$$33 = (9 + \sqrt{9})\sqrt{9} - \sqrt{9}$$
$$= \frac{99}{\sqrt{9}} \times \cdot\dot{9}$$
$$= \frac{99}{\sqrt{9}} \div \cdot\dot{9}$$
$$34 = \frac{99}{\sqrt{9}} + \cdot\dot{9}$$
$$35 = 9\sqrt{9} + 9 - \cdot\dot{9}$$
$$36 = 9 + 9 + 9 + 9$$
$$= \frac{99 + 9}{\sqrt{9}}$$

$$36 = 9\sqrt{9} + \sqrt{9 \times 9}$$
$$= 9\sqrt{9} + 9 \times \cdot\dot{9}$$
$$= 9\sqrt{9} + \frac{9}{\cdot\dot{9}}$$
$$= 9(\sqrt{9} + \cdot\dot{9}\dot{9})$$
$$37 = 9\sqrt{9} + 9 + \cdot\dot{9}$$
$$38 = 9\sqrt{9} + 9\cdot\dot{9} + \cdot\dot{9}$$

(Intractable)

$$39 = (9 + \sqrt{9})\sqrt{9} + \sqrt{9}$$
$$40 = (9 + \cdot\dot{9})(\sqrt{9} + \cdot\dot{9})$$

On the Solubility of Linear Algebraic Equations.—
(*a*) *It is proved in treatises on Algebra that the equations (in* three variables for brevity)

$$\left.\begin{aligned} a_1x + b_1y + c_1z + d_1 = 0, \\ a_2x + b_2y + c_2z + d_2 = 0, \\ a_3x + b_3y + c_3z + d_3 = 0, \end{aligned}\right\} \quad \text{..................... (1)}$$

have a unique solution given by

$$x \begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix} + \begin{vmatrix} d_1, & b_1, & c_1 \\ d_2, & b_2, & c_2 \\ d_3, & b_3, & c_3 \end{vmatrix} = 0; \text{ etc., } \text{.........(2)}$$

provided the determinant

$$\Delta \equiv \begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix} \text{.......................(3)}$$

does not vanish.

(*b*) It is also proved, from (2), that if the "degenerate" homogeneous system

$$\left.\begin{aligned} a_1x + b_1y + c_1z = 0, \\ a_2x + b_2y + c_2z = 0, \\ a_3x + b_3y + c_3z = 0, \end{aligned}\right\} \text{........................(4)}$$

has a *non-null* solution (*i.e.* a solution in which the variables are not all zero), then $\Delta = 0$.

(*c*) The converse of the latter theorem (viz., that, *if* $\Delta = 0$, *the system* (4) *has a non-null solution*), is true, and frequently required; but it is not so easy to prove as the direct theorem (*b*), and indeed in many current text-books it is not proved at all.

(*d*) For we must distinguish between a formal algebraical solution and an arithmetical solution.

Consider, for example, the theorem:

Two homogeneous linear equations in three variables have always at least one non-null solution.

Let the equations be

$$\left.\begin{aligned} a_1x + b_1y + c_1z = 0, \\ a_2x + b_2y + c_2z = 0. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Here it is not enough to point to the solution

$$(b_1c_2 - b_2c_1,\ c_1a_2 - c_2a_1,\ a_1b_2 - a_2b_1),$$

for these values of x, y, z may all vanish.

And in fact it does not seem possible to write down a formal solution which will be non-null whatever the special numerical values of the coefficients may be.

All the same the theorem is both true and important.

(*e*) The solution of many problems, and the proof of many theorems, in Algebra, Coordinate Geometry, Higher Analysis, and Mathematical Physics can be made to depend on the solution of a system of n non-homogeneous linear algebraic equations in n variables. In special cases special methods may be more instructive, and sometimes easier, but the wide applicability of this method of "undetermined coefficients" makes it an extremely valuable weapon for a student to have in his armoury.

It is of course always necessary to prove that the system of equations is "independent and consistent," or, in other words, that it has a unique solution. To the student familiar with determinants it suggests itself at once that this is equivalent, in accordance with (*a*), to proving that the determinant of the equations does not vanish. As a rule, however, a direct proof of the non-evanescence of the determinant is impracticable. But he knows also, or assumes as the converse of the theorem (*b*), that, if the "degenerate" system has no non-null solution, Δ cannot vanish, this being equivalent to (*c*). He may therefore complete the process by proving the "inconsistency" of the degenerate system, as it is usually easy to do.

(*f*) Now, apart from the point about the theorem (*c*) being often left unproved, it is evident that in the above train of thought the determinant is introduced quite adventitiously, and it seems desirable to have a proof which brings the unique solubility of the non-homogeneous system into immediate relationship with the inconsistency of the degenerate system.

The matter given below would make a useful and instructive lesson in algebraic principles for a class who have no acquaintance with determinants.

THEOREM I.

A system of $n-1$ *homogeneous linear equations in* n *variables has at least one non-null solution.*

(i) $n=2$. Let the equation be

$$ax+by=0 \qquad (6)$$

If $b=0$, $(0, 1)$ is a non-null solution; if $b \neq 0$, $(b, -a)$ is one.

(ii) $n=3$. Let the equations be

$$\begin{cases} a_1x+b_1y+c_1z=0, & (7) \\ a_2x+b_2y+c_2z=0. & (8) \end{cases}$$

If c_1 and c_2 are both zero, $(0, 0, 1)$ is a non-null solution.

