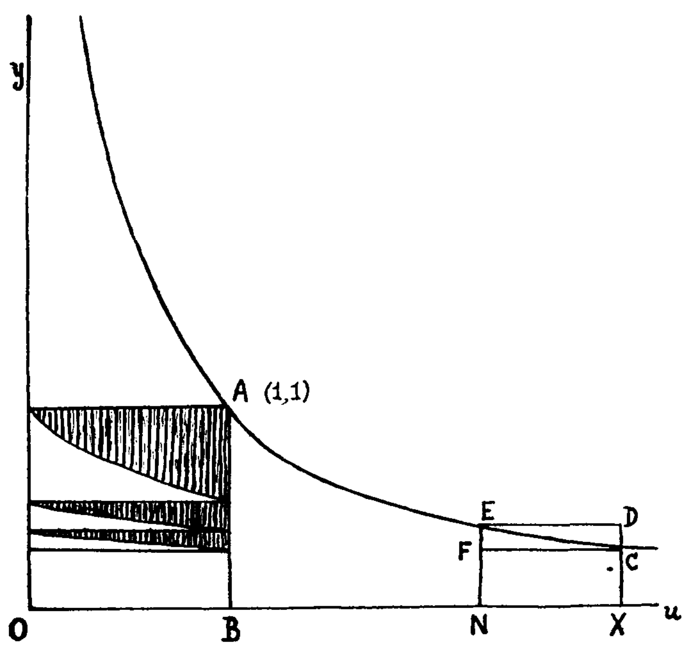
An Elementary Derivation of the Exponential Limit and of Euler's Constant

By H. W. TURNBULL.

By defining a logarithm as

$$\log a = \int_1^a \frac{du}{u}, \quad a > 0, \tag{1}$$

we may visualise the function as the area under the curve



$y = \dfrac{1}{u}$, $u > 0$, measured to the right from the zero value at the ordinate AB. The fundamental properties follow at once from (1): for if $u = av$, then

$$\log b = \int_1^b \frac{dv}{v} = \int_a^{ab} \frac{du}{u} = \int_1^{ab} \frac{du}{u} - \int_1^a \frac{du}{u} = \log ab - \log a.$$

Also if $u = v^n$, we get

$$\log a^n = \int_1^{a^n} \frac{du}{u} = \int_1^a n \frac{dv}{v} = n \log a.$$

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Now consider the area $ECXN$ below the arc EC, where $ON = n$, $OX = n + x$, so that

$$NE = \frac{1}{n} = XD, \quad XC = \frac{1}{n+x} = NF.$$

This area is evidently $\log(n+x) - \log n$. It is also equal to the rectangle $NXDE$ less the curvilinear triangle ECD, which is more than half the rectangle $EDCF$ $\left(\text{since } \frac{d^2 y}{du^2} > 0\right)$. Thus

$$\log(n+x) - \log n = \frac{x}{n} - \theta\left(\frac{1}{n} - \frac{1}{n+x}\right)x,$$

or

$$n \log \frac{n+x}{n} = x - \frac{\theta x^2}{n+x},$$

where $\frac{1}{2} < \theta < 1$.

Now let the inverse function $\exp b = a$ of the logarithmic function $b = \log a$ be defined; and we have at once

$$\left(1 + \frac{x}{n}\right)^n = \exp\left(x - \frac{\theta x^2}{n+x}\right), \quad \tfrac{1}{2} < \theta < 1,$$

for all $x > 0$ and for positive integral values of n. Hence

$$\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = \exp x.$$

Similarly if $x < 0$, provided that $n + x > 0$. But this last condition does not affect the passage to the limit.

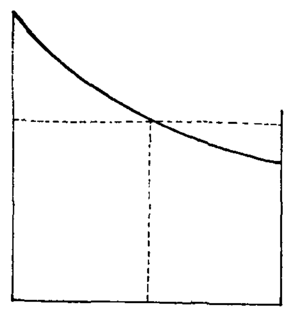
Also, by taking $x = 1$, $n = 1, 2, 3, \ldots, p$, it follows that the expression

$$S = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{p} - \log(p+1)$$

is given by the sum of p such curvilinear triangles EDC standing above consecutive sections of arc starting at A. This sum increases with p since each triangle has a positive area. Also if the triangles are moved parallel to the axis of u they can all be assembled within the rectangle OBA whose area is unity. The *total* shaded area exceeds the unshaded at every value of p. Hence S tends to a limit γ between $\frac{1}{2}$ and 1 as p tends to infinity. The addition of a term $\frac{1}{p+1}$ to S leaves the limit unchanged: and this proves Euler's theorem

$$\lim_{m \to \infty} \left(1 + \tfrac{1}{2} + \ldots + \frac{1}{m} - \log m\right) = \gamma.$$

By partitioning each strip into narrower strips further results may be obtained. For instance if each strip is divided into two parts of equal width, and the heights of the new ordinates are taken for those of the approximating rectangles, the triangular errors are now replaced by pairs of small triangles the differences of whose areas are required. In each pair the left area, since $\dfrac{d^2y}{dx^2} > 0$, is



greater than the right. For example, if the ordinates are $u = n$, $u = n + \frac{1}{2}$, $u = n + 1$, then

$$\log\left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}} = 1 + \frac{1}{8n\,(n+1)} - \tfrac{1}{4}\left(\frac{\epsilon}{n} + \frac{\epsilon'}{n+1}\right),$$

where $\epsilon > 0$, $\epsilon' > 0$ and both tend to zero when $n \to \infty$. It follows that

$$\left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}}$$

converges more rapidly to e than if the $\frac{1}{2}$ is omitted from the exponent. Similarly, if $x > 0$, it may be shewn that

$$\left(1 + \frac{x}{n}\right)^{n+x/2} = e^x + \frac{x^3}{8n\,(n+x)} \Big/ e^{\epsilon x^2/4n}$$

approximates more rapidly to e^x: and that

$$\gamma_p = 1 + \tfrac{1}{2} + \ldots\ldots + \frac{1}{p} - \log\left(p + \tfrac{1}{2}\right)$$

is a better approximation to γ. For example $\gamma_1 = \cdot 594\ldots$, $\gamma_2 = \cdot 583\ldots$, $\gamma_4 = \cdot 579\ldots$, while $\gamma = \cdot 577\ldots$.

Finally, let $ON = u$, $OX = v$ and $v = u^2 > 1$ in the original figure. Then, since the rectangle $EDXN > ECXN$,

$$\log v - \log u < \frac{v-u}{u}.$$

But $\log u = \frac{1}{2}\log v$. Hence, if $v > 1$,

$$\log v < 2\sqrt{v} - 2,$$

that is

$$\frac{\log v}{v} < \frac{2}{\sqrt{v}} - \frac{2}{v} \to 0 \text{ as } v \to \infty .$$

Put $v = x^{\alpha}$, $\alpha > 0$. Then also

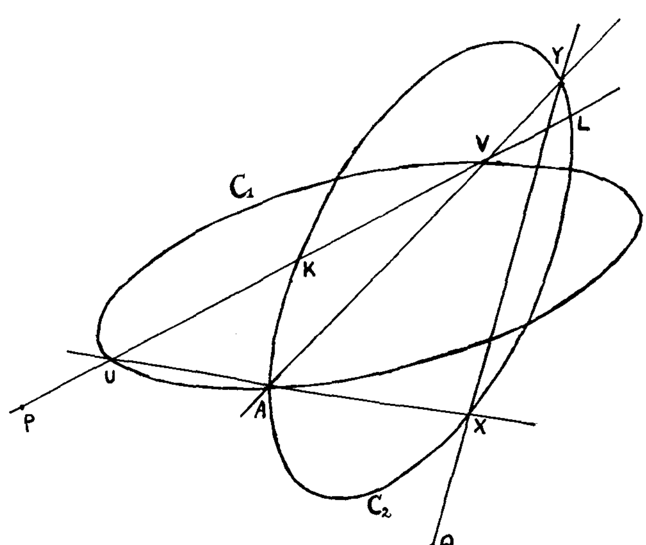
$$\frac{\log x}{x^{\alpha}} \to 0 \text{ as } x \to \infty .$$

This shows that $\log x$ tends to infinity $\left(\text{with } \Sigma \frac{1}{n}\right)$ more slowly than any positive power of x, as x tends to infinity.

A Synthetic Derivation of the Class of the Φ Conic

By H. E. Daniels.

The theorem that a line cutting a pair of conics in four harmonically separated points envelopes a conic, called the Φ conic, is a familiar result which admits of a simple proof by analytical methods. A synthetic proof, however, if we exclude the use of (2, 2) correspondences, is rather elusive. I have not been able to find such a proof in any book, and the only one published as far as I am aware is that set as a question in the 1934 Mathematical Tripos, due to Mr F. P. White. The proof written out below is rather more direct and may therefore be worth recording.



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