If not, say $c_1 \neq 0$. The equations are then "equivalent" to

$$\begin{cases} a_1x+b_1y+c_1z=0, & (9) \\ c_1(a_2x+b_2y+c_2z)-c_2(a_1x+b_1y+c_1z)=0 & (10) \end{cases}$$

(10) is an equation in the two variables x and y which by (i) has at least one non-null solution (X, Y); and, since $c_1 \neq 0$, (9) gives a definite value Z for z when X, Y are put for x, y.

(X, Y, Z) is a non-null solution of (7), (8).

(iii) $n=4$. Let the equations be

$$\begin{cases} a_1x+b_1y+c_1z+d_1w=0, & (11) \\ a_2x+b_2y+c_2z+d_2w=0, & (12) \\ a_3x+b_3y+c_3z+d_3w=0. & (13) \end{cases}$$

If d_1, d_2 and d_3 are all zero, $(0, 0, 0, 1)$ is a non-null solution.

If not, say $d_1 \neq 0$. The equations are then equivalent to

$$\begin{cases} a_1x+b_1y+c_1z+d_1w=0, & \ldots\ldots(14) \\ d_1(a_2x+b_2y+c_2z+d_2w)-d_2(a_1x+b_1y+c_1z+d_1w)=0, & \ldots\ldots(15) \\ d_1(a_3x+b_3y+c_3z+d_3w)-d_3(a_1x+b_1y+c_1z+d_1w)=0. & \ldots\ldots(16) \end{cases}$$

(15), (16) are two equations in the three variables x, y, z which by (ii) have at least one non-null solution (X, Y, Z); and since $d_1 \neq 0$, (14) gives a definite value W for w when X, Y, Z are put for x, y, z.

(X, Y, Z, W) is a non-null solution of (11), (12), (13).

It is obvious that we can proceed step by step in this way indefinitely. Hence the theorem.

Remark.—In (iii), (λX, λY, λZ, λW) is a non-null solution, where λ is arbitrary. It is not necessarily, however, the most general non-null solution. Similarly in the case of n variables.

Example 1. *Theorem.—At least one homogeneous linear relation subsists among $n+1$ homogeneous linear functions of n variables.*

For the relation

$$\lambda(a_1x+b_1y+c_1z)+\mu(a_2x+b_2y+c_2z)+\nu(a_3x+b_3y+c_3z) \\ +\rho(a_4x+b_4y+c_4z)\equiv 0 \quad (17)$$

is equivalent to the three equations

$$\lambda a_1+\mu a_2+\nu a_3+\rho a_4=0, \text{ etc.},$$

which by Theorem I. has always at least one solution in which λ, μ, ν, ρ are not all zero.

Example 2. *Theorems.—If a homogeneous linear relation subsists among n homogeneous linear functions of n variables, then values of the variables not all zero exist for which all the functions vanish; and conversely.*

For the *direct* theorem, suppose

$$\lambda(a_1x+b_1y+c_1z)+\mu(a_2x+b_2y+c_2z)+\nu(a_3x+b_3y+c_3z)\equiv 0, \quad (18)$$

where λ, μ, ν are not all zero, say $\nu \neq 0$.

Take x, y, z, not all zero, so that (*Th.* I.)

$$\begin{cases} a_1x+b_1y+c_1z=0, \\ a_2x+b_2y+c_2z=0. \end{cases}$$

Since $\nu \neq 0$, (18) then gives $a_3x+b_3y+c_3z=0$.

For the *converse*, suppose

$$\left.\begin{aligned} a_1X + b_1Y + c_1Z = 0, \\ a_2X + b_2Y + c_2Z = 0, \\ a_3X + b_3Y + c_3Z = 0, \end{aligned}\right\} \qquad (19)$$

where X, Y, Z are not all zero, say $Z \neq 0$.

As in *Ex.* 1, take λ, μ, ν, ρ so that

$$\lambda(a_1x + b_1y + c_1z) + \mu(a_2x + b_2y + c_2z) + \nu(a_3x + b_3y + c_3z) + \rho z \equiv 0. \quad (20)$$

Put X, Y, Z for x, y, z in (20) and we find $\rho = 0$.

Example 3. *Theorem.—In Cartesian coordinates, at least one curve whose equation is of the form*

$$a(x^2 + y^2) + 2gx + 2fy + c = 0 \qquad (21)$$

passes through any three points; where a, g, f *are not all zero.*

Observe that we cannot say without qualification that there is only one such curve, even if we premise that the three given points are distinct; *e.g.* the three points $(-iy_1, y_1)$, $(-iy_2, y_2)$, $(-iy_3, y_3)$ lie on the curve $x^2 + y^2 + \lambda(x + iy) = 0$, where the coefficient λ is arbitrary.

(*To be continued.*) See P. 137

JOHN DOUGALL

Notes on Algebraic Inequalities.—

1. It is worthy of remark that there is a method of proving algebraic inequalities which is very generally applicable, and which furnishes proofs of great directness and completeness.

The method consists in *expressing the difference* $A - B$ *in a manifestly positive form*, when we have to prove $A > B$.

The first inequality occurring in school Algebra is usually proved in this way. We have to show that if a, b, and x are positive quantities $\frac{a+x}{b+x}$ is nearer to 1 than $\frac{a}{b}$. We have

$$\frac{a+x}{b+x} - \frac{a}{b} = \frac{x(b-a)}{b(b+x)}.$$

The two cases $a > b$ and $b > a$ are then considered separately, and the conclusion drawn.

https://doi.org/10.1017/S1757748900001018 Published online by Cambridge University Press